Class on 24 Aug:

QR Factorization:

Thm Let $A \in M_n(\mathbb{R})$ be invertible. Then there exists an orthogonal matrix Q and an upper triangular matrix R s.t. $A = QR$. This decomposition is unique upto scaling of rows by ± 1. (may not be unique when A is non-singular).

Pf: Uniqueness: Let there be two decompositions Q_1, R_1 and Q_2, R_2.

$$Q_1 R_1 = Q_2 R_2$$

$$\underbrace{Q_2^{-1} Q_1}_{\text{orthogonal}} = \underbrace{R_2 R_1^{-1}}_{\text{upper } \Delta^r} = T$$

This is only possible when the matrix T is diagonal with each diagonal entry ± 1.

Existence: Let $\vec{a_1}, \dots, \vec{a_n}$ be column vectors of A.

$\vec{u_1} := \vec{a_1}$, $\quad \vec{u_2} := \vec{a_2} - (\text{projection of } \vec{a_2} \text{ onto } \vec{u_1})$

$\vec{u_3} := \vec{a_3} - (\text{proj. of } \vec{a_3} \text{ onto } \vec{u_2} + \text{proj. of } \vec{a_3} \text{ onto } \vec{u_3})$

Let
$$\vec{u_1} = \vec{a_1}$$
$$\vec{u_2} = \vec{a_2} - r_{12}\vec{u_1} \cancel{- r_{23}\vec{u_2}}$$
$$\vdots$$
$$\vec{u_n} = \vec{a_n} - r_{1n}\vec{u_1} - r_{2n}\vec{u_2} - \dots - r_{n-1,n}\vec{u_{n-1}}$$

$\therefore$
$$\vec{a_1} = \vec{u_1}$$
$$\vec{a_2} = \vec{u_2} + r_{12}\vec{u_1}$$
$$\vec{a_3} = \vec{u_3} + r_{13}\vec{u_1} + r_{23}\vec{u_2}$$
$$\vdots$$
$$\vec{a_n} = \vec{u_n} + r_{1n}\vec{u_1} + \dots + r_{n-1,n}\vec{u_{n-1}}$$

In matrix form:

$$[\vec{a_1} | \vec{a_2} | \dots | \vec{a_n}] = [\vec{u_1} | \vec{u_2} | \dots | \vec{u_n}] \begin{bmatrix} 1 & r_{12} & r_{13} & \dots & r_{1n} \\ 0 & 1 & r_{23} & & \vdots \\ \vdots & & 1 & \ddots & r_{n-1,n} \\ 0 & & & & 1 \end{bmatrix}$$

$$= \underbrace{\left[\frac{\vec{u_1}}{\|\vec{u_1}\|} \middle| \frac{\vec{u_2}}{\|\vec{u_2}\|} \middle| \dots \middle| \frac{\vec{u_n}}{\|\vec{u_n}\|} \right]}_{Q} \underbrace{\begin{bmatrix} \|\vec{u_1}\| & \|\vec{u_1}\| r_{12} & \|\vec{u_1}\| r_{13} & \dots \\ 0 & \|\vec{u_2}\| & \|\vec{u_2}\| r_{23} & \\ & 0 & \|\vec{u_3}\| & \\ & & & \ddots \\ & 0 & & & \|\vec{u_n}\| \end{bmatrix}}_{R}$$

*Disadvantages of Gram-Schmidt process:
- Small imprecisions in the calculation of inner products can accumulate and lead to 'effective' loss of orthogonality.

Other Q-R Decompositions:

Find a sequence of Q_i's st $\underbrace{Q_k \cdots Q_1}_{\text{Precisely the transpose of the 'Q' we are looking for.}} A = R$.

We can find these from a sequence of "row" operations which eliminate some entries below the main diagonal.

Two types of orthogonal transformations: rotations and reflections[1]

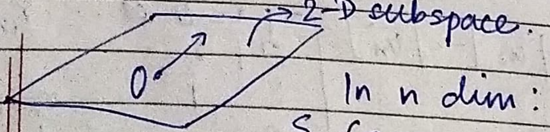
Givens Rotations:

$$\Omega^{[i,j]} := \begin{array}{c} \\ 1 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & \cdots & i & \cdots & j & \cdots & & n \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & & & & & & \\ \vdots & & \ddots & & & & & \\ 0 & \cdots & & \cos\theta & \cdots & \sin\theta & & \\ \vdots & & & 0 & & 0 & & \\ 0 & \cdots & & -\sin\theta & & \cos\theta & & \\ & & & 0 & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} \end{array}$$

This represents a rotation of span $[e_i, e_j]$.

$$\left[\begin{array}{c|ccc} a_{11} & a_{12} & & \\ a_{21} & \vdots & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{array}\right]$$

If $a_{11} \neq 0$ (using permutations)



In n dim:

$\{(0, x_2, \dots, x_n) : x_i \in \mathbb{R},\ 2 \leq i \leq n-1\}$.

is an $n-1$ dimensional subspace

[1] This distinction is clear in 3X3 matrices, but in higher dimensions ~~they~~ it becomes unclear.

We want θ s.t. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \end{bmatrix}$

gives $a_{11}\cos\theta + a_{21}\sin\theta = 0 \Rightarrow \cot\theta = \dfrac{-a_{21}}{a_{11}}$.

This method is useful for sparse matrices.
(Since you have to find Q_i for each non-zero entry below the main diagonal, $\sim\binom{n}{2}$ calculations. - this number is much lesser for sparse matrices).

Householder Reflections:

Take $\vec{v} \neq \vec{0} \in \mathbb{R}^n$.

Reflection along $\underbrace{[\vec{v}]}_{\text{Span of } \vec{v}}$ takes $\vec{v}$ to $\vec{v}$,
$\vec{x} \in \vec{v}^\perp$ to $-\vec{x}$.

Similarly, reflection along $[\vec{v}^\perp]$ takes $\vec{v}$ to $-\vec{v}$,
$\vec{x} \in \{\vec{v}\}^\perp$ to $\vec{x}$.

$$H_{\vec{v}} = I - \frac{2\vec{v}\vec{v}^T}{\|\vec{v}\|^2}$$

Fact :- S is a reflection iff $\dfrac{I+S}{2}$ is a projection.

- Given a reflection operator S, you can recover the subspace along which reflection occurs by looking at the eigenspace corresponding to eigenvalue 1.
- Reflections are self-adjoint.

Thm: Let $R \in M_n(\mathbb{R})$. The R is a reflection iff $\dfrac{I+R}{2}$ is a projection.

Pf: $R^2 = I \Leftrightarrow \left(\dfrac{I+R}{2}\right)^2 = \dfrac{I+R^2+2R}{4} = \dfrac{I+R}{2}$.

Assume $\|\vec{v}\| = 1$. Then:

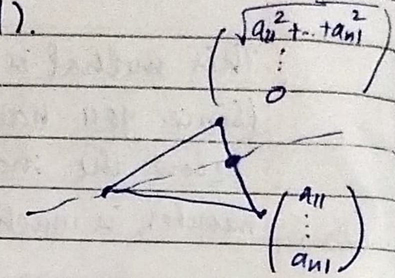
Projection onto $\{\vec{v}\}^\perp = I - \vec{v}\vec{v}^T$ ~~$(\|\vec{v}\|=1)$~~.

Reflection about $\{\vec{v}\}^\perp = 2(I - \vec{v}\vec{v}^T) - I$

$= I - 2\vec{v}\vec{v}^T = H_{\vec{v}}$

We want Q_i's s.t. $Q_k \cdots Q_1 A = R$.
Find $H_{\vec{v}}$ to convert $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$ to $\begin{bmatrix} r_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ---(2)

(If this is possible then $|r_1| = \| \vec{a}_{*1} \|$).



Choose $\vec{v_1} = \begin{bmatrix} a_{11} + \sqrt{a_{11}^2 + a_{21}^2 + \dots + a_{n1}^2} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$

$$H_{\vec{v_1}} = Q_{11} = I - \frac{2\vec{v_1}\vec{v_1}^T}{\|\vec{v_1}\|^2}.$$

Once a_{*1} is converted to the form stated in (2), then $H_{\vec{v_2}}$ is chosen for:

$$\begin{bmatrix} r_1 & * & * & * \\ 0 & \boxed{\begin{matrix} r_2 \\ 0 \\ \vdots \\ 0 \end{matrix}} & & \\ \end{bmatrix}$$

converting a smaller column to the form $\begin{bmatrix} r_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n-1 \times 1}$.

$\therefore H_{\vec{v_n}} \cdots H_{\vec{v_1}} A = R$.
You prove this by induction.

- This method is good for non-sparse matrices too.

Cor: Every orthogonal matrix on $M_n(\mathbb{R})$ can be decomposed into the product of 'n' reflections. (Cartan–Dieudonné Theorem)

In the complex case: $R^2 = I$, $R = R^*$ hold.
Can we write unitary matrices as the product of 'n' reflections?
Take $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = R_1 \dots R_n$. ($\omega$: cube root of unity)

det of RHS $= \pm 1$, det of LHS $= \omega^2$ which ~~and~~ are not equal.

Thm (Radjavi): A unitary matrix U in $M_n(\mathbb{C})$ is the product of four reflections iff $\det(U) = \pm 1$.
(says that the only obstruction to writing a unitary matrix as a product of reflections is the one imposed by the det).

Thm (Halmos-Kakutani): Every unitary operator in $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$: infinite dimensional) can be decomposed as the product of four reflections.

Exc. Let $\alpha: X \to X$ be a bijection. Then there are bijections β_1, β_2 s.t. $\beta_1^2 = Id$, $\beta_2^2 = Id$ (called involutions) and $\alpha = \beta_1 \circ \beta_2$.

Take G: locally compact (topological) group
(loc compact: means every pt has a nhbd whose closure is compact).

Examples:
1. $G = (\mathbb{R}^n, +)$ with Euclidean topology.
2. $G = O_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ with subsp. top.
3. $G = (\mathbb{Z}, +)$ with discrete top.
4. $G = (S^1, \circ)$ with subsp. top.

Defn: A Radon measure μ on the Borel σ-algebra of G is said to be a left-invariant Haar measure if $\mu(g \cdot E) = \mu(E)$ for all measurable sets E in σ_G.

Thm: Every locally compact group has a left-invariant measure (unique upto positive scalar multiples).

Examples: 3. $G = (\mathbb{Z}, +)$ with discrete top: the counting measure is a left-invariant Haar measure.
4. $G = (S^1, \circ)$ with subsp. top: Lebesgue measure

Uniform Sampling from G:
If $S = \{g_1, \dots, g_n\}$ is a sample, then it is "uniform" if

$$\#(E \cap S) \approx \#(g \cdot E \cap S),$$

where E is a nhbd of identity in G.