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Rotations and quaternions: $Q \cong \mathbb{R}^4$ as a real vector space.

Prove that if $\gamma_1, \gamma_2 \in Q$, then $\overline{\gamma_1 \gamma_2} = \overline{\gamma_2}\,\overline{\gamma_1}$.

$\alpha = \gamma(xi + yj + zk)\overline{\gamma}$ with $|\gamma| = 1$ is a pure quaternion:

$\overline{\alpha} = \gamma(-xi - yj - zk)\overline{\gamma} = -\alpha$. ie., $\alpha + \overline{\alpha} = 0$. Hence α is a pure quater-nion.

Theorem: Suppose $\gamma = a + bi + cj + dk$ with $\gamma \neq 0$ and $|\gamma| = 1$. Then, R_γ is a rotation about the axis determined by the vector (b,c,d) with angle of rotation $\theta = 2\cos^{-1} a = 2\sin^{-1}\sqrt{b^2+c^2+d^2}$.

Proof: Step 1: R_γ preserves norm: $|R_\gamma(x)| = |x|$.

Step 2: R_γ has eigenvector (b,c,d) with eigenvalue 1.

$R_\gamma(bi + cj + dk) = \gamma(bi + cj + dk)\overline{\gamma} = bi + cj + dk \longrightarrow ①$

Since ① holds iff γ commutes with $(bi + cj + dk)$ but this holds as $a \in Z(Q)$ and $\pm(bi + cj + dk)$ commutes with $bi + cj + dk$.

Step 3: Compute angle of rotation

Case 1: at least one of $b, c \neq 0$.

Let $\vec{w} = ci - bj + 0k$. Then, $\vec{w} \perp (b,c,d)$.

Case 2: If $b = c = 0$ (and $d \neq 0$. Note if $b = c = d = 0$, then $a = \pm 1$, in which case R_γ = identity).

$$\cos\theta = \frac{\vec{w} \cdot R_\gamma \vec{w}}{|w|^2} \quad \text{(due to step 1)}$$

$$= a^2 - b^2 - c^2 - d^2 = 2a^2 - 1. \Rightarrow a^2 = \frac{1+\cos\theta}{2} = \cos^2(\theta/2)$$

$$\Rightarrow \cos(\theta/2) = a$$

$$\Rightarrow \theta = 2\cos^{-1} a.$$

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Remark: $S^3 \subseteq \mathbb{R}^4$ with the operation of quaternian multiplication satisfies the axioms of a group. Let $SO(3)$ denote the set of rotations on 3-space $\mathbb{R}^3$. $\varphi: S^3 \longrightarrow SO(3)$, $\gamma \longmapsto R_\gamma$ has kernel $= \{1, -1\}$, and hence $S^3/\{1,-1\} \cong SO(3)$ since φ is surjective.

S^3 is a double cover of $SO(3)$.

<u>The Hopf map / Hopf fibration:</u> $h : S^3 \longrightarrow S^2$

$(a,b,c,d) \mapsto (a^2+b^2-c^2-d^2,\ 2(ad+bc),\ 2(bd-ac))$

$\gamma \mapsto R_\gamma\,(1,0,0)$

<u>Exercise</u>: Show that the two descriptions are the same.

Hopf map is the generator of the 3^{rd} homotopy gp of S^2.

<u>Question</u>: What is the pre-image of $(1,0,0)$ under h?

<u>Ans</u>: $(\cos\theta, \sin\theta, 0, 0)$.

<u>Exercise</u>: Show that pre-image of any point in S^2 under h is a unit circle on S^3.

$$\begin{tikzcd} & S^1 \times S^2 \arrow[d, "\pi"] \\ S^2 \arrow[r, "id"] \arrow[ur, dashed, "x \mapsto (1,x)"] & S^2 \end{tikzcd} \qquad \begin{tikzcd} & S^3 \arrow[d, "h"] \\ S^2 \arrow[r, "id"] \arrow[ur, dashed, "\varphi\,??"] & S^2 \end{tikzcd} \qquad H_k(S^n) = \begin{cases} \mathbb{Z} \ ; & k = n \\ 0 \ ; & k \neq n. \end{cases}$$

$V_1(\mathbb{C}^2) \xrightarrow[h]{\text{Hopf map}} Gr_1(\mathbb{C}^2)$

looks like S^3 looks like S^2.

S^2 is homeomorphic to $Gr_1(\mathbb{C}^2)$. $x = (z,\theta) \in S^2$ (in cylindrical coordinates)

$-1 \le z \le 1$, $\theta \in S^1$

$$i(z,\theta) = \frac{1}{2}\begin{pmatrix} 1+z & \theta(1-z^2)^{1/2} \\ \bar{\theta}(1-z^2) & 1-z \end{pmatrix}$$

$$\begin{tikzcd} & S^3 \arrow[r, tail] \arrow[d, "h"] & V_1(\mathbb{C}^2) \arrow[d, "h"] \\ S^2 \arrow[r, "id"] \arrow[ur, dotted, bend left] & S^2 \arrow[r, dotted] \arrow[ur, dotted] & Gr_1(\mathbb{C}^2) \end{tikzcd}$$

X - Hausdorff topological space.

$X = S^2 \longrightarrow i(z,\theta)$ is a rank 1 projection matrix

(1) No multiplicity obstruction

(2) No covering space obstruction.

Singular Value Decomposition:

Theorem: If A is a real $m\times n$ matrix, then there exist orthogonal matrices $U = [u_1|\cdots|u_m] \in O_m(\mathbb{R})$ and $V = [v_1|\cdots|v_n] \in O_n(\mathbb{R})$ s.t. $U^TAV = \Sigma = \operatorname{diag}(\sigma_1,\dots,\sigma_p)$ where $p = \min\{m,n\}$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$.

$\hookrightarrow$ ($m\times n$ matrix)

Proof: A^TA is positive semidefinite $n\times n$ matrix. So $\exists\, V \in O_n(\mathbb{R})$ s.t.
$$V^T(A^TA)V = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & & O \\ & \ddots & \\ O & & \sigma_n^2 \end{pmatrix}$$

$$\Rightarrow (A^TA)[v_1|\cdots|v_n] = [v_1|\cdots|v_n]\begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix} = [\sigma_1^2 v_1|\cdots|\sigma_n^2 v_n].$$

$$\Rightarrow (A^TA)v_i = \sigma_i^2 v_i.$$

Consider $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Let $u_i = \frac{1}{\sigma_i} Av_i$.

Then, $\langle u_i, u_i\rangle = \frac{1}{\sigma_i^2}\langle Av_i, Av_i\rangle = \frac{1}{\sigma_i^2}\langle A^TAv_i, v_i\rangle = 1.$

$\langle u_i, u_r\rangle = \frac{1}{\sigma_i\sigma_r}\langle A^TAv_i, v_r\rangle = \frac{1}{\sigma_r}\langle v_i, v_r\rangle = 0$. since $V \in O_n(\mathbb{R})$

ie, $Av_i = \sigma_i u_i$. Hence, $AV = U\operatorname{diag}(\sigma_1,\dots,\sigma_p)$.

$\Rightarrow U^TAV = \operatorname{diag}(\sigma_1,\dots,\sigma_p)$ where σ_1^2 = largest eigen value of A^TA
σ_2^2 = 2^{nd} largest eigen value of A^TA.
$\vdots$
and if $m \geq k > n$, $\sigma_k^2 = 0$.

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Some applications: $f \in L^2(\mathbb{R}^n)$, $\tau_a : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, translation

- differentiation is the infinitesimal generator of translation.

$$\frac{f(x+h) - f(x)}{h} = \frac{\tau_{-h}f - f}{h}.$$

Theorem: (Eckhart – Young, 1936) Let $A = U\Sigma V^T$ be the SVD of A. $\Sigma_i = \operatorname{diag}(0,0,\dots,\sigma_i,0,\dots 0)$ where σ_i is in the i^{th} position. Then,

(1) $U\Sigma_1V^T$ is the best rank 1 approximation of A.
ie, $\|A - U\Sigma_1V^T\| \leq \|A - X\|$ $\forall\, X$ of rank 1.

(2) $U\Sigma_1V^T + \cdots + U\Sigma_rV^T$ $(1 \leq r \leq \operatorname{rank}(A))$ is the best rank r approximation of A. ie, $\|A - U\Sigma_1V^T - \cdots - U\Sigma_rV^T\| \leq \|A - X\|$ $\forall\, X$ of rank r.