

12/8/22

Rotations and quaternions:  $Q \cong \mathbb{R}^4$  as a real vector space.

Prove that if  $r_1, r_2 \in Q$ , then  $\overline{r_1 r_2} = \overline{r_2} \overline{r_1}$ .

$\alpha = r(x_i + y_j + z_k)\overline{r}$  with  $|r|=1$  is a pure quaternion:

$\overline{\alpha} = r(-x_i - y_j - z_k)\overline{r} = -\alpha$ . i.e.,  $\alpha + \overline{\alpha} = 0$ . Hence  $\alpha$  is a pure quaternion.

Theorem: Suppose  $r = a + bi + cj + dk$  with  $r \neq 0$  and  $|r|=1$ . Then,  $R_r$  is a rotation about the axis determined by the vector  $(b, c, d)$  with angle of rotation  $\theta = 2\cos^{-1}a = 2\sin^{-1}\sqrt{b^2 + c^2 + d^2}$ .

Proof: Step 1:  $R_r$  preserves norm:  $|R_r(x)| = |x|$ .

Step 2:  $R_r$  has eigenvector  $(b, c, d)$  with eigenvalue 1.

$$R_r(bi + cj + dk) = r(bi + cj + dk)\overline{r} = bi + cj + dk \rightarrow \textcircled{1}$$

since  $\textcircled{1}$  holds iff  $r$  commutes with  $(bi + cj + dk)$  but this holds as  $a \in Z(Q)$  and  $\pm(bi + cj + dk)$  commutes with  $bi + cj + dk$ .

Step 3: Compute angle of rotation

Case 1: at least one of  $b, c \neq 0$ .

Let  $\vec{w} = ci - bj + dk$ . Then,  $\vec{w} \perp (b, c, d)$ .

Case 2: If  $b = c = 0$  (and  $d \neq 0$ ). Note if  $b = c = d = 0$ , then  $a = \pm 1$ , in which case  $R_r = \text{identity}$ .

$$\cos \theta = \frac{\vec{w} \cdot R_r \vec{w}}{|\vec{w}|^2} \quad (\text{due to step 1})$$

$$= \frac{a^2 - b^2 - c^2 - d^2}{2} = 2a^2 - 1. \quad \Rightarrow a^2 = \frac{1 + \cos \theta}{2} = \cos^2(\theta/2)$$

$$\Rightarrow \cos(\theta/2) = a$$

$$\Rightarrow \theta = 2\cos^{-1}a.$$

\* ————— \*

Remark:  $S^3 \subseteq \mathbb{R}^4$  with the operation of quaternion multiplication satisfies the axioms of a group. Let  $SO(3)$  denote the set of rotations on 3-space  $\mathbb{R}^3$ .  $\varphi: S^3 \rightarrow SO(3)$  has kernel =  $\{1, -1\}$ , and hence  $S^3 / \{1, -1\} \cong SO(3)$  since  $\varphi$  is surjective.

$S^3$  is a double cover of  $SO(3)$ .

The Hopf map / Hopf fibration:  $h: S^3 \rightarrow S^2$   
 $(a, b, c, d) \mapsto (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac))$   
 $r \mapsto \mathbb{R}_v(1, 0, 0)$

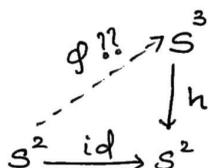
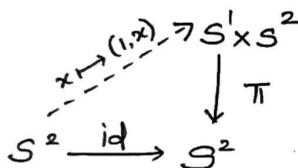
Exercise: Show that the two descriptions are the same.

Hopf map is the generator of the 3<sup>rd</sup> homotopy gp of  $S^2$ .

Question: What is the pre-image of  $(1, 0, 0)$  under  $h$ ?

Ans:  $(\cos \theta, \sin \theta, 0, 0)$ .

Exercise: Show that pre-image of any point in  $S^2$  under  $h$  is a unit circle on  $S^3$ .



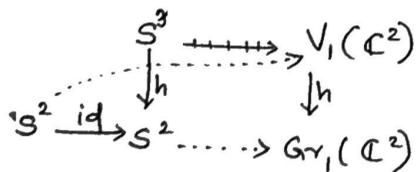
$$H_k(S^n) = \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n \end{cases}$$

$V_1(\mathbb{C}^2) \xrightarrow[h]{\text{Hopf map}} \text{Gr}_1(\mathbb{C}^2)$   
 looks like  $S^3$                       looks like  $S^2$

$S^2$  is homeomorphic to  $\text{Gr}_1(\mathbb{C}^2)$ .  $x = (z, \theta) \in S^2$  (in cylindrical coordinates)

$-1 \leq z \leq 1, \theta \in S^1$

$$i(z, \theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+z & \theta(1-z)^{1/2} \\ \bar{\theta}(1-z) & 1-z \end{pmatrix}$$



$X$  - Hausdorff topological space.

$X = S^2 \rightarrow i(z, \theta)$  is a rank 1 projection matrix

- (1) No multiplicity obstruction
- (2) No covering space obstruction.

## Singular Value Decomposition:

Theorem: If  $A$  is a real  $m \times n$  matrix, then there exist orthogonal matrices  $U = [u_1 | \dots | u_m] \in O_m(\mathbb{R})$  and  $V = [v_1 | \dots | v_n] \in O_n(\mathbb{R})$  s.t.  $U^T A V = \Sigma$ ,  $\text{diag}(\sigma_1, \dots, \sigma_p)$  where  $p = \min\{m, n\}$ , and  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$ .  
 $\hookrightarrow$  (max matrix)

Proof:  $A^T A$  is positive semidefinite  $n \times n$  matrix. So  $\exists V \in O_n(\mathbb{R})$  s.t.  $V^T (A^T A) V = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix}$   
 $\Rightarrow (A^T A) [v_1 | \dots | v_n] = [v_1 | \dots | v_n] \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix} = [\sigma_1^2 v_1 | \dots | \sigma_n^2 v_n]$   
 $\Rightarrow (A^T A) v_i = \sigma_i^2 v_i$ .

Consider  $A: \mathbb{R}^n \rightarrow \mathbb{R}^m$  let  $u_i = \frac{1}{\sigma_i} A v_i$ .

Then,  $\langle u_i, u_i \rangle = \frac{1}{\sigma_i^2} \langle A v_i, A v_i \rangle = \frac{1}{\sigma_i^2} \langle A^T A v_i, v_i \rangle = 1$ .

$\langle u_i, u_r \rangle = \frac{1}{\sigma_i \sigma_r} \langle A^T A v_i, v_r \rangle = \frac{1}{\sigma_r} \langle v_i, v_r \rangle = 0$ . since  $V \in O_n(\mathbb{R})$

ie,  $A v_i = \sigma_i u_i$ . Hence,  $AV = U \text{diag}(\sigma_1, \dots, \sigma_p)$ .

$\Rightarrow U^T A V = \text{diag}(\sigma_1, \dots, \sigma_p)$  where  $\sigma_1^2 =$  largest eigen value of  $A^T A$   
 $\sigma_2^2 =$  2<sup>nd</sup> largest eigen value of  $A^T A$ .  
and if  $m \geq k > n$ ,  $\sigma_k^2 = 0$ .

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Some applications:  $f \in L^2(\mathbb{R}^n)$ ,  $\tau_a: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ , translation.

• differentiation is the infinitesimal generator of translation.

$$\frac{f(x+h) - f(x)}{h} = \frac{\tau_{-h} f - f}{h}$$

Theorem: (Eckhart - Young, 1936) Let  $A = U \Sigma V^T$  be the SVD of  $A$ .

$\Sigma_i = \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)$  where  $\sigma_i$  is in the  $i^{\text{th}}$  position. Then,

(1)  $U \Sigma_1 V^T$  is the best rank 1 approximation of  $A$ .

ie,  $\|A - U \Sigma_1 V^T\| \leq \|A - X\| \forall X$  of rank 1.

(2)  $U \Sigma_1 V^T + \dots + U \Sigma_r V^T$  ( $1 \leq r \leq \text{rank}(A)$ ) is the best rank  $r$  approximation of  $A$ . ie,  $\|A - U \Sigma_1 V^T - \dots - U \Sigma_r V^T\| \leq \|A - X\| \forall X$  of rank  $r$ .