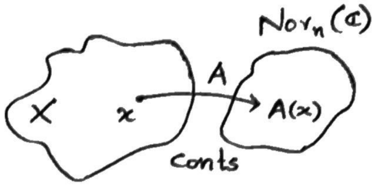
10/8/22

Question: Does there exist a φ satisfying the following diagram?

$$\begin{array}{ccc} Nor_n(\mathbb{C}) & \xrightarrow{\varphi} & \mathcal{U}_n(\mathbb{C}) \times \mathcal{D}_n(\mathbb{C}) & (U, D) \\ \uparrow A \text{ conts.} & & \downarrow & \downarrow \\ X \text{ (Hausdorff space)} & & Nor_n(\mathbb{C}) & UDU^* \end{array}$$



$$M_n(\mathbb{C}) \longrightarrow \mathbb{C}^n_{sym}$$

matrix $\longmapsto$ (unordered) n-tuple of eigenvalues.

<u>Example</u>: $A(\varepsilon) = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}$; $\varepsilon \geq 0$ and $A(-\varepsilon) = \begin{bmatrix} -\varepsilon/2 & -\varepsilon/2 \\ -\varepsilon/2 & -\varepsilon/2 \end{bmatrix}$; $\varepsilon > 0$

Eigenspace corresponding to 0 is $\left\langle\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\right\rangle$ and $\left\langle\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}\right\rangle$ respectively.

<u>Obstructions to diagonalization</u>:

1. Multiplicity: which allows eigenspaces of $A(x)$; $x \in X$ to have discontinuities.
2. Covering spaces: for the characteristic polynomial of $A(x)$.
3. Bundles over X: which may obstruct the conts. choice of eigenvectors. in prescribed eigenspace of $A(x)$.

<u>Remark</u>: Diagonalization meaning:

$$\begin{array}{ccc} & Nor_n(\mathbb{C}) & \\ {}^{A}\nearrow & & \searrow \\ X & \dashrightarrow & U_n(\mathbb{C}) \times D_n(\mathbb{C}) \end{array}$$

$U \in C(X, M_n(\mathbb{C}))$ is unitary means $UU^* = I = U^*U$ pointwise.

Consider $\chi_{A(x)}(t) = t^n - a_1(x)\,t^{n-1} + \dots + (-1)^n a_n(x).$; $a_1, \dots, a_n : X \longrightarrow \mathbb{C}$.

$\chi_{D(x)}(t) = (t - \lambda_1(x))(t - \lambda_2(x)) \cdots (t - \lambda_n(x)).$ $\longrightarrow$ ①

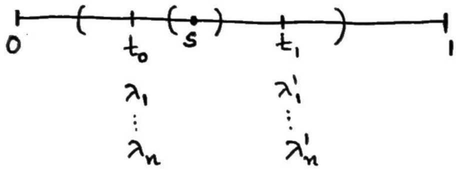
<u>Example</u>: $X = S^1 \subseteq \mathbb{C}$, $A(z) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$, $z \in S^1$. $\chi_{A(z)}(t) = t^2 - z$.

A is not diagonalizable, although there is no multiplicity obstruction.

If ① holds for every polynomial over $C(X)$, we say that $C(X)$ is algebraically closed.

Theorem: $[0,1]$ has the above property. Let Λ be a continuous map from $[0,1]$ into the space $\mathbb{C}^n_{sym}$. Then there exist n complex-valued continuous functions $\lambda_1(t), \dots, \lambda_n(t)$ such that $\Lambda(t) = \{\lambda_1(t), \dots, \lambda_n(t)\}$ for each $t \in [0,1]$.

Proof: Use Induction: Let $K \subseteq [0,1]$ be such that $\forall t \in K$, all n elements of $\Lambda(t)$ (unordered tuple) are identical. Then, K is a closed subset of $[0,1]$. Let $L = [0,1] \setminus K$ and suppose that $t_0 \in L$ and $x \in \Lambda(t_0)$ with multiplicity k where $0 < k < n$. Since Λ is continuous for t close to t_0, $\Lambda(t)$ splits into two groups. By induction hypothesis, each of these groups has a continuous selection in a nghd of t_0.

Idea: 

for an s in the intersection, $\{\lambda_1(s), \dots, \lambda_n(s)\}$ need not be same as $\{\lambda_1'(s), \dots, \lambda_n'(s)\}$ but can be reordered to map it compatible.

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Exercise: Show that every totally disconnected space has this property.

Alternate Proof: $\mathcal{F} = \{(U, S_U) :$ U is an open subset of $[0,1]$ and S_U is a continuous function of $\lambda_1, \dots, \lambda_n \}$.

$\mathcal{F}$ is non-empty. Define $(U, S_U) \leq (V, S_V)$ if $U \subseteq V$ and $S_U = S_V|_U$.

Let $\{(U_i, S_{U_i}) : i \in I$, totally ordered set$\}$ be a chain.

Then $U = \bigcup_{i \in I} U_i$, $S_U(x) = S_{U_j}(x)$ for $x \in U_j$ defines an upper bound for the chain. Hence by Zorn's Lemma, $\mathcal{F}$ has a maximal element.

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Corollary: $C[0,1]$ is algebraically closed.

Proof: $p(t,\lambda) := \lambda^n - a_1(t)\lambda^{n-1} + \cdots + (-1)^n a_n(t)$; $t \in [0,1]$ and
$\Lambda(t_0) =$ roots of $p(t_0,\lambda)$ with multiplicity.

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Grassmannian: $Gr(k,n)$ - space of k-dimensional subspaces of $\mathbb{C}$

By giving a topological (smooth, etc.) structure to $Gr(k,n)$, it is possible to talk about a conts. (smooth, etc.) choice of subspaces.

2nd view point: set of (orthogonal) projection matrices of rank k in $M_n(\mathbb{C})$ and we use norm-topology inherited from $M_n(\mathbb{C})$.

3rd view point: $\mathcal{U}_n(\mathbb{C})$ acts on $Gr(k,n)$ transitively $U \in \mathcal{U}_n(\mathbb{C})$ takes k-dimensional subspaces to k-dimensional subspaces.

$$Gr(k,n) \cong \mathcal{U}_n(\mathbb{C}) / \mathcal{U}_k(\mathbb{C}) \times \mathcal{U}_{n-k}(\mathbb{C}).$$

(transfer smooth structure).

Example: Reducing high-dimensional data to lower dimensions.
Find "best" 25-dim. supspace.
↳ (score associated with each subspace).

Stiefel manifold: $V_k(\mathbb{C}^n)$ - space of all orthonormal k-frames on $\mathbb{C}^n$.
k-frame → n-tuple of k-orthonormal vectors in $\mathbb{C}^n$.

$$V_k(\mathbb{C}^n) \cong \mathcal{U}_n(\mathbb{C}) / \mathcal{U}_{n-k}(\mathbb{C}) \longrightarrow Gr(k,n)$$

Rotations and Quaternions: To describe a rotation, we need an axis and an angle. Composition of two rotations is a rotation.

W. R. Hamilton: $Q \xrightarrow{\text{isomorphic}} \mathbb{R}^+$
$(a,b,c,d) \rightarrow a + bi + cj + dk$ where $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$ and $ki = j$. $\|(a,b,c,d)\| = \sqrt{a^2+b^2+c^2+d^2}$ and $\|r_1 r_2\| = \|r_1\| \|r_2\|$

A quaternion r determines a linear mapping $R_r : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
pure quaternions

For $r \neq 0$, $R_r(x,y,z) = r(xi+yj+zk)r^{-1}$

If $r = \pm 1$, R_r is the identity mapping.

Theorem: For $r = (a,b,c,d)$ with $\|r\| = 1$, R_r is a rotation about the axis determined by (b,c,d) with angle of rotation $\sigma = 2\cos^{-1}(a)$.

Note that $R_r \circ R_s = R_{rs}$

$$\begin{aligned} R_r \circ R_s(x,y,z) &= R_r\left(s(xi+yj+zk)s^{-1}\right) \\ &= rs(xi+yj+zk)s^{-1}r^{-1} \\ &= rs(xi+yj+zk)(rs)^{-1} \\ &= R_{rs}(x,y,z). \end{aligned}$$