3/8/22 :

$M_n(\mathbb{C})$ - $n \times n$ matrices with complex entries.
$A : \mathbb{C}^n \to \mathbb{C}^n$ is thought as $A \in M_n(\mathbb{C})$.

Schur triangularization Thm: Every $A \in M_n(\mathbb{C})$ is unitarily equivalent to an upper triangular matrix

Proof: Choose an eigenvalue λ_1 & corresponding eigenvector v_1 with norm 1.

$$\begin{bmatrix} \lambda_1 & \\ 0 & \boxed{} \\ 0 & \\ 0 & \end{bmatrix}$$

. Next, use Gramshidt to get $\langle v_1 \rangle \subseteq \operatorname{span}\{v_1, v_2\} \subseteq \cdots$ each invariant and get upper triangular. U- unitary.

Corollary: Every normal matrix in $M_n(\mathbb{C})$ is unitarily equivalent to a diag. matrix.

HW
* Every normal upper triangular matrix is diagonal.

$Nor_n(\mathbb{C})$ - $n \times n$ normal matrices.

$$\begin{tikzcd} Nor_n(\mathbb{C}) \arrow[r, "\varphi"] \arrow[dr, dotted, "id"'] & U_n(\mathbb{C}) \times D_n(\mathbb{C}) \arrow[d, "{(U,D) \mapsto UDU^*}"] \\ & Nor_n(\mathbb{C}) \end{tikzcd}$$

Can φ be chosen to be conts?

Frobenious norm $= tr(B^*A) = \|A\|_2^2$.

$$\begin{tikzcd} H_n(\mathbb{C}) \arrow[r, "?\, f"] \arrow[dr, "id"'] & U_n(\mathbb{C}) \times D_n(\mathbb{C}) \arrow[d] \\ & H_n(\mathbb{C}) \end{tikzcd}$$

Yes. But No for normal

Continuity of Roots of Polynomials (Chap-6 in Bhatia).

$\mathbb{C}^n_{sym} \to$ n-tuples in $\mathbb{C}^n$ modulo ordering.
$(z_1, \dots, z_n) \sim (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ endowed with quotient topology. $\sigma \in S_n$.

An optimal matching distance. (o.m.d)

$$d(\vec{\lambda}, \vec{\mu}) = \min_{\sigma \in S_n} \left(\max_{1 \le j \le n} |\lambda_j - \mu_{\sigma(j)}| \right)$$

HW ✱ Show that the quotient topology on $\mathbb{C}^n_{sym}$ and the metric topology generated by optimal matching distance is the same.

Universal property of quotient topology.

(X, τ) - top. $g: X \to Z$ conts s.t. $a \sim b \Rightarrow g(a) = g(b)$.
$\exists!$ conts map f

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "g"'] & X/\sim \arrow[d, "f"] \\ & Z \end{tikzcd}$$

s.t. $g = f \circ q$.

$p(z) = z^n - a_1 z^{n-1} + \dots + (-1)^n a_n$
roots of $p = \alpha_1, \alpha_2, \dots, \alpha_n$.

$\Rightarrow a_1 = \sum \alpha_j, \quad a_2 = \sum_{i \neq j} \alpha_i \alpha_j \quad \dots \quad a_n = \alpha_1 \cdots \alpha_n$.

$S: \mathbb{C}^n \to \mathbb{C}^n, \quad S((\alpha_1 \cdots \alpha_n)^t) = (a_1 \cdots a_n)^t$
S is a conts surjective map.

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] \arrow[dr, "S"'] & \mathbb{C}^n \arrow[r] \arrow[d, "\tilde{S}"] & \mathbb{C}^n/\sim \\ & \mathbb{C}^n & \end{tikzcd}$$

sym

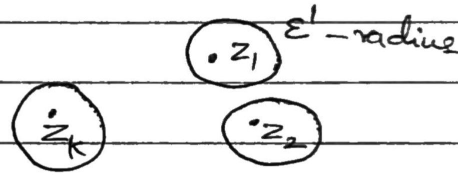
Then $\tilde{S}$ — conts, surjective.

Proposition: $\tilde{S}$ is a homeomorphism btw. $\mathbb{C}^n_{sym}$ & $\mathbb{C}^n$.

Proof: For $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $|a_j - b_j| < \delta$ $\forall j$, then the o.m.d btw roots of corresponding monic polynomials is smaller than ε.

$z_1, z_2, \ldots, z_k$ — distinct roots of p.

$$\Gamma = \bigcup_{j=1}^{k} \partial B(z_j, \varepsilon').$$



$$\eta = \inf_{z \in \Gamma} |p(z)| > 0.$$

Γ - compact. $\Rightarrow \exists\, \delta > 0$ s.t. if q is a monic polyn. with coefficients b_j, and $|a_j - b_j| < \delta$ $\forall j$, then $|p(z) - q(z)| < \eta$ $\forall z \in \Gamma$.

$$< |p(z)|.$$

(Rouche's Thm: If $|g(z)| < |f(z)|$ on ∂K (simple closed curve), then $f, f+g$ have same no. of roots inside K.)

p, q have same no. of roots inside $B(z_j, \varepsilon)$.

$$d(\{\alpha_1, \ldots, \alpha_n\}, \{\beta_1, \ldots, \beta_n\}) < \varepsilon'.$$

$\tilde{S}$ is a homeomorphism.

* ——— *

Corollary: $M_n(\mathbb{C}) \longrightarrow \mathbb{C}^n_{sym}$

$A \longmapsto$ "unordered" tuple of eigenvalues.

is conts.

HW

* Show that $\mathbb{R}^n_{sym} \longrightarrow \mathbb{R}^n$

$\{\alpha_1, \dots, \alpha_n\} \longmapsto (\alpha_{(1)}, \alpha_{(2)}, \dots, \alpha_{(n)})$

is conts.

Eg: $A(\varepsilon) = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}$, $A(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A(-\varepsilon) = \begin{pmatrix} -\varepsilon/2 & -\varepsilon/2 \\ -\varepsilon/2 & -\varepsilon/2 \end{pmatrix}$