Advanced Linear Algebra - M. Math.

Assignment 3 — 1st Semester 2022-2023

Due date: October 07, 2022 (in class)
Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

A real-valued function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur-convex* if whenever $\vec{x} \prec \vec{y}$, we have $\varphi(\vec{x}) \le \varphi(\vec{y})$. It is said to be *Schur-concave* if $-\varphi$ is Schur-convex.

1. (20 points) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function such that $f(0) = 0$.
   (a) (3 points) Show that f is sub-additive: $f(a+b) \le f(a) + f(b)\ \forall a, b \in \mathbb{R}_+$
   (b) (7 points) Let $\Phi : \mathbb{R}_+^{2n} \to \mathbb{R}_+$ be defined as
   $$\Phi(\vec{x}, \vec{y}) = \sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j), \text{ for } \vec{x}, \vec{y} \in \mathbb{R}_+^n.$$
   Show that Φ is Schur-concave.
   (c) (5 points) Show that the function
   $$F(x) = \sum_{j=1}^{n} f(|x_j|),$$
   is sub-additive on $\mathbb{R}^n$.
   (d) (5 points) Show that for $\vec{x}, \vec{y} \in \mathbb{R}^n$:
   $$\sum_{j=1}^{n} \frac{|x_j + y_j|}{1 + |x_j + y_j|} \le \sum_{j=1}^{n} \frac{|x_j|}{1 + |x_j|} + \sum_{j=1}^{n} \frac{|y_j|}{1 + |y_j|},$$
   $$\sum_{j=1}^{n} \log(1 + |x_j + y_j|) \le \sum_{j=1}^{n} \log(1 + |x_j|) + \sum_{j=1}^{n} \log(1 + |y_j|).$$
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2. (15 points) (a) (5 points) (The minimax principle for singular values) For $A \in M_n(\mathbb{C})$ and $1 \leq j \leq n$, show that

$$\begin{aligned} s_j(A) &= \max_{V:\dim V = j} \Big(\min_{\vec{x} \in V. \|\vec{x}\|=1} \|A\vec{x}\| \Big) \\ &= \min_{V:\dim V = n-j+1} \Big(\max_{\vec{x} \in V. \|\vec{x}\|=1} \|A\vec{x}\| \Big) \end{aligned}$$

   (b) (10 points) For $A \in M_n(\mathbb{C})$ and $H \in M_n(\mathbb{C})$ of rank k, show that

$$s_j(A) \geq s_{j+k}(A+H), 1 \leq j \leq n-k.$$

3. (10 points) Prove that a translation-invariant probability measure on the unitary group, $U_n(\mathbb{C})$, of $M_n(\mathbb{C})$ is invariant under transposition and under conjugation, that is, if U is Haar-distributed, so are both U^T and U^*.

4. (15 points) (a) (5 points) Show that $U_n(\mathbb{C}) = \mathrm{ext}\big(\{A \in M_n(\mathbb{C}) : \|A\| \leq 1\}\big)$, where $ext(K)$ denotes the set of extreme points of a convex body K.

   (b) (5 points) Let $A \in M_n(\mathbb{C})$ with $\|A\| < 1$. For every unitary matrix $U \in U_n(\mathbb{C})$, show that $U + A$ is invertible.

   (c) (5 points) Let $A \in M_n(\mathbb{C})$ with $\|A\| \leq 1$. Show that there are unitaries $U_1, U_2 \in U_n(\mathbb{C})$ such that

$$A = \frac{U_1 + U_2}{2}.$$

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