Advanced Linear Algebra - M. Math.

Assignment 2 — 1st Semester 2022-2023

Due date: September 30, 2022 (in class)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) Let $A \in M_n(\mathbb{C})$. Show that there is a unique partial isometry $V \in M_n(\mathbb{C})$ such that $A = VP$ for some symmetric positive-semidefinite matrix $P \in M_n(\mathbb{C})$, and $\operatorname{rank}(A) = \operatorname{rank}(V)$. Is the corresponding P also uniquely determined?

2. (10 points) Let $A, B \in M_n(\mathbb{C})$. Show the following:

    1. $\det(I_n + AB) = \det(I_n + BA)$;
    2. $\operatorname{rank}(I_n + AB) = \operatorname{rank}(I_n + BA)$.

3. (10 points) (a) (10 points) If $\lambda_j^{\downarrow}(A)$ denotes the eigenvalues of an Hermitian matrix $A \in M_n(\mathbb{C})$ arranged in decreasing order, then for $1 \leq k \leq n$, we have

    $$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \max \sum_{j=1}^{k} \langle x_j, Ax_j \rangle,$$

    where the maximum is taken over all orthonormal k-tuples of vectors $(x_1, \ldots, x_k)$ in $\mathbb{C}^n$. (Hint: Use Schur's majorization theorem about diagonals of Hermitian matrices.)

    (b) (5 points) (Ky Fan's maximum principle) Consider Hermitian matrices $A, B \in M_n(\mathbb{C})$. Show that for $1 \leq k \leq n$, we have

    $$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A+B) \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}(A) + \sum_{j=1}^{k} \lambda_j^{\downarrow}(B).$$

4. (10 points) Show that for $A, B \in M_n(\mathbb{C})$ and $1 \leq k \leq n$, we have

    $$\sum_{j=1}^{k} s_j(A+B) \leq \sum_{j=1}^{k} s_j(A) + \sum_{j=1}^{k} s_j(B),$$

    where $s_j(\cdot)$ denotes the jth singular value.

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5. (15 points) Let $P_1, \ldots, P_m$ be a family of mutually orthogonal projections in $\mathbb{C}^n$ such that $\oplus_{j=1}^m P_j = I$. The linear mapping $\mathcal{C} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $A \mapsto \sum_{j=1}^m P_j A P_j$ defines a conditional expectation of $M_n(\mathbb{C})$ onto the unital $*$-subalgebra consisting of block-diagonal matrices (the blocks corresponding to P_j's), and is called a **pinching** of A. In fact, every conditional expectation on $M_n(\mathbb{C})$ arises as a pinching operation.

   (a) (10 points) Show that for every pinching $\mathcal{C}$, we have

   $$s(\mathcal{C}(A)) \prec_w s(A),$$

   where $s(\cdot)$ denotes the vector of singular values. (Hint: Show this for $m = 2$.)

   (b) (5 points) Prove that if $A \in M_n(\mathbb{C})$ is a Hermitian matrix, then $\lambda(\mathcal{C}(A)) \prec \lambda(A)$, where $\lambda(\cdot)$ denotes the vector of eigenvalues (counted with multiplicity) arranged in decreasing order.

   (In fact, we will later see in this course that for every completely positive map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and a Hermitian matrix $A \in M_n(\mathbb{C})$, $\lambda(\Phi(A)) \prec \lambda(A)$. The Hadamard determinant inequality, the Hadamard-Fischer determinant inequality are corollaries of this general majorization inequality involving vectors of eigenvalues.)

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