Advanced Linear Algebra - M. Math.

Assignment 1 — 1st Semester 2022-2023

Due date: September 16, 2022 (in class)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) Show that if A is a skew-symmetric matrix in $M_n(\mathbb{R})$ (that is, $A^T = -A$), then $I - A$ is nonsingular and the matrix $(I - A)^{-1}(I + A)$ is orthogonal. With justification, find all orthogonal matrices in $O_n(\mathbb{R})$ that arise in the above manner from a skew-symmetric matrix in $M_n(\mathbb{R})$.

2. (10 points) (a) (5 points) Show that matrices with distinct eigenvalues are dense in $M_n(\mathbb{C})$.

   (b) (5 points) Show that any matrix in $M_{m,n}(\mathbb{R})$ is the limit of a sequence of full-rank matrices.

3. (10 points) (a) (5 points) For $\alpha \in \mathbb{R}$, what is the nearest rank-one matrix to $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ in the Frobenius norm? Provide justification for your answer.

   (b) (5 points) Show that, for $A \in M_{m,n}(\mathbb{R})$, we have $\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_{op}$. ($\|\cdot\|_F$ denotes the Frobenius norm, and $\|\cdot\|_{op}$ denotes the usual operator norm.)

4. (10 points) (Least squares minimization over a solid sphere) Given $A \in M_{m,n}(\mathbb{R})$ with $m \geq n, \vec{b} \in \mathbb{R}^m$, and $\alpha > 0$, solve the following optimization problem.

$$\arg\min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{b}\|_2$$

   subject to

$$\|x\|_2 \leq \alpha.$$

   (Hint: Use the SVD of A.)

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5. (10 points) Let $A \in M_n(\mathbb{R})$ and P be the permutation obtained by reversing the order of the rows in I_n.
   (a) (4 points) Show that if $R \in M_n(\mathbb{R})$ is upper-triangular, then $L = PRP$ is lower-triangular.
   (b) (6 points) Show how to compute an orthogonal $Q \in M_n(\mathbb{R})$ and a lower-triangular $L \in M_n(\mathbb{R})$ so that $A = QL$, assuming the availability of a procedure for computing the QR factorization.
6. (10 points) Let $n \geq 2$. Let $B_n(\mathbb{R})$ denote the set of upper-triangular matrices in $M_n(\mathbb{R})$. Show that it is not possible to find continuous functions $Q : M_n(\mathbb{R}) \to O_n(\mathbb{R}), R : M_n(\mathbb{R}) \to B_n(\mathbb{R})$ such that $Q(A)R(A) = A$ for all $A \in M_n(\mathbb{R})$. (In other words, the QR-factorization cannot be done in a continuous manner over all of $M_n(\mathbb{R})$.)

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