1) Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions :

(a) $\frac{Q}{K_0} = 1, u(0) = T_1, u(L) = T_2$

(b) $\frac{Q}{K_0} = x^2, u(0) = T, \frac{\partial u}{\partial x}(L) = 0$

Ans:

The heat equation for a one-dimensional rod with constant thermal properties is given by,

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q,$$

where $u \to$ temperature distribution,
$c \to$ specific heat,
$\rho \to$ density of material,
$Q \to$ density of generated energy from heat source.

For the equilibrium temperature distribution, one has that $\frac{\partial u}{\partial t} = 0$. Thus we must have :

(a) $\frac{\partial^2 u}{\partial x^2} = -\frac{Q}{K_0} = -1$. Thus $u(x) = -\frac{x^2}{2} + C_1 x + C_2$ after equilibrium temperature distribution is attained. From the boundary conditions, we have $u(0) = C_2 = T_1$, and $u(L) = -\frac{L^2}{2} + C_1 L + T_1 = T_2$. As a result, $C_1 = \frac{1}{L}(T_2 - T_1 + \frac{L^2}{2}) = \frac{T_2 - T_1}{L} + \frac{L}{2}$, and

$$u(x) = -\frac{x^2}{2} + (\frac{T_2 - T_1}{L} + \frac{L}{2})x + T_1.$$

(b) $\frac{\partial^2 u}{\partial x^2} = -\frac{Q}{K_0} = -x^2$. Thus $u(x) = -\frac{x^4}{12} + C_1 x + C_2$, and $\frac{\partial u}{\partial x} = -\frac{x^3}{3} + C_1$ after equilibrium temperature distribution is attained. From the boundary conditions, we have $\frac{\partial u}{\partial x}(L) = -\frac{L^3}{3} + C_1 = 0$, and $u(0) = C_2 = T$. As a result, $C_1 = \frac{L^3}{3}$ and $C_2 = T$, and

$$u(x) = -\frac{x^4}{12} + \frac{L^3}{3}x + T.$$

2) For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions?

(a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1, u(x,0) = f(x), \frac{\partial u}{\partial x}(0,t) = 1, \frac{\partial u}{\partial x}(L,t) = \beta,$

(b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta, u(x,0) = f(x), \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0.$

Ans: For the equilibrium temperature distribution (if one exists), one has that $\frac{\partial u}{\partial t} = 0$.

(a) $\frac{\partial^2 u}{\partial x^2} = -1$. Thus $u(x) = -\frac{x^2}{2} + C_1 x + C_2$ and $\frac{\partial u}{\partial x} = -x + C_1$. From the boundary conditions, we have that $\frac{\partial u}{\partial x}(0,t) = C_1 = 1$ and $\frac{\partial u}{\partial x}(L,t) = -L + C_1 = \beta$. Thus there is an equilibrium temperature distribution if and only if $-L+1 = \beta$. Integrating both sides of the heat equation for the whole length of the rod, we obtain that for all times t',

$$\int_0^L \frac{\partial u}{\partial t}\, dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\, dx + \int_0^L 1\, dx = \frac{\partial u}{\partial x}(L,t') - \frac{\partial u}{\partial x}(0,t') + L = \beta - 1 + L = 0$$

Thus the integral $w(t) := \int_0^L u(x,t)\, dx$ is constant with respect to time, and $w(0)$ is equal to $w(t_{eq})$ where t_{eq} denotes time when equilibrium distribution is attained. As a result, $\int_0^L f(x)\, dx = \int_0^L (-\frac{x^2}{2} + x + C_2)\, dx = -\frac{L^3}{6} + \frac{L^2}{2} + C_2 L$ and $C_2 = \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}(\int_0^L f(x)\, dx)$
Thus there is an equilibrium temperature distribution if and only if $\beta = 1 - L$ and in that case, the distribution is given by

$$u(x) = -\frac{x^2}{2} + x + \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}(\int_0^L f(x)\, dx)$$

(b) $\frac{\partial^2 u}{\partial x^2} = -x + \beta$. Thus $u(x) = -\frac{x^3}{6} + \frac{\beta x^2}{2} + C_1 x + C_2$ and $\frac{\partial u}{\partial x} = -\frac{x^2}{2} + \beta x + C_1$. From the boundary conditions, we have that $\frac{\partial u}{\partial x}(0,t) = C_1 = 0$ and $\frac{\partial u}{\partial x}(L,t) = -\frac{L^2}{2} + \beta L + C_1 = 0$. Thus $-\frac{L^2}{2} + \beta L = 0 \Rightarrow \beta = \frac{L}{2}$. Thus there is an equilibirum temperature distribution if and only if $\beta = \frac{L}{2}$. Integrating both sides of the heat equation for the whole length of the rod, we obtain that for all times t',

$$\int_0^L \frac{\partial u}{\partial t}\, dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\, dx + \int_0^L (x - \beta)\, dx = \frac{\partial u}{\partial x}(L,t') - \frac{\partial u}{\partial x}(0,t') + \frac{L^2}{2} - \beta L = 0$$

Thus the integral $w(t) := \int_0^L u(x,t)\, dx$ is constant with respect to time, and $w(0)$ is equal to $w(t_{eq})$ where t_{eq} denotes time when equilibrium distribution is attained. As a result, $\int_0^L f(x)\, dx = \int_0^L (-\frac{x^3}{6} + \frac{Lx^2}{4} + C_2)\, dx = -\frac{L^4}{24} + \frac{L^4}{12} + C_2 L$ and $C_2 = \frac{1}{L}(\int_0^L f(x)\, dx) - \frac{L^3}{24}$
Thus there is an equilibrium temperature distribution if and only if $\beta = \frac{L}{2}$ and in that case, the distribution is given by

$$u(x) = -\frac{x^3}{6} + \frac{Lx^2}{4} + \frac{1}{L}(\int_0^L f(x)\, dx) - \frac{L^3}{24}$$

3) Consider the heat equation,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0.$$

If the temperature distribution is initially

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x \leq \frac{L}{2} \\ 2 & \text{if } \frac{L}{2} < x < L \end{cases}$$

(a) Solve for $u(x,t)$.

(b) What is the total heat energy in the rod as a function of time?

Ans:

(a) If $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})$, then $u(x,t) = \sum_{n=1}^{\infty} A_n \exp^{-k(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L})$. For the given initial distribution, the coefficients A_n are given by

$$A_n = \frac{2}{L}\int_0^L u(x,0)\sin(\frac{n\pi x}{L})\ dx = \frac{2}{L}(\int_0^{\frac{L}{2}} \sin(\frac{n\pi x}{L})\ dx + \int_{\frac{L}{2}}^L 2\sin(\frac{n\pi x}{L})\ dx)$$

$$= \frac{2}{n\pi}(1-\cos(\frac{n\pi}{2}) + 2\cos(\frac{n\pi}{2}) - 2\cos(n\pi)) = \frac{2}{n\pi}(1+\cos(\frac{n\pi}{2}) - 2\cos(n\pi))$$

$$= \begin{cases} \frac{6}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } 4|n \\ \frac{-4}{n\pi} & 4|n-2 \end{cases}$$

(b) Total heat energy $= \int_0^L c\rho u A\ dx = c\rho A \sum_{n=1}^{\infty} A_n \exp^{-k(\frac{n\pi}{L})^2 t} \int_0^L \sin(\frac{n\pi x}{L})\ dx = c\rho A \sum_{n=1}^{\infty} A_n \exp^{-k(\frac{n\pi}{L})^2 t}\ \frac{L}{n\pi}(1-(-1)^n)$.

4) Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions:

(a) $\phi(0) = 0, \phi(1) = 0$

(b) $\frac{d\phi}{dx}(0) = 0, \phi(L) = 0$

Ans: Let ϕ be defined on the interval $[a,b]$. Integrating by parts, we get

$$\int_a^b \frac{d^2\phi}{dx^2}\cdot\phi\ dx + \int_a^b \frac{d\phi}{dx}\cdot\frac{d\phi}{dx}\ dx = \phi(b)\frac{d\phi}{dx}(b) - \phi(a)\frac{d\phi}{dx}(a)$$

From the kind of boundary conditions described in the problem, we have

$$\int_a^b \frac{d^2\phi}{dx^2}\cdot\phi\ dx + \int_a^b \frac{d\phi}{dx}\cdot\frac{d\phi}{dx}\ dx = 0.$$

If ϕ is an eigenfunction corresponding to the eigenvalue λ, then

$$-\lambda\int_a^b \phi\cdot\phi\ dx = -\int_a^b \frac{d\phi}{dx}\cdot\frac{d\phi}{dx}\ dx.$$

Thus $\lambda \geq 0$ and equality holds if and only if ϕ is a constant function on $[a,b]$ (as $\int_a^b \frac{d\phi}{dx}\cdot\frac{d\phi}{dx}\ dx = 0 \Rightarrow \frac{d\phi}{dx} \equiv 0 \Rightarrow \phi \equiv constant$) .

(a) If $\lambda = 0$, we must have $\phi \equiv 0$ and thus 0 is not an eigenvalue.

<u>$\lambda > 0$</u> The characteristic polynomial for the second order linear ODE is $x^2 + \lambda$ whose roots are $\pm\iota\sqrt{\lambda}$. Thus $\phi(x) = c_1 \exp(\iota\sqrt{\lambda}) + c_2 \exp(-\iota\sqrt{\lambda})$. For $c_3 = c_1 + c_2, c_4 = \iota(c_1 - c_2)$, we have that $\phi(x) = c_3 \cos(\sqrt{\lambda}x) + c_4 \sin(\sqrt{\lambda}x)$. We may assume that c_3, c_4 are not both zero as that would yield the trivial solution i.e. $\phi \equiv 0$.

We check the boundary conditions to determine c_3, c_4. As $\phi(0) = 0$, we must have $c_3 = 0$. From $\phi(1) = 0$, we get that $c_4 \sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = n\pi, n = 1, 2, 3 \cdots \Rightarrow \lambda = (n\pi)^2, n = 1, 2, 3, \cdots$. For $\lambda = (n\pi)^2$, the corresponding eigenfunction is $\sin(n\pi x)$.

(b) If $\lambda = 0$, we must have $\phi \equiv 0$ and thus 0 is not an eigenvalue.

<u>$\lambda > 0$</u> The characteristic polynomial for the second order linear ODE is $x^2 + \lambda$ whose roots are $\pm\iota\sqrt{\lambda}$. Thus $\phi(x) = c_1 \exp(\iota\sqrt{\lambda}) + c_2 \exp(-\iota\sqrt{\lambda})$. For $c_3 = c_1 + c_2, c_4 = \iota(c_1 - c_2)$, we have that $\phi(x) = c_3 \cos(\sqrt{\lambda}x) + c_4 \sin(\sqrt{\lambda}x)$. We may assume that c_3, c_4 are not both zero as that would yield the trivial solution i.e. $\phi \equiv 0$.

We check the boundary conditions to determine c_3, c_4. As $\frac{d\phi}{dx}(0) = -\sqrt{\lambda}c_3 \sin(0) + \sqrt{\lambda}c_4 \cos(0) = 0$, we must have $c_4 = 0$. From $\phi(L) = 0$, we get that $c_3 \cos(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda}L = \frac{(2n-1)\pi}{2}, n = 1, 2, 3, \cdots \Rightarrow \lambda = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3, \cdots$. For $\lambda = (\frac{(2n-1)\pi}{2L})^2$, the corresponding eigenfunction is $\cos(\frac{(2n-1)\pi}{2L}x)$.

5) Show that there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

subject to the boundary conditions

$$\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0.$$

<u>Ans:</u> Refer to the first part of previous question.

6) Solve Laplace's equation inside a rectangle $0 \leq x \leq L, 0 \leq y \leq H$, with the following boundary condition : $\frac{\partial u}{\partial x}(0, y) = 0, \frac{\partial u}{\partial x}(L, y) = 0, u(x, 0) = 0, u(x, H) = f(x)$

<u>Ans:</u> First we look for product solutions which are of the form $u(x, y) = \phi(x)g(y)$. From Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \phi''(x)g(y) + \phi(x)g''(y) = 0 \Rightarrow -\frac{\phi''(x)}{\phi(x)} = \frac{g''(y)}{g(y)} = \lambda$. Without loss of generality, we may assume $\lambda \geq 0$.

Thus we have two second order linear ODEs.

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \frac{d^2 g}{dy^2} - \lambda g = 0.$$

For $\lambda > 0$, the general solutions are $\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ and $g(y) = c_3 \cosh\sqrt{\lambda}y + c_4 \sinh\sqrt{\lambda}y$. As we are looking for non-zero solutions, using the boundary conditions we get $\phi'(0) = 0, \phi'(L) = 0, g(0) = 0$. Note that $\phi'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$ and thus $\phi'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0$ and $\phi'(L) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi, n = 1, 2, 3, \cdots \Rightarrow \lambda = (\frac{n\pi}{L})^2, n = 1, 2, 3\cdots$. As $g(0) = 0$, we have that $c_3 = 0$ and thus $g(y) = \sinh\frac{n\pi y}{L}$ is a solution to the second ODE, corresponding to $\lambda = (\frac{n\pi}{L})^2$.

If $\lambda = 0$, then the solution is of the form $(c_1x+c_2)(c_3y+c_4)$. From the boundary conditions, $u(x, y) = y$ is a product solution.

Thus the product solutions are of the form $y, \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}, n = 1, 2, 3, \cdots$. An infinite linear combination of the product solutions would yield $u(x, y) = a_0y+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$. It must be the case that $u(x, H) = f(x) = a_0H+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}\sinh\frac{n\pi H}{L}$. If f is a continuous function, the Fourier cosine series of f, converges to f pointwise. Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}$. Comparing it with the expression for $u(x, H)$, we conclude that $a_0H = A_0, a_n\sinh\frac{n\pi H}{L} = A_n \Rightarrow a_0 = \frac{A_0}{H}, a_n = \frac{A_n}{\sinh\frac{n\pi H}{L}}$.

Thus the solution to the above boundary value problem is $A_0\frac{y}{H}+\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\frac{\sinh\frac{n\pi y}{L}}{\sinh\frac{n\pi H}{L}}$, where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

7) Compute the Fourier cosine series of $f(x) = \sin\frac{\pi x}{L}, 0 \le x \le L$.

<u>Ans:</u> The Fourier cosine series of f is $a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$ where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L \sin\frac{\pi x}{L}\,dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L \frac{1}{2}(\sin\frac{(1+n)\pi x}{L} + \sin\frac{(1-n)\pi x}{L})\,dx$$
$$= \frac{1}{L}(\frac{L}{n\pi}(1-\cos(1+n)\pi) + \frac{L}{n\pi}(1-\cos(1-n)\pi)) = \frac{2}{n\pi}(1-(-1)^{n+1})$$

8) Compute the Fourier sine series of $f(x) = 1, 0 < x \le L$ and $f(0) = 0$.

<u>Ans:</u> The Fourier sine series of f is $\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$ where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}(1-\cos n\pi)$$