1) There is a rectangular metal sheet with constant thermal properties and no heat sources. The temperature distribution $u(x,y,t)$ satisfies the following equation

$$\frac{\partial u}{\partial t} = \nabla^2 u$$

Find the equilibrium temperature distribution if the boundary conditions are given by,

$$\frac{\partial u}{\partial x}(0,y,t) = 0, \frac{\partial u}{\partial x}(L,y,t) = 0, u(x,0,t) = 0, u(x,H,t) = \cos\frac{n\pi x}{L}.$$

2) Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

subject to the initial condition $u(x,y,0) = \alpha(x,y)$. Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

$$\frac{\partial u}{\partial x}(0,y,t) = 0, \frac{\partial u}{\partial x}(L,y,t) = 0, \frac{\partial u}{\partial y}(x,0,t) = 0, \frac{\partial u}{\partial y}(x,H,t) = 0$$

3) If

$$f(x) = \begin{cases} 0 & \text{if } |x| > a \\ 1 & \text{if } |x| < a \end{cases}$$

determine the Fourier transform of $f(x)$.

4) If $F(\omega) = \exp(-|\omega|\alpha), (\alpha > 0)$, determine the inverse Fourier transform of $F(\omega)$.

5) Consider the wave equation with a restoring vertical force of αu $(\alpha < 0)$.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \alpha u.$$

Solve the initial value problem

$$u(0,t) = 0, u(L,t) = 0, u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration?

6) Consider the one-dimensional heat equation for nonconstant thermal properties

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}[K_0(x)\frac{\partial u}{\partial x}]$$

with the initial condition $u(x,0) = f(x)$. Assuming the eigenfunctions are known, solve the initial value problem, with boundary conditions,

$$\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0.$$

Evaluate $\lim_{t\to\infty} u(x,t)$.

7) Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0, \frac{d\phi}{dx}(0) = 0, \frac{d\phi}{dx}(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

8) Consider the wave equation for a vibrating rectangular membrane $0 < x < L, 0 < y < H$,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial conditions

$$u(x, y, 0) = 0, \frac{\partial u}{\partial t}(x, y, 0) = \alpha(x, y)$$

Solve the initial value problem if

$$u(0, y, t) = 0, u(L, y, t) = 0, \frac{\partial u}{\partial y}(x, 0, t) = 0, \frac{\partial u}{\partial y}(x, H, t) = 0.$$

9) Consider the eigenvalue problem on $[0, L] \times [0, H]$,

$$\nabla^2\phi + \lambda\phi = 0$$

$$\frac{\partial \phi}{\partial x}(0, y) = 0, \phi(x, 0) = 0$$

$$\frac{\partial \phi}{\partial x}(L, y) = 0, \phi(x, H) = 0.$$

If $L = H$ show that most eigenvalues have more than one eigenfunction.

10) Solve for $u(r, t)$ if it satisfies the circularly symmetric heat equation

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

subject to the conditions

$$u(a, t) = 0, u(r, 0) = f(r).$$

Analyze the limit as $t \to \infty$.