1) a. This is a direct application of the multiplication principle. We have 3 choices from the first column, 4 choices from the second column, 5 choices for the second column and 2 choices for the last column. In total, we have $3 \times 4 \times 5 \times 2 = 120$ possibilities for the lunch special.

   b. If one decides to forego meat, the first column only has one choice. And we are only considering the first three columns as we decided to cancel the dessert. So the number of possibilities in this case is $1 \times 4 \times 5 = 20$.

2) If there are n questions, there are $n!$ ways of ordering the questions on the quiz. We want enough number of questions so that all 34 students get different questions. Thus we are looking for the minimum value of n such that $n! > 34$. $1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120$. Hence, one should have atleast 5 questions on the quiz.

3. a. Counting the number of ways 5 heads can occur is the same as counting the number of ways of choosing 5 positions in the sequence of 10 tosses. Thus, the answer is $\binom{10}{5} = 252$.

   The wording for the second part may be a bit ambiguous. If one interpretes it as exactly 5 heads occurring in the sequence and that too consecutively, then consider them as one bunch called, say $(5H)$. Now we have 5 tails and we are trying to fit the bunch $(5H)$ in between them. There are 6 positions to do that. For instance, $(5H)TTTTT, T(5H)TTTT, TT(5H)TTT, \cdots$.

   The ratio is $\frac{6}{252} = \frac{1}{42} = 0.02381$(approx.)

b. For the nth level, if n is odd, there are two candidates in the middle for the biggest number. If n is even, the biggest number is the one in the middle. This is a consequence of Pascal's identity, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, If we have an increasing sequence, $a \leq b \leq c \leq d \leq \cdots$, clearly $a + b \leq b + c \leq c + d \leq \cdots$. Using Pascal's identity to compute the entries of next level of Pascal's triangle, we inductively note that it is symmetric and also increases upto the middle entry(ies).

c. The total number of possibilities in the sequence of 6 coin tosses $= 2^6 = 64$. The number of ways to get 3 or 4 heads $= \binom{6}{4} + \binom{6}{3} = 15 + 20 = 35$. The remaining number of possibilities $= 64 - 35 = 29$. As $35 > 29$, this is not a fair game. There are more chances of me winning.

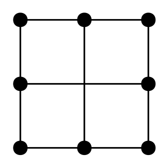
4. The floor function $\lfloor \cdot \rfloor$ takes a real number as input and spits out the greatest integer less than that number. For instance, $\lfloor 3.4 \rfloor = 3, \lfloor -3.4 \rfloor = -4, \lfloor 2.999 \rfloor = 2$

   a. pigeons $\rightarrow$ the socks selected from drawer, pigeonholes $\rightarrow$ color of the socks (red, black). If one chooses 3 socks randomly from the drawer, we have 3 pigeons and 2 pigeonholes and thus, we must have atleast $\lfloor \frac{3}{2} \rfloor + 1 = 2$ of the same color.

b. pigeons $\to$ Skittles in the bag, pigeonholes $\to$ color of the Skittles (purple, yellow green, orange). There are 21 pigeons and 4 pigeonholes, so there must be one pigeonhole with atleast $\lfloor \frac{21}{4} \rfloor + 1 = 6$ i.e. atleast one color which corresponds to 6 Skittles.

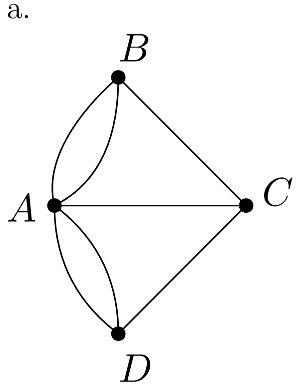
c. pigeons $\to$ birthdays of Penn students, pigeonholes $\to$ all possible birthdays (366 including 29th of February). There are 10000 pigeons and 366 pigeonholes. Hence there is a group of $\lfloor 10000/366 \rfloor + 1 = 28$ students who have the same birthday.

d. We split the unit square into four congruent squares with side $\frac{1}{2}$. pigeons $\to$ the points, pigeonholes $\to$ the smaller squares of side $\frac{1}{2}$.



There are 5 pigeons and 4 pigeonholes, thus there must be one smaller square with two points. Those two points have distance less than or equal to the length of the diagonal of the square which is $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{1}{\sqrt{2}}$.

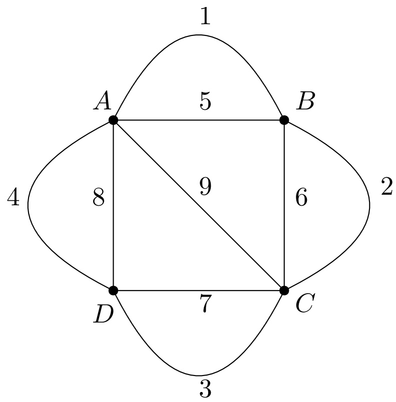
5. a.



Vertex	Degree
A	5
B	3
C	3
D	3

As there are 4 vertices with odd degree, there can neither be an Eulerian circuit nor an Eulerian path.

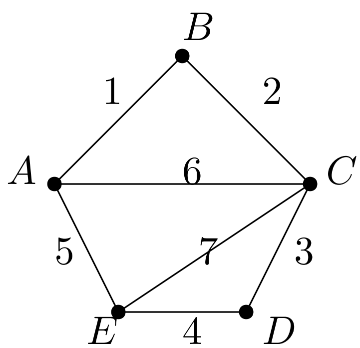
b.



Vertex	Degree
A	5
B	4
C	5
D	4

As the graph has a vertex with odd degree, there cannot be any Eulerian circuit. But there is an Eulerian path, given by following the edges $1-2-3-4-5-6-7-8-9$. Note that it starts at A and ends at C both of which are vertices with odd degree.

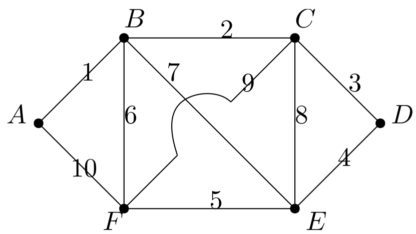
c.



Vertex	Degree
A	3
B	2
C	4
D	2
E	3

As the graph has a vertex with odd degree, there cannot be any Eulerian circuit. But there is an Eulerian path, given by following the edges $1-2-3-4-5-6-7$. Note that it starts at A and ends at E both of which are vertices with odd degree.

d.



Vertex	Degree
A	2
B	4
C	4
D	2
E	4
F	4

As all vertices of the graph have even degree, there is a Eulerian circuit (which is also a Eulerian path) given by following the edges $1-2-3-4-5-6-7-8-9-10-1$.