1. Let P denote the set of students who like pizza and let D denote the set of students who like donuts. We are interested in finding the cardinality of $P \cap D$. We know that $\#(P) = 14$ and $\#(D) = 10$. By the inclusion-exclusion principle,

$$\#(P \cup D) = \#(P) + \#(D) - \#(P \cap D)$$

We are interested in computing $\#(P \cap D)$ which is equal to $\#(P) + \#(D) - \#(P \cup D)$. But the data given in the question is not sufficient to determine the number of students who like either pizza or donuts or both. There might be some students who like neither. Thus the answer is $14+10-\#(P \cup D) = 24-\#(P \cup D)$, which may range from 3 to 10 depending on how many students like neither pizza nor donuts (which may range from 7 to 0 respectively).

2. (i) Let A be the set of numbers from 1 to 30 divisible by 3, and B be the set of numbers from 1 to 30 divisible by 5. The cardinality of A is given by $\frac{30}{3} = 10$ and the cardinality of B is given by $\frac{30}{5} = 6$. The set $A \cap B$ consists of numbers divisible by both 3 and 5. Every such number must be divisible by 15 and conversely, every number divisible by 15 is divisible by 3 ande 5. The cardinality of $A \cap B$ is thus $\frac{30}{15} = 2$. By the inclusion-exclusion principle, $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B) = 10 + 6 - 2 = 14$.

   Thus there are $30 - 14 = 16$ numbers from 1 to 30 not divisible by 3 or 5.

   (ii) One method for this problem would be to use the inclusion-exclusion principle for three sets. An easier method would be to notice that any number not divisible by 2 is also not divisible by 4. So 4 is redundant and all we need to do is find numbers not divisible by 2 or 7.

   Let A be the set of numbers from 1 to 50 divisible by 2, and B be the set of numbers from 1 to 50 divisible by 7. The cardinality of A is $\frac{50}{2} = 25$ and the cardinality of B is $\frac{49}{7} = 7$. Similar to the previous exercise, a number is divisible by 3 and 7 if and only if it is divisible by 14. The cardinality of $A \cap B$ is $\frac{42}{14} = 3$. By the inclusion-exclusion principle, $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B) = 25 + 7 - 3 = 29$.

   Thus there are $50 - 29 = 21$ numbers from 1 to 50 not divisible by 2 or 7.

   (iii) To be done manually.

3. This procedure is no different from what we discussed in class once one realizes that this is just considering the expansion of a real number in $(0, 1)$, in base 100 and repeating the diagonal argument which will work for any base ≥ 3.

$$1 \leftrightarrow 0.67198679\cdots$$

$$2 \leftrightarrow 0.84556150\cdots$$

$$3 \leftrightarrow 0.11682251\cdots$$

$$\vdots$$

The first few decimal places of m look like $0.222244\cdots$. It cannot equal any number in the list as it does not match any number in the list at atleast two decimal places.

4. The nth odd number is given by the formula $2n-1$. The sum of the first n odd numbers, $1+3+\cdots+2n-1$, is equal to n^2.

   We set up the problem for a proof by induction in the following manner. Let $P(n): 1+3+\cdots+2n-1=n^2$ be the nth proposition. We would like to prove that $P(n)$ is true for all natural numbers n.

   Base case: $P(1): 2\times 1-1=1^2$, which is clearly true.

   Induction Hypothesis: Let $P(k)$ be true for some natural number k. In other words, $1+3+\cdots+2k-1=k^2$.

   We would like to deduce that $P(k+1): 1+3+\cdots+2k-1+2(k+1)-1=(k+1)^2$ is true.

   From the induction hypothesis,$1+3+\cdots+2k-1=k^2$. Adding $2(k+1)-1=2k+1$ on both sides of the equation in $P(k)$ we get, $1+3+\cdots+2k-1+2(k+1)-1=k^2+2(k+1)-1=k^2+2k+1=(k+1)^2$. Thus $P(k)\Rightarrow P(k+1)$.

   By the principle of mathematical induction, our proof is done.

5. For natural numbers n greater than or equal to 4, we have the nth proposition $P(n): n! > 2^n$.

   Base case: $P(4): 4! > 2^4$. $P(4)$ is true as $24 > 16$.

   Induction Hypothesis: Let $P(k)$ be true for some natural number $k \geq 4$. In other words, $k! > 2^k$.

   We would like to deduce that $P(k+1): (k+1)! > 2^{k+1}$ is true.

   From the induction hypothesis, $k! > 2^k$. Multiplying both sides by $k+1$, we have the inequality $(k+1)\cdot k! > (k+1)2^k$. But as $k+1 > 2$, the right hand side of the inequality is bigger than $2\cdot 2^k = 2^{k+1}$. Thus $(k+1)! > 2^{k+1}$.

   By the principle of mathematical induction, our proof is done.