Midterm Exam Solutions

July 22, 2013

1

Consider the ellipse given by the parametrization below :

$$x = \cos t, y = 2\sin t, 0 \le t \le 2\pi.$$

Use this parametrization to find the surface area of ellipsoid swept out by revolving the ellipse about the x-axis.

Solution:
We need only consider $[0, \pi]$ as domain for t to eliminate the risk of computing twice the surface area.(as the ellipse is symmetric about x-axis)

$$S = \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = 4\pi \int_0^{\pi} \sin t \sqrt{1 + 3\cos^2 t}\, dt = \frac{4\pi}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1+u^2}\, du$$

(substituting $u = \sqrt{3}\cos t$). Now, susbtituting $u = \tan\theta$, we have that the integral is

$$= \frac{4\pi}{\sqrt{3}} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^3\theta\, d\theta = \frac{4\pi}{\sqrt{3}} \cdot \frac{1}{2} (\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{4\pi}{\sqrt{3}}(2\sqrt{3} + \log(2+\sqrt{3}))$$

Thus, the surface area of the ellipsoid is $\boxed{8\pi + \frac{4\pi}{\sqrt{3}} \log(2+\sqrt{3})}$.

The integral of $\sec^3\theta$ is done via integration by parts on $u = \sec\theta, v = \sec^2\theta$.

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \tan\theta(\sec\theta\tan\theta)\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$\Rightarrow 2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \log|\sec\theta + \tan\theta|$$

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2

Find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.

Solution:
A smooth parametrization of the given parabola is $x(t) = t, y(t) = t^2, -\infty <$

$t < \infty$.
$\vec{v}(t) = \hat{\imath} + 2t\hat{\jmath}$.
The unit tangent vector, $\vec{T}(t) = \frac{1}{\sqrt{1+4t^2}}\hat{\imath} + \frac{2t}{\sqrt{1+4t^2}}\hat{\jmath}$ and

$$\frac{d\vec{T}}{dt} = \frac{-4t}{(1+4t^2)\sqrt{1+4t^2}}\hat{\imath} + \frac{2}{(1+4t^2)\sqrt{1+4t^2}}\hat{\jmath}$$

Use the formula for curvature

$$\kappa = \frac{1}{|\vec{v}(t)|}\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{(1+4t^2)\sqrt{1+4t^2}}$$

So curvature at the point $(1,1)$, which corresponds to $t = 1$ is $\boxed{\frac{2}{5\sqrt{5}}}$.

3

Prove that for any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ we have

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}.$$

<u>Solution</u>:
$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
$\vec{v} \times (\vec{w} \times \vec{u}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}$
$\vec{w} \times (\vec{u} \times \vec{v}) = (\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v}$
Adding the above three expressions gives us the desired result.
There are many ways to prove the vector identities.
The brute-force way is to compute both sides in terms of their components and see that they match.

One may also notice that both sides of the identity are multilinear (linear in each variable, when the others are kept fixed). If we check that the identity is true for $\vec{u} = \hat{\imath}, \vec{v} = \hat{\jmath}, \vec{w} = \hat{k}$ or $\vec{u} = \hat{\imath}, \vec{v} = \hat{\imath}, \vec{w} = \hat{\jmath}$(and consequently, for symmetric expressions in terms of $\hat{\imath}, \hat{\jmath}, \hat{k}$), we are done. Checking it for the above special cases is trivial.

Yet another proof can be given by noticing that if $\vec{u}$ and $\vec{v}$ are not parallel vectors, any vector in $\mathbb{R}^3$ may be written in the form $a\vec{u} + b\vec{v} + c(\vec{u} \times \vec{v}), a, b, c \in \mathbb{R}$. Thus one need only prove the identity for the special cases $\vec{w} = \vec{u}, \vec{w} = \vec{v}, \vec{w} = \vec{u} \times \vec{v}$.

4

Find the length of the astroid $x = \cos^3 t, y = \sin^3 t, 0 \le t \le 2\pi$.

<u>Solution</u>:

$$\frac{dx}{dt} = -3\cos^2 t \sin t, \frac{dy}{dt} = 3\sin^2 t \cos t$$

The formula for length of a curve parametrized by $(x(t), y(t))$ from a to b (where the parametrization satisfies certain conditions) is given by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Length of first-quadrant portion =

$$\int_0^{\frac{\pi}{2}} 3\cos t \sin t\, dt = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2t\, dt = \frac{3}{2}$$

Thus, the length of the astroid $= 4 \times \frac{3}{2} = 6$.

5

A person standing at the top of a vertical sea cliff whose height above sea level is 180 m throws a rock horizontally at a speed of 10 m/s towards the sea. How far from the base of the cliff will the rock hit the water ? (You may neglect the effects of air resistance. You may assume that the acceleration due to gravity is $g = 10$ m/s^2)

Solution:
Initially, the vertical component of velocity is 0. After time t, the distance covered by the rock in the vertical downward direction is $h = \frac{1}{2}gt^2$. Distance covered in the horizontal direction in the same time is $10t = 10\sqrt{\frac{2h}{g}}$.
Thus, distance from base of cliff when the rock hits the water is $10\sqrt{\frac{2\times 180}{10}}\, m = 60\, m$.

6

Write the equivalent Cartesian equations for the given polar equations:

1. $r\cos\theta = 3$
2. $r^2 \sin 2\theta = 4$
3. $r = 1 + 2r\cos\theta$

Solution:
$x = r\cos\theta, y = r\sin\theta$

1. $x = 3$
2. $r^2\sin 2\theta = 4 \Rightarrow r^2(2\sin\theta\cos\theta) = 4 \Rightarrow (r\cos\theta)(r\sin\theta) = 2 \Rightarrow xy = 2$
3. $r = 1 + 2r\cos\theta \Rightarrow r^2 = (1+2x)^2 \Rightarrow x^2 + y^2 = 1 + 4x^2 + 4x$
   $\Rightarrow y^2 - 3x^2 - 4x - 1 = 0$

7

Does the line $L : x = 3t + 10, y = -2t - 4, z = -t + 6$ intersect the plane $x + y + z = 4$? If yes, find the point(s) of intersection. If no, justify why that is the case.

Solution:
$3\hat{\imath} - 2\hat{\jmath} - \hat{k}$ is orthogonal to $\hat{\imath} + \hat{\jmath} + \hat{k}$, which can be seen by computing the dot product. Thus the line is parallel to the plane which means, either it completely lies in the plane or it doesn't intersect the plane at all.
Plugging in $t = 0$, we see the point $(10, -4, 6)$ lies on the line L but not in the plane. (as $10 - 4 + 6 \neq 4$)
Thus, L does not intersect the plane at all.

Alternatively, we can try to solve for the point of intersection. $(3t + 10) + (-2t - 4) + (-t + 6) = 4 \Rightarrow (3 - 2 - 1)t + 12 = 4 \Rightarrow 12 = 4$ (which is absurd). Thus, there is no value of t for which $(3t + 10, -2t - 4, -t + 6)$ lies in the plane. In other words, L does not intersect the plane.