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A NOTE ON PROJECTION EQUIVALENCE IN VON NEUMANN ALGEBRAS

Dedicated to the memory of Israel Halperin, the guiding spirit of Canadian Functional Analysis and a fearless fighter for the freedom of scientists everywhere.

RICHARD V. KADISON

Presented by George Elliott, FRSC

Abstract. A new structural result in the comparison theory of projections for von Neumann algebras is proved: two monotone-increasing nets of projections indexed by the same directed set have unions that are equivalent when pairs of projections with the same index are equivalent. The same is not true, in general, for intersections of monotone-decreasing nets of projections. Counterexamples are given indicating limitations on extensions, variants, and methods for proving that result.

1. **Introduction.** The basic technique introduced by Murray and von Neumann [MvN] in the analysis of "factors", those von Neumann algebras whose centers consist of scalar multiples of the identity operator I, is that of "comparison" of the projections* in the von Neumann algebra $\mathcal{R}$. Analysis of a spectral-theoretic nature assures us of the existence of many projections E in $\mathcal{R}$ ($E = E^*$ and $E = E^2$); the norm closure of their linear span in $\mathcal{B}(\mathcal{H})$, the family (algebra) of all bounded linear operators on the Hilbert space $\mathcal{H}$, is $\mathcal{R}$. Murray and von

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*From personal conversations with Murray, von Neumann, and Halperin (to whose memory this article is dedicated), this author has concluded that a remark to von Neumann by Halperin was fundamental to the development of the "comparison theory" of projections. Halperin, as a young mathematician sent to the Institute for Advanced Study in Princeton (1935) for early guidance by von Neumann, was invited to be a party to the very first discussions von Neumann had with Murray. Murray, too, had been sent to von Neumann as a young post-doc. In those early discussions, von Neumann was setting Murray on the project that was to become the subject they called "rings of operators", and more specifically, "the theory of factors". The day following Halperin's remark, von Neumann mentioned to Halperin, excitedly, so I am told, that it seemed to lead to significant consequences. He requested Halperin's permission to use that suggestion. The permission was granted, of course. That situation is reminiscent of von Neumann's request to B. O. Koopman, a day after Koopman had made his observation to von Neumann that a measure-preserving transformation gives rise to a unitary operator on the L_2-Hilbert space of the measure (which became known, for a period, as "the Koopman method") that von Neumann be allowed to use it. Von Neumann had proved his "mean-ergodic theorem" with the aid of Koopman's construction a few hours after hearing of it from Koopman. Evidently, von Neumann was punctilious where others' ideas were involved.

Neumann define "equivalence" of projections E and F in $\mathcal{R}$ ("equivalence mod $\mathcal{R}$" written as $E \sim F$) by comparing the ranges of E and F via operators in $\mathcal{R}$. More specifically, $E \sim F \mod \mathcal{R}$, when there is a V in $\mathcal{R}$ that maps $E(\mathcal{H})$ isometrically onto $F(\mathcal{H})$ and maps $(I-E)(\mathcal{H})$ to (0). They write '$E \precsim F$' when $E \sim F_0 \le F$. The relation '$\precsim$' is a partial ordering of the equivalence classes of projections in $\mathcal{R}$ under $\sim$. (The only non-trivial aspect of that assertion is the equivalence of E and F when $E \precsim F$ and $F \precsim E$, which involves a "Cantor–Schröder–Bernstein" argument in a Hilbert-space setting.) This partial ordering is a linear ordering exactly when $\mathcal{R}$ is a factor.

"Additivity" of equivalence, is easy to establish: If $\{E_a\}_{a\in\mathbb{A}}$ and $\{F_a\}_{a\in\mathbb{A}}$ are each orthogonal families of projections in $\mathcal{R}$ such that $E_a \sim F_a$, then $\sum E_a \sim \sum F_a$. It provides the key to passing from algebraic properties of equivalence to properties deduced from the algebraic assertions that require an analytic bridge.

Another route to such an analytic bridge proceeds through increasing nets of projections and their strong-operator limits. If $\{E_a\}_{a\in\mathbb{A}}$ and $\{F_a\}_{a\in\mathbb{A}}$ are two such nets indexed and directed by the same directed set $\mathbb{A}$ (they are "increasing" in the sense that $E_a \le E_{a'}$ and $F_a \le F_{a'}$ when $a \le a'$) and $E_a \sim F_a$ for each a in $\mathbb{A}$, are the unions E of $\{E_a\}$ and F of $\{F_a\}$ equivalent in $\mathcal{R}$? Offhand, this would seem to follow from an easy conversion of this "monotone" analytic bridge to the "additive" analytic bridge. Specifically, dealing with increasing sequences for illustrative purposes, the union E of $E_1, E_2, \dots$ is the sum of the orthogonal set E_1, $E_2 - E_1$, $E_3 - E_2$, $\dots$ of projections E_1, $\{E_{j+1} - E_j\}_{j\in\mathbb{N}}$. The problem with this "conversion" is that, even though $E_j \sim F_j$, $E_{j+1} \sim F_{j+1}$, $E_j \le E_{j+1}$, and $F_j \le F_{j+1}$, the projections $E_{j+1} - E_j$ and $F_{j+1} - F_j$ need not be equivalent; one may be finite (in $\mathcal{R}$) while the other is infinite (in $\mathcal{R}$). At the same time, the seemingly "dual" problem of the equivalence of the intersections of termwise equivalent projections in decreasing sequences has a negative answer; one intersection may be infinite and the other may be finite of any dimension or the first and second may be finite with any given dimensions. To see this, let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and E_j be the projection with range spanned by $\{e_1, \dots, e_k, e_{k+j}, e_{k+j+1}, \dots\}$. Then $\bigwedge_{j=1}^{\infty} E_j$ is the projection F_k with range spanned by $\{e_1, \dots, e_k\}$. Of course, F_k and $F_{k'}$ are not equivalent in $\mathcal{B}(\mathcal{H})$ when $k \ne k'$, yet all the projections E_j have dimension $\aleph_0$ and are equivalent in $\mathcal{B}(\mathcal{H})$.

In Section 2 we shall see that equivalence holds in the case of unions (Theorem 3). We restrict attention, though, to the case of a countably decomposable von Neumann algebra to avoid an elaborate analysis in terms of higher infinite cardinals. In Remark 4, we shall note some limitations on extensions of this result.

On a number of occasions, we support our reasoning by reference to results in [KR1]– [KR4]. These references will be made by citing the number of the result alone. For example, 'Theorem 8.2.8' refers to the result numbered '8' in Section 2 of Chapter 8 of [KR2], while Exercise 7.6.6 refers to the exercise numbered '6' in Section 6 of Chapter 7 as that exercise appears with its complete solution in [KR4].

2. **Equivalence of limits.** It was noted in the preceding section that pairwise equivalent decreasing sequences of projections need not have equivalent intersections. Since the intersections are the strong-operator limits of the sequences, that warns us that the strong-operator limits, necessarily projections, of pairwise equivalent sequences of projections need not be equivalent, in general. There is something positive to be said in that direction when $\mathcal{R}$ is finite.

PROPOSITION 1. *If $\{E_a\}_{a\in\mathbb{A}}$ and $\{F_a\}_{a\in\mathbb{A}}$ are nets of projections in the finite von Neumann algebra $\mathcal{R}$ that are strong-operator convergent to E and F, respectively, and $E_a \sim F_a$ for each a in $\mathbb{A}$, then $E \sim F$.*

PROOF. Let τ be the center-valued trace on $\mathcal{R}$. From Theorem 8.2.8, τ is ultraweakly continuous on $\mathcal{R}$. For each x and y in the Hilbert space on which $\mathcal{R}$ acts, $\{\langle E_a x, y\rangle\}_{a\in\mathbb{A}}$ converges to $\langle Ex, y\rangle$. Thus $\{E_a\}_{a\in\mathbb{A}}$ and $\{F_a\}_{a\in\mathbb{A}}$ are weak-operator convergent to E and F, respectively. Since the ultraweak and weak-operator topologies coincide on the unit ball in $\mathcal{R}$, $\{E_a\}_{a\in\mathbb{A}}$ and $\{F_a\}_{a\in\mathbb{A}}$ are ultraweakly convergent to E and F, respectively. Thus $\{\tau(E_a)\}_{a\in\mathbb{A}}$ and $\{\tau(F_a)\}_{a\in\mathbb{A}}$ converge ultraweakly to $\tau(E)$ and $\tau(F)$, respectively. Since $E_a \sim F_a$, $\tau(E_a) = \tau(F_a)$ for each a in $\mathbb{A}$, whence $\tau(E) = \tau(F)$. From Theorem 8.4.3(vi), $E \sim F$. $\square$

REMARK 2. The preceding result is not valid, in general, when all the projections of the sequences and the limits are of finite (relative) dimension but $\mathcal{R}$ is not a finite von Neumann algebra. To see this, let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, $\mathcal{R}$ be $\mathcal{B}(\mathcal{H})$ and E_{kn} $(n > k)$ be the $k+1$-dimensional projection with range spanned by $e_1, \dots, e_k, e_n$. The strong-operator limit of $\{E_{kn}\}$ as n tends to infinity is E_k, the projection with range spanned by $e_1, \dots, e_k$, since $E_{kn}e_j = e_j$ when $j \in \{1, \dots, k, n\}$ and $E_{kn}e_j = 0$ otherwise. Thus $E_{kn}e_j \to E_k e_j$ as $n \to \infty$ for each j. As $\{e_j\}_{j\in\mathbb{N}}$ generates a dense linear submanifold of $\mathcal{H}$ and $\{E_{kn}, E_k\}$ is a bounded subset of $\mathcal{B}(\mathcal{H})$, $\{E_{kn}\}$ is strong-operator convergent to E_k as $n \to \infty$. (See [KR1, p. 114].)

Now, let F_{kn} $(n > k-1)$ be the $k+1$-dimensional projection with range spanned by $e_1, \dots, e_{k-1}, e_n, e_{n+1}$. By our earlier reasoning, $\{F_{kn}\}$ is strong-operator convergent to F_k, where F_k is the projection with range spanned by $e_1, \dots, e_{k-1}$. Of course, $E_{kn} \sim F_{kn}$ for all n and E_k is not equivalent to F_k (in $\mathcal{B}(\mathcal{H})$).

With the caution inspired by our examples, we turn to the equivalence question when $\mathcal{R}$ is not assumed to be finite.

THEOREM 3. *Let $\{E_a\}_{a\in\mathbb{A}}$ and $\{F_a\}_{a\in\mathbb{A}}$ be nets of projections indexed by the directed set $\mathbb{A}$ and monotone increasing with respect to the usual projection (operator) ordering. If $E_a \sim F_a$ in $\mathcal{R}$, a countably decomposable von Neumann algebra, for each a in $\mathbb{A}$, then $E \sim F$ where $E = \bigvee_{a\in\mathbb{A}} E_a$ and $F = \bigvee_{a\in\mathbb{A}} F_a$.*

Proof. From Proposition 6.2.8, the central carriers of E_a and F_a are the same (since $E_a \sim F_a$). From Proposition 5.5.3, the central carriers of E and F are, respectively, the unions of the central carriers of $\{E_a\}$ and $\{F_a\}$. Thus E and F have the same central carrier P.

If there is a non-zero subprojection P' of P, central in $\mathcal{R}$, such that $P'E_a$ is finite for all a (whence $P'F_a$ is finite for all a since $P'E_a \sim P'F_a$), let $\{P_b\}_{b\in\mathbb{B}}$ be a maximal orthogonal family of such subprojections and P_0 be its sum. Then P_0E_a and P_0F_a are finite for each a from Lemma 6.3.6. In addition, if P'' is a non-zero central subprojection of $P - P_0$ $(= P_0')$, then $P''E_a$ and, hence, $P''F_a$ are infinite for some a. Thus $P''E$ and $P''F$ are infinite. It follows that $P_0'E$ and $P_0'F$ are properly infinite (with the same central carrier P_0'). From Corollary 6.3.5, $P_0'E \sim P_0'F$ since $\mathcal{R}$ is countably decomposable.

It remains to show that $P_0E \sim P_0F$. Changing notation, we may assume that all E_a and F_a are finite in $\mathcal{R}$. If E and F are infinite, then there are (unique) non-zero, central projections P_1 and P_2 in $\mathcal{R}$ such that P_1E and P_2F are properly infinite, while $(I - P_1)E$ and $(I - P_2)F$ are finite (from Proposition 6.3.7). We show that $P_1 = P_2$ and that both of E and F are infinite if one is.

Since $\mathcal{R}$ is countably decomposable, a maximal orthogonal family of non-zero projections in $\mathcal{R}$ cyclic under $\mathcal{R}'$ must be countable. Let $G_1, G_2, \dots$ be such a family with unit cyclic vectors $x_1, x_2, \dots$, respectively. By maximality, $G_1 + G_2 + \cdots = I$. Let ω be $\sum_{j=1}^{\infty} 2^{-j}\omega_{x_j}|\mathcal{R}$. Then ω is a normal state of $\mathcal{R}$ with support I (from Exercise 7.6.1), and ω is a faithful state of $\mathcal{R}$ from Exercise 7.6.6. Thus the GNS representation π of $\mathcal{R}$ constructed from ω is faithful (from Exercise 4.6.15(i)). From Proposition 7.1.15, π is a normal (faithful) representation of $\mathcal{R}$ with a cyclic unit vector x_0 such that $\omega_{x_0} \circ \pi = \omega$. Thus $\omega_{x_0}|\pi(\mathcal{R})$ is a faithful normal state of the von Neumann algebra $\pi(\mathcal{R})$. If $Ax_0 = 0$ for some A in $\pi(\mathcal{R})$, then $\langle A^*Ax_0, x_0\rangle = 0$, and $A^*A = 0$. Hence $A = 0$, and x_0 is a separating vector for $\pi(\mathcal{R})$. Passing to the representation π of $\mathcal{R}$, we may assume that $\mathcal{R}$ has a separating unit vector x_0.

Since $\{E_a\}_{a\in\mathbb{A}}$ is strong-operator convergent to E, $P_1E_ax_0 \to P_1Ex_0$. We may choose an increasing sequence from $\{P_1E_a\}$, say, $P_1E_{a(1)}, P_1E_{a(2)}, \dots$, such that $P_1E_{a(j)}x_0$ tends to P_1Ex_0 as $j \to \infty$. Since each $P_1E_{a(j)}$ has norm not exceeding 1, x_0 is generating for $\mathcal{R}'$, and $P_1E_{a(j)}A'x_0$ tends to $P_1EA'x_0$ for each A' in $\mathcal{R}'$, $P_1E_{a(j)}$ converges to P_1E in the strong-operator topology. Thus $P_1E = P_1E_{a(1)} + \sum_{j=1}^{\infty}(P_1E_{a(j+1)} - P_1E_{a(j)})$. Moreover, $P_1E_{a(1)} \sim P_1F_{a(1)}$ and $P_1(E_{a(j+1)} - E_{a(j)}) \sim P_1(F_{a(j+1)} - F_{a(j)})$ from Exercise 6.9.26. Thus

$$\begin{aligned} P_1E &= P_1E_{a(1)} + \sum_{j=1}^{\infty}(P_1E_{a(j+1)} - P_1E_{a(j)}) \\ &\sim P_1F_{a(1)} + \sum_{j=1}^{\infty}(P_1F_{a(j+1)} - P_1F_{a(j)}) \le P_1F. \end{aligned}$$

It follows that F is infinite if E is. By symmetry, E is infinite if F is. Since P_1E is properly infinite by construction, P_1F is properly infinite, and $P_1 \le P_2$ (by definition of P_2). By symmetry, $P_2 \le P_1$ and $P_1 = P_2$ $(= P)$. Since PE and PF have the same central carrier and $\mathcal{R}$ is countably decomposable, the properly infinite projections PE and PF are equivalent in $\mathcal{R}$ (from Corollary 6.3.5).

We show, next, that $(I-P)E \sim (I-P)F$ in $\mathcal{R}$. From Theorem 6.3.8, the union of $(I-P)E$ and $(I-P)F$ is a finite projection G in $\mathcal{R}$. Thus $G\mathcal{R}G$ is a finite von Neumann algebra containing each $(I-P)E_a$ and $(I-P)F_a$. Moreover, $\{(I-P)E_a\}$ converges to $(I-P)E$ and $\{(I-P)F_a\}$ converges to $(I-P)F$ over a, in the strong-operator topology on $G\mathcal{R}G$. From Proposition 1, $(I-P)E \sim (I-P)F$ in $G\mathcal{R}G$ and, hence, in $\mathcal{R}$. □

Remark 4. The result of Theorem 3 does not seem to lend itself to much extension when we are in an "infinite environment". Suppose, for example, that $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$. Let E_j and F_j be, respectively, the projections in $\mathcal{B}(\mathcal{H})$ with ranges spanned by $\{e_1,\dots,e_j\}$ and $\{e_{j+1},\dots,e_{2j}\}$. Then $E_j \sim F_j$ in $\mathcal{B}(\mathcal{H})$, $\{E_j\}$ is monotone increasing, under the usual projection ordering, to I, while $\{F_j\}$ is strong-operator convergent to 0.

This example can be altered to yield one in which $\vee_{j=1}^{\infty} E_j$ is a finite projection. For that, we work in a factor $\mathcal{M}$ of type II_∞ on $\mathcal{H}$. Let $G_1, G_2, \dots$ be an orthogonal family of infinite projections in $\mathcal{M}$ with sum I. Let $\mathcal{D}$ be a dimension function on $\mathcal{M}$ based on a (finite) "unit" projection in $\mathcal{M}$ (so that unit projection has dimension 1). We can find an increasing sequence $\{E_j\}$ in $\mathcal{M}$ of subprojections of G_1 such that $\mathcal{D}(E_j) = \frac{j}{j+1}$. Then $\vee_{j=1}^{\infty} E_j = E$, where $\mathcal{D}(E) = 1$. Now, let F_j be a subprojection of G_j in $\mathcal{M}$ such that $\mathcal{D}(F_j) = \frac{j}{j+1}$. Then $E_j \sim F_j$ in $\mathcal{M}$ and $\{F_j\}$ is strong-operator convergent to 0 (by an argument very close to that of the corresponding observation in Remark 2).

In both of these examples, the sequences $\{E_j\}$ and $\{F_j\}$ are monotone with respect to the ordering "$\precsim$". In neither of the examples are the strong-operator limits equivalent and in the second example, the unions are not equivalent as well. □

Remark 5. In the (primitive) case of a factor $\mathcal{M}$ acting on a separable Hilbert space $\mathcal{H}$, when the nets involved are monotone increasing sequences of projections $\{E_j\}_{j=1,2,\dots}$, $\{F_j\}_{j=1,2,\dots}$, a (relatively) simple argument can be given to show that $\bigvee_{j=1}^{\infty} E_j$ $(= E) \sim \bigvee_{j=1}^{\infty} F_j$ $(= F)$. We sketch that argument. Since $E_j \sim F_j$, for each j, by assumption, if either E_j or F_j is infinite, for some j, the other is, as well, by Proposition 6.3.2, and both E and F are infinite. In that case, $E \sim F$, from Corollary 6.3.5. So, we may assume that all E_j and F_j are finite. In this case, $E_j \vee F_j$ $(= G_j)$ is finite (Theorem 6.3.8), whence $G_j\mathcal{M}G_j$ is a finite factor (Exercise 6.9.15(iii)) and $E_j - E_{j-1} \sim F_j - F_{j-1}$ in $\mathcal{M}$. Let V_j be a partial isometry in $\mathcal{M}$ with initial and final projections $E_j - E_{j-1}$ and $F_j - F_{j-1}$, respectively, where we define E_0 and F_0 to be 0. Then $\sum_{j=1}^{\infty} V_j$ converges in the strong-operator topology on $\mathcal{B}(\mathcal{H})$ to a partial isometry in $\mathcal{M}$ with initial projection E and final projection F.

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Mathematics Department
University of Pennsylvania
Philadelphia, PA 19104-6395
U.S.A.
email: kadison@math.upenn.edu

and

Mathematics Department
Louisiana State University
Baton Rouge, LA 70803
U.S.A.