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Non-commutative Conditional Expectations and their Applications

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ABSTRACT. The motivation for viewing certain idempotent linear mappings from an operator algebra onto a subalgebra as "non-commutative conditional expectations" is explained starting from the classical, measure-theoretic meaning of "conditioned expectation." The basic theory and several of the applications of non-commutative conditional expectations are studied in the operator-algebra framework.

1. Introduction

This article is an extended version of a fifty-minute lecture delivered to a Special Session of the American Mathematical Society on January 15, 2003 in Baltimore, MD. My goal in that lecture was to explain how the classical, or what we shall refer to as, "commutative," case of measure and probability theory, with particular emphasis on the concept of *conditional expectation* can be made *non-commutative* and why it is important to do that. We include some of the beautiful results that have been proved in the non-commutative case and a few of the applications of non-commutative, conditional expectations. In many cases, the results along the way to a main result are new or new formulations of an older result. In most cases, there are new or "updated" proofs. Some of the concluding material, on "Schur Inequalities," is part of work-in-progress with W. B. Arveson. References such as "Corollary 8.3.12" are to the correspondingly numbered result in **[K-R I,II,III,IV]**. We use the notation of **[K-R]** as well.

2. Background and preliminaries

We begin with a background discussion that establishes much of our notation and many of the definitions we need.

We deal with a complex Hilbert space $\mathcal{H}$ on which $\langle x, y\rangle$ denotes the inner product of x and y and $\|x\|(= \langle x, x\rangle^{\frac{1}{2}})$ is the length or *norm* of the vector x. We use notation of the form T to denote a linear transformation of $\mathcal{H}$ into $\mathcal{H}$

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(referred to as *an operator on* $\mathcal{H}$). Recall that such a T is continuous if and only if $\sup\{\|Tx\| : \|x\| \leq 1\} = \|T\| < \infty$, in which case, we say that T is *bounded* and $\|T\|$ is its *bound* or *norm*. Moreover, $T \to \|T\|$ is a norm on $\mathcal{B}(\mathcal{H})$, the family of all bounded T on $\mathcal{H}$. The family $\mathcal{B}(\mathcal{H})$ is an algebra under the operations, $(aA + B)x = a(Ax) + Bx$ and $(AB)x = A(Bx)$, of addition and multiplication, respectively.

The metric topology defined by the norm on $\mathcal{B}(\mathcal{H})$ is referred to as the *norm topology*.

The adjoint A^* of an operator A in $\mathcal{B}(\mathcal{H})$ is the unique operator in $\mathcal{B}(\mathcal{H})$, satisfying $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for each pair of vectors x and y in $\mathcal{H}$. A subset $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$ is said to be *self-adjoint* when $\mathcal{F} = \mathcal{F}^*$, where $\mathcal{F}^* = \{T^* : T \in \mathcal{F}\}$. The principal structure we study, and the basis for our "non-commutative" extensions of classical analytic and measure-theoretic concepts is the *C^*-algebra.* It is a norm-closed, self-adjoint subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$. For the purposes of this article, we may assume that $I \in \mathcal{A}$, where $Ix = x$ for all x in $\mathcal{H}$.)

There are many reasons for studying operator algebras. One of the first motivations for their study is their role as complex, group algebras for infinite groups, discrete and topological. If you want to study the structure and representations of a topological group, the operator algebra as group algebra provides a powerful tool [**Se47**]. An operator algebra is the main component of the most natural mathematical model of a quantum mechanical system. At the same time, operator algebras are a prominent tool in the study of families of measure (and measurability)-preserving transformations.

Aside from studying the properties of a general C^*-algebra, the main approach to understanding the structure of C^*-algebras is to describe what families of C^*-algebras defined by certain common properties are like. Let me illustrate this, first, by describing abelian (that is, commuting) C^*-algebras.

THEOREM. *Each abelian C^*-algebra $\mathcal{A}$ is isomorphic to the algebra $C(X)$ of complex valued continuous functions on a compact-Hausdorff space X (under pointwise operations). Two such algebras are isomorphic iff the associated compact-Hausdorff spaces are homeomorphic. Each $C(X)$ is isomorphic to some abelian C^*-algebra $\mathcal{A}$.*

This theorem can be drawn directly from a result by Stone [**St40**] or from one in the famous 1943 Gelfand–Neumark paper [**G-N43**].

Another class of C^*-algebras of central importance are constructed from a countably infinite, discrete group. Referring back to our introductory discussion, they are an *operator-algebra group algebra* for that group.

Let G be a countable (discrete) group and $\mathcal{H}$ be $l_2(G)$, that is

$$\left\{\varphi : \sum_{g\in G} |\varphi(g)|^2 < \infty\right\}, \qquad \langle\varphi, \psi\rangle = \sum_{g\in G} \varphi(g)\overline{\psi(g)}.$$

Let $(L_g\varphi)(g')$ be $\varphi(g^{-1}g')$ $(\varphi \in \mathcal{H})$. Then L_g is a unitary operator. Let $C_{\mathrm{r}}^*(G)$ (the *reduced C^*-group algebra of* G) be the norm closure of the algebra generated by $\{L_g\}$.

For our measure-theoretic purposes it is necessary to introduce the *strong-operator topology* on $\mathcal{B}(\mathcal{H})$. It is the topology in which convergence of $\{A_n\}$ to A means $A_nx \to Ax$ for each x in $\mathcal{H}$. Let $\mathcal{L}_G$ be the strong-operator closure of the algebra generated by $\{L_g\}$. The algebra $\mathcal{L}_G$ is an example of a *von Neumann algebra* — a C^*-algebra that is strong-operator closed. Among the von Neumann algebras, those whose centers consist of scalar multiples of I are called *factors.* The general von Neumann algebra is not quite a direct sum of factors, rather, a "direct integral." (See Chapter 14 of **[K-R2]**.) In a series of papers [**M-vN36**],[**M-vN37**],[**M-vN43**],[**vN40**] from 1936 to 1943, Murray and von Neumann studied these factors intensively. They separated them into three main types. The factors of type I are those that have a minimal idempotent. Thus $\mathcal{B}(\mathcal{H})$ is a type I factor, where the projections on one-dimensional subspaces are minimal idempotents.

THEOREM (Murray–von Neumann). *Each type I factor is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

If $\mathcal{H}$ has dimension n, where n is a finite or infinite cardinal, we say that the factor is of type I_n. One of the classes of factors that Murray and von Neumann discovered, the factors of type II_1, has fascinating properties. Those factors have no minimal idempotents and admit a trace-like functional (a linear functional τ such that, for all A and B, $\tau(AB) = \tau(BA)$). The behavior of those II_1 factors resembles that of $M_n(\mathbb{C})$ in many ways, among others, they are simple algebras — but of course, they have infinite linear dimension. We have examples of them at hand.

THEOREM. *$\mathcal{L}_G$ is a factor iff all conjugacy classes in G but $\{e\}$ are infinite. In this case, $\mathcal{L}_G$ is a factor of type II_1.*

The free (non-abelian) group $\mathcal{F}_n$ on $n(> 1)$ generators and Π, the group of "finite" permutations of the integers, are examples of these *i.c.c groups.*

THEOREM. *$\mathcal{L}_{\mathcal{F}_n}$ is not isomorphic to $\mathcal{L}_\Pi$.*

If x_g is the function that is 1 at the group element g and 0 at each other element of the group G, then x_g is a generating and separating unit vector for the von Neumann algebra $\mathcal{L}_G$ and the functional defined by: $A \to \langle Ax_g, x_g\rangle$ is a (faithful, normal) tracial state on $\mathcal{L}_G$. The vector x_g is referred to as a *unit trace vector* for $\mathcal{L}_G$ in this case. The set $\{x_g : g \in G\}$ is an orthonormal basis for $l_2(G)$.

In the case of abelian von Neumann algebras measure theory enters the picture via deep results of von Neumann [**vN31**]. Let (S, μ) be a σ-finite measure space, $\mathcal{H}$ be $L_2(S, \mu)$, and f be an essentially bounded measurable function on S. Define $M_f(g)$ to be $f \cdot g$ for each g in $\mathcal{H}$. Let $\mathcal{A}$ be the set $\{M_f\}$. Then $\mathcal{A}$ is an abelian von Neumann algebra in no larger abelian subalgebra of $\mathcal{B}(\mathcal{H})$. We say that $\mathcal{A}$ is *maximal abelian.* Let us look at some specific examples.

If S has a finite or countably infinite number of points, say n, and each point has a positive measure (each is an *atom*), we write $\mathcal{A}_n$ or $\mathcal{A}_{\mathrm{d}}$ for $\mathcal{A}$. If S is $[0,1]$ and μ is Lebesgue measure, we write $\mathcal{A}_{\mathrm{c}}$ for $\mathcal{A}$. If S is $[0,1]$ (with Lebesgue measure) + a finite or (countably) infinite number of atoms, we write $\mathcal{A}_{\mathrm{c}} \oplus \mathcal{A}_n$ or $\mathcal{A}_{\mathrm{c}} \oplus \mathcal{A}_{\mathrm{d}}$ for $\mathcal{A}$. The atoms in S correspond to the minimal projections in $\mathcal{A}$. Moreover, n is the number of atoms in S, if there are any.

THEOREM. *Each abelian von Neumann algebra on a separable Hilbert space is isomorphic to one of $\mathcal{A}_n$, $\mathcal{A}_{\mathrm{c}}$, or $\mathcal{A}_{\mathrm{c}} \oplus \mathcal{A}_n$. Each maximal abelian von Neumann algebra is unitarily equivalent to one of these.*

From these results, measure spaces and the associated algebras of bounded measurable functions can be studied totally within the framework of abelian von Neumann algebras. In this framework, the random variables, that is, the measurable functions, on the probability space correspond to operators in the abelian von Neumann algebra. Where is the measure in the operator framework? What we have is the integration process associated with that measure — a linear functional ρ on the algebra $\mathcal{A}$ that assigns 1 to the identity operator I (that corresponds to the probability space having total measure 1) and assigns a non-negative real number to each positive A in $\mathcal{A}$ (corresponding to a positive measure). Such a functional is called a *state* of $\mathcal{A}$. The same definition and terminology apply to all C^*-algebras for that matter. For a von Neumann algebra $\mathcal{R}$, we usually want ρ to satisfy a strong-operator continuity condition (corresponding to countable additivity of the measure). Specifically, we are primarily interested in *normal* states ρ of $\mathcal{R}$: those such that $\rho(\sum_{a\in\mathbb{A}} E_a) = \rho(E)$, where $\{E_a\}_{a\in\mathbb{A}}$ is an orthogonal family of projections in $\mathcal{R}$ and $E = \sum_{a\in\mathbb{A}} E_a$.

3. Conditional expectations

To recall, the "expectation" or "expected value" of a random variable f or an observable A is the average or 'mean' of a "large" number of values of f or measurements of A at points of the measure (sample) space, or with the dynamical system in a given state. The average is taken with reference to the measure μ on the total space, hence is $\int f\,d\mu$. The expectation of one random variable f "conditioned" by another g is, again, the average value of f, but at (sample) points at which g fulfills the prescribed conditions. In the most primitive instance, the condition may be that g have a given value λ. Say, the set of points at which that occurs is S. Then the "conditioned" expectation of f for that condition is $\mu(S)^{-1}\int_S f\,d\mu$. If we do this for the various values g may assume, we partition the space X into (disjoint) sets on each of which we calculate the expectation of f. The result is then a function f_0, constant where g is constant — so f_0 is a function of g and lies in the algebra $\mathcal{A}(g)$ generated by g (and 1). When g takes on only a finite set of values (that is, g is a "step function"), f_0 is a polynomial in g. If we start with a function f constant where g is constant, then f_0 is f. Thus the mapping $f \to f_0$ is idempotent. It is linear and positive (that is, $f_0 \geq 0$ when $f \geq 0$). Notice, too, that if we multiply an arbitrary random variable f by a function h in $\mathcal{A}(g)$ and form the "conditioned" expectation $(hf)_0$ of hf, then $(hf)_0 = hf_0$, for on each of the "level" sets of g we have multiplied the expectation of f by the (constant) value assumed by h. Moreover, $\int f\,d\mu = \int f_0\,d\mu$, from the definition of f_0 and since the distinct level sets are disjoint.

Of course, we could "condition" f by several random variables $g_1, g_2, \ldots$, or an arbitrary family, or a subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}$ of all (bounded) random variables. It is appropriate, now, to define the entire process we have been discussing, the mapping from random variables in $\mathcal{B}$ to those in a subalgebra $\mathcal{A}$, by the features we have noted, as a *conditional expectation* (from $\mathcal{B}$ onto $\mathcal{A}$). It adds no difficulty to make this definition for the non-commutative case (that is, the general case — so, including the commutative case) and for a C^*-algebra as well.

Definition. A positive, linear, mapping Φ of a von Neumann algebra $\mathcal{S}$ onto a von Neumann subalgebra $\mathcal{R}$ ($\mathcal{S}$ and $\mathcal{R}$ may be general C^*-algebras as well) is said to be a *conditional expectation* (of $\mathcal{S}$ onto $\mathcal{R}$) when $\Phi(I) = I$ and $\Phi(R_1SR_2) = R_1\Phi(S)R_2$ if $R_1, R_2 \in \mathcal{R}$ and $S \in \mathcal{S}$.

From some basic operator-algebra theory, a positive-linear mapping taking I to I, such as Φ, has norm 1 (as a mapping between normed spaces). If we choose S and R_2 to be I in the preceding definition, it follows that Φ is the identity on $\mathcal{R}$, and is, therefore, an idempotent mapping of $\mathcal{S}$ onto $\mathcal{R}$.

PROPOSITION 1. *Suppose that $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras acting on a Hilbert space $\mathcal{H}$, $\mathcal{R} \subseteq \mathcal{S}$, and Φ is a conditional expectation from $\mathcal{S}$ into $\mathcal{R}$. For all S in $\mathcal{S}$,*

(i) $\Phi(S^*) = \Phi(S)^*$
(ii) $\Phi(S)^*\Phi(S) \le \Phi(S^*S)$
(iii) $\|\Phi(S)\| \le \|S\|$.

PROOF. (i) Each self-adjoint element H of $\mathcal{S}$ has the form $H_1 - H_2$, where $H_1, H_2 \in \mathcal{S}^+$. Since Φ is a positive linear mapping, $\Phi(H_1)$, $\Phi(H_2) \in \mathcal{R}^+$ and $\Phi(H)(= \Phi(H_1) - \Phi(H_2))$ is self-adjoint. Each element S of $\mathcal{S}$ has the form $H + iK$, where H and K are self-adjoint elements of $\mathcal{S}$. From the preceding paragraph, $\Phi(H)$ and $\Phi(K)$ are self-adjoint. Thus

$$\Phi(S)^* = [\Phi(H) + i\Phi(K)]^* = \Phi(H) - i\Phi(K) = \Phi(S^*).$$

(ii) When $R \in \mathcal{R}(\subseteq \mathcal{S})$ and $S \in \mathcal{S}$, we have $(S-R)^*(S-R) \ge 0$, and thus

$$\begin{aligned} 0 \le \Phi((S-R)^*(S-R)) &= \Phi(S^*S - R^*S - S^*R + R^*R) \\ &= \Phi(S^*S) - R^*\Phi(S) - \Phi(S)^*R + R^*R. \end{aligned}$$

When R is the element $\Phi(S)$ of $\mathcal{R}$, we obtain $0 \le \Phi(S^*S) - \Phi(S)^*\Phi(S)$, so $\Phi(S)^*\Phi(S) \le \Phi(S^*S)$.

(iii) From (ii), and since $S^*S \le \|S\|^2 I$, we have

$$\Phi(S)^*\Phi(S) \le \Phi(S^*S) \le \|S\|^2\Phi(I) = \|S\|^2 I,$$

and thus $\|\Phi(S)\|^2 = \|\Phi(S)^*\Phi(S)\| \le \|S\|^2$. Hence $\|\Phi(S)\| \le \|S\|$. □

PROPOSITION 2. *Let $\mathcal{U}$ be the unitary group in a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$. Suppose that $\mathrm{co}_{\mathcal{R}}(T)^-$ meets the commutant $\mathcal{R}'$, for each T in $\mathcal{B}(\mathcal{H})$, where $\mathrm{co}_{\mathcal{R}}(T)^-$ denotes the weak-operator closure of the convex hull $\mathrm{co}_{\mathcal{R}}(T)$ of the set $\{UTU^* : U \in \mathcal{U}\}$. Let $\mathcal{M}$ be the set of all positive linear mappings $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\varphi(I) = I$, $\varphi(R_1'TR_2') = R_1'\varphi(T)R_2'$, and $\varphi(T) \in \mathrm{co}_{\mathcal{R}}(T)^-$, when $T \in \mathcal{B}(\mathcal{H})$ and $R_1', R_2' \in \mathcal{R}'$. Let $\mathcal{D}(\subseteq \mathcal{M})$ be the set of all mappings $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that can be defined by an equation of the form $\alpha(T) = \sum_{j=1}^{k} a_jU_jTU_j^*$, where $U_1, \dots, U_k \in \mathcal{U}$ and $a_1, \dots, a_k$ are positive scalars with sum 1. Then*

(i) *$\varphi(R') = R'$ when $\varphi \in \mathcal{M}$ and $R' \in \mathcal{R}'$; moreover, $\varphi_1 \circ \varphi_2 \in \mathcal{M}$ when $\varphi_1, \varphi_2 \in \mathcal{M}$;*
(ii) *$\mathcal{M}$ can be viewed as a closed subset of the topological space $\prod_{T\in\mathcal{B}(\mathcal{H})} X_T$, where X_T is $\mathrm{co}_{\mathcal{R}}(T)^-$ with the weak-operator topology.*
(iii) *If $T_0 \in \mathcal{B}(\mathcal{H})$ and $A_0' \in \mathrm{co}_{\mathcal{R}}(T_0)^- \cap \mathcal{R}'$, then $A_0' = \psi(T_0)$ for some ψ in $\mathcal{M}$.*
(iv) *If $T_1, \dots, T_n \in \mathcal{B}(\mathcal{H})$, there is a φ in $\mathcal{M}$ such that $\varphi(T_1), \dots, \varphi(T_n) \in \mathcal{R}'$.*
(v) *For each finite subset $\mathbb{F}$ of $\mathcal{B}(\mathcal{H})$, let $\mathcal{M}_{\mathbb{F}}$ be $\{\varphi \in \mathcal{M} : \varphi(T) \in \mathcal{R}'$if $T \in \mathbb{F}\}$. The family of all such sets $\mathcal{M}_{\mathbb{F}}$ has non-empty intersection.*

(vi) *There is a conditional expectation Φ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$, with the property that $\Phi(T) \in \mathrm{co}_{\mathcal{R}}(T)^- \cap \mathcal{R}'$ for each T in $\mathcal{B}(\mathcal{H})$. If $T_0 \in \mathcal{B}(\mathcal{H})$ and $A_0' \in \mathrm{co}_{\mathcal{R}}(T_0)^- \cap \mathcal{R}'$, then Φ can be chosen so that $\Phi(T_0) = A_0'$.*

PROOF. (i) If $\varphi \in \mathcal{M}$ and $R' \in \mathcal{R}'$, we have $\varphi(R') = \varphi(R'II) = R'\varphi(I)I = R'$. If $\varphi_1, \varphi_2 \in \mathcal{M}$, then $\varphi_1 \circ \varphi_2$ is a positive linear mapping from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$, and $\varphi_1(\varphi_2(I)) = \varphi_1(I) = I$. Also, $\varphi_1(\varphi_2(R_1'TR_2')) = \varphi_1(R_1'\varphi_2(T)R_2') = R_1'\varphi_1(\varphi_2(T))R_2'$, whenever $R_1', R_2' \in \mathcal{R}'$ and $T \in \mathcal{B}(\mathcal{H})$. In order to complete the proof that $\varphi_1 \circ \varphi_2 \in \mathcal{M}$, we have to show that $\varphi_1(\varphi_2(T)) \in \mathrm{co}_{\mathcal{R}}(T)^-$ for each T in $\mathcal{B}(\mathcal{H})$. Since $\varphi_2(T) \in \mathrm{co}_{\mathcal{R}}(T)^-$ and $\varphi_1(\varphi_2(T)) \in \mathrm{co}_{\mathcal{R}}(\varphi_2(T))^-$, it suffices to show that $\mathrm{co}_{\mathcal{R}}(T_1)^- \subseteq \mathrm{co}_{\mathcal{R}}(T)^-$ when $T_1 \in \mathrm{co}_{\mathcal{R}}(T)^-$. Now it is apparent that $USU^* \in \mathrm{co}_{\mathcal{R}}(T)$ whenever $U \in \mathcal{U}$ and $S \in \mathrm{co}_{\mathcal{R}}(T)$. From this, together with the weak-operator continuity of the mapping $A \to UAU^*$, it follows that $USU^* \in \mathrm{co}_{\mathcal{R}}(T)^-$ whenever $U \in \mathcal{U}$ and $S \in \mathrm{co}_{\mathcal{R}}(T)^-$. In particular, $\mathrm{co}_{\mathcal{R}}(T)^-$ contains the set $\{UT_1U^* : U \in \mathcal{U}\}$, and so contains its weak-operator closed convex hull $\mathrm{co}_{\mathcal{R}}(T_1)^-$, when $T_1 \in \mathrm{co}_{\mathcal{R}}(T)^-$. This completes the proof that $\varphi_1 \circ \varphi_2 \in \mathcal{M}$ when $\varphi_1, \varphi_2 \in \mathcal{M}$.

(ii) The product space Π is the set of all mappings $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\pi(T) \in \mathrm{co}_{\mathcal{R}}(T)^-$ for each T in $\mathcal{B}(\mathcal{H})$, with the coarsest topology that makes each of the "coordinate mappings" $\pi \to \pi(T) : \Pi \to \mathcal{B}(\mathcal{H})$ continuous relative to the weak-operator topology on $\mathcal{B}(\mathcal{H})$. Since the bounded closed set X_T is weak-operator compact, it follows from Tychonoff's theorem that Π is compact. The set $\mathcal{M}$ consists of those elements φ of Π that satisfy the conditions $\varphi(I) - I = 0$, $\varphi(aS+bT) - a\varphi(S) - b\varphi(T) = 0$, $\varphi(R_1'TR_2') - R_1'\varphi(T)R_2' = 0$, $\varphi(H) \in \mathcal{B}(\mathcal{H})^+$, when $H \in \mathcal{B}(\mathcal{H})^+$, $S, T \in \mathcal{B}(\mathcal{H})$, $R_1', R_2' \in \mathcal{R}'$, and $a, b \in \mathbb{C}$. Since each of the mappings $\pi \to \pi(I) - I$, $\pi \to \pi(aS + bT) - a\pi(S) - b\pi(T)$, $\pi \to \pi(R_1'TR_2') - R_1'\pi(T)R_2'$, and $\pi \to \pi(H)$, from Π into $\mathcal{B}(\mathcal{H})$ (with the weak-operator topology) is continuous, $\mathcal{M}$ is the intersection of a family of sets, each one of which is the inverse image of a closed set (either $\{0\}$ or $\mathcal{B}(\mathcal{H})^+$) under a continuous mapping. Hence $\mathcal{M}$ is a closed subset of Π, and is therefore compact in the relative topology.

(iii) If $T_0 \in \mathcal{B}(\mathcal{H})$ and $A_0' \in \mathrm{co}_{\mathcal{R}}(T_0)^- \cap \mathcal{R}'$, there is a net $\{S_j\}$ in $\mathrm{co}_{\mathcal{R}}(T_0)$ that is weak-operator convergent to A_0'. For each index j, there exists an element α_j of $\mathcal{D}$ such that $S_j = \alpha_j(T_0)$. Since $\mathcal{M}$ is compact, the net $\{\alpha_j\}$ has a subnet $\{\alpha_{j_k}\}$ that converges to an element ψ of $\mathcal{M}$. Since the "coordinate mapping" $\pi \to \pi(T_0)$ is continuous,

$$\psi(T_0) = \lim_k \alpha_{j_k}(T_0) = \lim_k S_{j_k} = \lim_j S_j = A_0'.$$

(iv) Given T_1 in $\mathcal{B}(\mathcal{H})$, suppose A_1' is in $\mathrm{co}_{\mathcal{R}}(T_1)^- \cap \mathcal{R}'$. By (iii), $A_1' = \varphi(T_1)$ for some φ in $\mathcal{M}$. This proves the stated result in the case in which $n = 1$. Now suppose that r is a positive integer, and the stated result has been proved in the case in which $n = r$. Accordingly, given $T_1, \dots, T_{r+1}$ in $\mathcal{B}(\mathcal{H})$, there is an element φ_0 of $\mathcal{M}$ such that $\varphi_0(T_1), \dots, \varphi_0(T_r) \in \mathcal{R}'$. If $A' \in \mathrm{co}_{\mathcal{R}}(\varphi_0(T_{r+1}))^- \cap \mathcal{R}'$, from (iii), $\psi(\varphi_0(T_{r+1})) = A'$ $(\in \mathcal{R}')$ for some ψ in $\mathcal{M}$. Moreover, $\psi(\varphi_0(T_j)) = \varphi_0(T_j)$ $(\in \mathcal{R}')$ when $1 \le j \le r$, by (i). If φ is $\psi \circ \varphi_0$ (in $\mathcal{M}$), we have $\varphi(T_j) \in \mathcal{R}'$ $(j = 1, \dots, r+1)$. This completes the proof by induction of the result stated in (iv).

(v), (vi) For T in $\mathcal{B}(\mathcal{H})$, the subset $\{\varphi \in \mathcal{M} : \varphi(T) \in \mathcal{R}'\}$ of $\mathcal{M}$ is closed, since $\mathcal{R}'$ is (weak-operator) closed in $\mathcal{B}(\mathcal{H})$ and the mapping $\varphi \to \varphi(T) : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is continuous. Thus $\mathcal{M}_{\mathbb{F}}$ (a finite intersection of sets of the type just considered) is closed in $\mathcal{M}$, for each finite subset $\mathbb{F}$ of $\mathcal{B}(\mathcal{H})$. Moreover, $\mathcal{M}_{\mathbb{F}}$ is not empty, by (iv). Since $\mathcal{M}_{\mathbb{F}_1} \cap \mathcal{M}_{\mathbb{F}_2} = \mathcal{M}_{\mathbb{F}_1 \cup \mathbb{F}_2}$, the family $\{\mathcal{M}_{\mathbb{F}}\}$ of all such sets has the finite intesection property. Since $\mathcal{M}$ is compact, the intersection of all the sets $\mathcal{M}_{\mathbb{F}}$ is not empty. With Φ in this intersection, $\Phi \in \mathcal{M}$ and $\Phi(T) \in \mathcal{R}'$ for each T in $\mathcal{B}(\mathcal{H})$. Thus Φ is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$, with the property that $\Phi(T) \in \mathrm{co}_{\mathcal{R}}(T)^- \cap \mathcal{R}'$ for each T in $\mathcal{B}(\mathcal{H})$.

Given T_0 in $\mathcal{B}(\mathcal{H})$ and A_0' in $\mathrm{co}_{\mathcal{R}}(T_0)^- \cap \mathcal{R}'$, choose ψ as in (iii). Then $\Phi \circ \psi \in \mathcal{M}$, and $\Phi \circ \psi$ maps $\mathcal{B}(\mathcal{H})$ into $\mathcal{R}'$ (because Φ does so). Hence $\Phi \circ \psi$ is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$, $(\Phi \circ \psi)(T) \in \mathrm{co}_{\mathcal{R}}(T)^- \cap \mathcal{R}'$ for each T in $\mathcal{B}(\mathcal{H})$, and

$$(\Phi \circ \psi)(T_0) = \Phi(\psi(T_0)) = \Phi(A_0') = A_0'.$$

□

PROPOSITION 3. *Suppose that $\mathcal{R}$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, and there is a family $\{\mathcal{R}_a\}_{a \in \mathbb{A}}$ of finite-dimensional *subalgebras of $\mathcal{R}$ such that if $a, b \in \mathbb{A}$, then there is a c in $\mathbb{A}$ for which $\mathcal{R}_a \cup \mathcal{R}_b \subseteq \mathcal{R}_c$, and $\mathcal{R} = (\cup_{a \in \mathbb{A}} \mathcal{R}_a)^-$. For each A in $\mathcal{B}(\mathcal{H})$, $\mathrm{co}_{\mathcal{R}}(A)^-$ meets $\mathcal{R}'$. There is a conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$.*

PROOF. We may assume that $I \in \mathcal{R}_a$, whence the unitary group of $\mathcal{R}$ contains that of $\mathcal{R}_a$, for each a in $\mathbb{A}$. The unitary group of a finite-dimensional von Neumann algebra $\mathcal{S}$ has a finite subgroup whose linear span is $\mathcal{S}$. To see this, suppose first that $\mathcal{S}$ is a type I_n factor, and let $\{E_{jk} : j, k = 1, \dots, n\}$ be a self-adjoint system of matrix units for $\mathcal{S}$. With $S(n)$ the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$ and F the class of all mappings from $\{1, 2, \dots, n\}$ into $\{1, -1\}$, define $V(f, \pi) = \sum_{j=1}^{n} f(j) E_{\pi(j)j}$ where $f \in F$ and $\pi \in S(n)$. Then, the set $\{V(f, \pi) : f \in F, \pi \in S(n)\}$ is a finite subgroup $\mathcal{V}$ of the unitary group of $\mathcal{S}$. (In terms of matrices relative to $\{E_{jk}\}$, $\mathcal{V}$ is generated by the group of permutation matrices and the group of diagonal matrices with ± 1 at each diagonal entry.) The linear span of $\mathcal{V}$ contains each E_{jk}, and is, therefore, all of $\mathcal{S}$. Note that $\mathcal{V} = -\mathcal{V}$.

The general finite-dimensional von Neumann algebra $\mathcal{S}$ is (*isomorphic to) a finite direct sum $\sum_{j=1}^{m} \oplus \mathcal{S}_j$ of finite-dimensional factors $\mathcal{S}_1, \dots, \mathcal{S}_m$. From the preceding, the unitary group of $\mathcal{S}_j$ has a finite subgroup $\mathcal{V}_j$ $(= -\mathcal{V}_j)$ whose linear span is $\mathcal{S}_j$. Thus $\mathcal{V}$ $(= \{\sum_{j=1}^{m} \oplus V_j : V_j \in \mathcal{V}_j\})$ is a finite subgroup of the unitary group of $\mathcal{S}$, and has linear span $\mathcal{S}$.

We note next that, for each a in $\mathbb{A}$, $\mathrm{co}_{\mathcal{R}}(A)$ meets $\mathcal{R}_a'$. Let $\mathcal{V}$ be a finite subgroup of the unitary group $\mathcal{R}_a$, whose linear span is $\mathcal{R}_a$: and define T to be $n^{-1} \sum_{V \in \mathcal{V}} VAV^*$, where n is the order of $\mathcal{V}$. Then $T \in \mathrm{co}_{\mathcal{R}}(A)$ and, since left translation by an element W of $\mathcal{V}$ permutes $\mathcal{V}$, we have

$$WTW^* = n^{-1} \sum_{V \in \mathcal{V}} (WV)A(WV)^* = T$$

for each W in V. Thus $WT = TW$ for every W in $\mathcal{V}$. Hence $T \in \mathcal{R}_a' \cap \mathrm{co}_{\mathcal{R}}(A)$.

For each a in $\mathbb{A}$, the convex set $\mathcal{S}_a = \mathcal{R}_a' \cap \mathrm{co}_{\mathcal{R}}(A)$ is non-empty, from what we have just proved, and is weak-operator compact since it is closed and bounded. When $a, b \in \mathbb{A}$, we can choose c in $\mathbb{A}$ so that $\mathcal{R}_a \cup \mathcal{R}_b \subseteq \mathcal{R}_c$, and then $\mathcal{S}_a \cap \mathcal{S}_b \supseteq \mathcal{S}_c$. Thus the family $\{\mathcal{S}_a\}_{a \in \mathbb{A}}$ has the finite intersection property. Since $\mathcal{S}_a$ is compact,

$$\emptyset \neq \bigcap_{a \in \mathbb{A}} \mathcal{S}_a = \bigcap_{a \in \mathbb{A}} \mathcal{R}_a' \cap \mathrm{co}_{\mathcal{R}}(A)^- = \mathcal{R}' \cap \mathrm{co}_{\mathcal{R}}(A)^-.$$

□

One of the most profound uses of conditional expectations in the theory of operator algebras is the partial converse of A. Connes [**Co76**] to Proposition 3: Each factor of type II_1 on a separable Hilbert space $\mathcal{H}$ that is the range of a conditional expectation from $\mathcal{B}(\mathcal{H})$ is isomorphic to $\mathcal{L}_{\Pi}$.

PROPOSITION 4. *Let $\mathcal{R}$ be a von Neumann algebra of type I_n acting on a Hilbert space $\mathcal{H}$. There is a family $\{\mathcal{R}_a\}_{a\in\mathbb{A}}$ of finite-dimensional *subalgebras of $\mathcal{R}$ such that if $a, b \in \mathbb{A}$, then there is a c in $\mathbb{A}$ for which $\mathcal{R}_a \cup \mathcal{R}_b \subseteq \mathcal{R}_c$, and $\mathcal{R} = (\cup_{a\in\mathbb{A}}\mathcal{R}_a)^-$.*

PROOF. Consider, first, the case in which $\mathcal{R}$ is $n \otimes \mathcal{C}$, where $\mathcal{C}$ is an abelian von Neumann algebra. Let $\mathbb{K}$ be a set with cardinality n, so that each element of $n \otimes \mathcal{C}$ is represented by a matrix $(C_{j,k})_{j,k\in\mathbb{K}}$ with entries in $\mathcal{C}$. Let $\mathbb{A}$ be the set of all pairs $(\mathbb{F}, \mathcal{A})$, in which $\mathbb{F}$ is a finite subset of $\mathbb{K}$ and $\mathcal{A}$ is a finite-dimensional * subalgebra of $\mathcal{C}$. When $a = (\mathbb{F}, \mathcal{A}) \in \mathbb{A}$, let $\mathcal{R}_a$ be the set of all elements of $n \otimes \mathcal{C}$ with matrices $(C_{j,k})$ such that $C_{j,k} \in \mathcal{A}$ for all j and k in $\mathbb{K}$ and $C_{j,k} = 0$ unless $j, k \in \mathbb{F}$. From the proof of Corollary 8.3.12, the set of all finite-dimensional $*$-subalgebras of an abelian von Neumann algebra $\mathcal{C}$ is directed by the inclusion relation $\subseteq$, and has union norm-dense in $\mathcal{C}$. It follows that the set $\mathbb{A}$ is directed by the relation $\leq$, in which $(\mathbb{F}_1, \mathcal{A}_1) \leq (\mathbb{F}_2, \mathcal{A}_2)$ if and only if $\mathbb{F}_1 \subseteq \mathbb{F}_2$ and $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and that $\{\mathcal{R}_a : a \in \mathbb{A}\}$ is an increasing net of finite-dimensional $*$-subalgebras of $n \otimes \mathcal{C}$. For each finite subset $\mathbb{F}$ of $\mathbb{K}$, let $E(\mathbb{F})$ be the projection in $n \otimes \mathcal{C}$ whose matrix has I in the (k, k) position when $k \in \mathbb{F}$ and has 0 in all other entries. When $R \in n \otimes \mathcal{C}$, $E(\mathbb{F})RE(\mathbb{F})$ has a matrix $(C_{j,k})$ in which $C_{j,k} = 0$ unless $j, k \in \mathbb{F}$, and $C_{j,k} \in \mathcal{C}$ when $j, k \in \mathbb{F}$. Each of the finite set of non-zero elements $C_{j,k}$ can be approximated in norm, as closely as we please, by an element $A_{j,k}$ of some finite-dimensional $*$-subalgebra $\mathcal{A}_{j,k}$ of $\mathcal{C}$. Then, $E(\mathbb{F})RE(\mathbb{F})$ is approximated in norm by an element of $\mathcal{R}_a$, where $a = (\mathbb{F}, \mathcal{A})$ with $\mathcal{A}$ a finite-dimensional * subalgebra of $\mathcal{C}$ that contains $\bigcup_{j,k\in\mathbb{F}} \mathcal{A}_{j,k}$. It follows that the norm closure $(\bigcup_{a\in\mathbb{A}} \mathcal{R}_a)^=$, and also the weak-operator closure $(\bigcup_{a\in\mathbb{A}} \mathcal{R}_a)^-$, contains $E(\mathbb{F})(n \otimes \mathcal{C})E(\mathbb{F})$ for each finite subset $\mathbb{F}$ of $\mathbb{K}$. Since $\bigvee E(\mathbb{F}) = I$, we have $(\bigcup_{a\in\mathbb{A}} \mathcal{R}_a)^- = n \otimes \mathcal{C}$.

Given any type I_n von Neumann algebra $\mathcal{R}$, by Theorem 6.6.5 there is an abelian von Neumann algebra $\mathcal{C}$ and a $*$-isomorphism φ from $n \otimes \mathcal{C}$ onto $\mathcal{R}$. With $\mathcal{R}_a (a \in \mathbb{A})$ constructed as in the preceding paragraph, $\{\varphi(\mathcal{R}_a) : a \in \mathbb{A}\}$ is an increasing net of finite-dimensional $*$-subalgebras of $\mathcal{R}$. Now φ is isometric, and gives rise to a homeomorphism between the unit balls $(n \otimes \mathcal{C})_1$ and $(\mathcal{R})_1$ in the weak-operator topology, by Remark 7.4.4. By Kaplansky density, $(\bigcup_{a\in\mathbb{A}} \mathcal{R}_a)_1$ is weak-operator dense in $(n \otimes \mathcal{C})_1$; so $(\bigcup_{a\in\mathbb{A}} \varphi(\mathcal{R}_a))_1$ is dense in $(\mathcal{R})_1$, and $\bigcup_{a\in\mathbb{A}} \varphi(\mathcal{R}_a)$ is dense in $\mathcal{R}$. □

COROLLARY 5. *If $\mathcal{R}$ is a type I von Neumann algebra acting on a Hilbert space $\mathcal{H}$, then there is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$.*

PROOF. There is an orthogonal family $\{Q_k : k \in \mathbb{K}\}$ of central projections in $\mathcal{R}$, with sum I, and (for each k in $\mathbb{K}$) a cardinal $n(k)$ such that $\mathcal{R}Q_k$ is of type $\mathrm{I}_{n(k)}$. By the result of Proposition 4, there is a family $\mathcal{F}_k$ of finite-dimensional * subalgebras of $\mathcal{R}Q_k$ that is directed upward by the inclusion relation $\subseteq$, and has union weak-operator dense in $\mathcal{R}Q_k$. Given any finite subset $\{k(1), \ldots, k(m)\}$ of $\mathbb{K}$ and any choice of $\mathcal{R}_j$ in $\mathcal{F}_{k(j)}$ (for each $j = 1, \ldots, m$), the linear span $\mathcal{R}_1 + \cdots + \mathcal{R}_m$ of $\bigcup_{j=1}^m \mathcal{R}_j$ is a finite-dimensional * subalgebra of $\mathcal{R}$. The set of all such algebras $\mathcal{R}_1 + \cdots + \mathcal{R}_m$ is directed by $\subseteq$, with union ultraweakly dense in $\mathcal{R}$. From

Propositions 2 and 3, there is a conditional expectation Φ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$, such that $\Phi(T) \in \mathrm{co}_{\mathcal{R}}(T)^- \cap \mathcal{R}'$ for all T in $\mathcal{B}(\mathcal{H})$. □

PROPOSITION 6. *Suppose that $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{R}$. Then there are conditional expectations, Φ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}$, and Ψ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$. Moreover, there are conditional expectations, Φ_0 from $\mathcal{R}$ onto $\mathcal{A}$, and Ψ_0 from $\mathcal{R}$ onto $\mathcal{R} \cap \mathcal{A}'$.*

PROOF. As in the proof of Lemma 8.2.3, the von Neumann algebra $\mathcal{A}'$ is of type I. From Corollary 5, there is a conditional expectation Φ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}''(= \mathcal{A})$. The restriction $\Phi|\mathcal{R}$ is a conditional expectation from $\mathcal{R}$ onto $\mathcal{A}$.

Since $\mathcal{A}$ is of type I (in fact, of type I_1), there is a conditional expectation Ψ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$. Since $\mathrm{co}_{\mathcal{A}}(T)^-$ meets $\mathcal{A}'$, for each T in $\mathcal{B}(\mathcal{H})$ (Propositions 3 and 4), Ψ can be chosen in such a way that $\Psi(T) \in \mathrm{co}_{\mathcal{A}}(T)^- \cap \mathcal{A}'$ for each T in $\mathcal{B}(\mathcal{H})$, by Proposition 2. Let $\mathcal{U}(\subseteq \mathcal{R})$ be the unitary group of $\mathcal{A}$. When $R \in \mathcal{R}$, $\mathcal{R}$ contains the set $\{URU^* : U \in \mathcal{U}\}$, and therefore contains the weak-operator closed convex hull $\mathrm{co}_{\mathcal{A}}(R)^-$ of that set. Accordingly, $\Psi(R) \in \mathcal{R} \cap \mathcal{A}'$ for R in $\mathcal{R}$. Since Ψ is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$, it follows that the restriction $\Psi|\mathcal{R}$ is a conditional expectation Ψ_0 from $\mathcal{R}$ onto $\mathcal{R} \cap \mathcal{A}'$. □

THEOREM 7. *If $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras acting on a Hilbert space $\mathcal{H}$, such that $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{S}$ has a faithful normal tracial state τ, then, for each element S of $\mathcal{S}$, there is a unique element $\varphi(S)$ of $\mathcal{R}$ such that $\tau(SR) = \tau(\varphi(S)R)$ for each R in $\mathcal{R}$. The mapping $\varphi : \mathcal{S} \to \mathcal{R}$ defined is an ultraweakly continuous conditional expectation from $\mathcal{S}$ onto $\mathcal{R}$, and is faithful in the sense that $\varphi(S) \neq 0$ when $0 \neq S \in \mathcal{S}^+$.*

PROOF. If $H \in \mathcal{S}^+$, the equation $\rho(R) = \tau(HR)$ defines a positive normal linear functional ρ on $\mathcal{R}$ (for positivity, note that $\rho(R) = \tau(H^{1/2}RH^{1/2})$). With H replaced by $\|H\|I - H$, it follows that the mapping $R \to \tau(\|H\|R - HR) = \|H\|\tau(R) - \rho(R)$ is a positive linear functional on $\mathcal{R}$. Thus $0 \leq \rho \leq \|H\|\tau$. From Theorem 7.3.13, there is a K_0 in $(\mathcal{R}^+)_1$ such that $\rho(R) = \frac{1}{2}\|H\|\tau(K_0R + RK_0) = \|H\|\tau(K_0R)$ for each R in $\mathcal{R}$. This proves the first assertion when S is a positive element H, with $\varphi(H)$ the element $\|H\|K_0$ of $\mathcal{R}^+$.

An arbitrary $S \in \mathcal{S}$ can be expressed as a linear combination of four positive elements of $\mathcal{S}$. From what we have proved, there is an element S_0 of $\mathcal{R}$ such that

$$(\dagger) \qquad \tau(SR) = \tau(S_0R) \qquad (R \in \mathcal{R}).$$

If S_1 (in the von Neumann algebra $\mathcal{R}$) has the property just ascribed to S_0, then $\tau((S_1 - S_0)R) = 0$ for each R in $\mathcal{R}$; in particular, $\tau((S_1 - S_0)(S_1 - S_0)^*) = 0$. Since τ is faithful, $S_1 = S_0$. Hence there is a *unique* element S_0 of $\mathcal{R}$ that satisfies $(\dagger)$, and we define $\varphi(S)$ to be S_0.

Suppose that $S, S_1, S_2 \in \mathcal{S}$, $R_1, R_2 \in \mathcal{R}$, and $a_1, a_2 \in \mathbb{C}$. For each R in $\mathcal{R}$,

$$\begin{aligned}\tau(\varphi(a_1S_1 + a_2S_2)R) &= \tau((a_1S_1 + a_2S_2)R) = a_1\tau(S_1R) + a_2\tau(S_2R)\\ &= a_1\tau(\varphi(S_1)R) + a_2\tau(\varphi(S_2)R) = \tau((a_1\varphi(S_1) + a_2\varphi(S_2))R),\end{aligned}$$

and

$$\tau(\varphi(R_1SR_2)R) = \tau(R_1SR_2R) = \tau(SR_2RR_1) = \tau(\varphi(S)R_2RR_1) = \tau(R_1\varphi(S)R_2R).$$

In addition,

$$\tau(\varphi(I)R) = \tau(R) = \tau(IR).$$

From these equations, and the uniqueness we have proved

$$\varphi(a_1S_1 + a_2S_2) = a_1\varphi(S_1) + a_2\varphi(S_2), \varphi(R_1SR_2) = R_1\varphi(S)R_2$$

and $\varphi(I) = I$. Moreover, if $H \in \mathcal{S}^+$, it follows, from the first part of the proof, that $\varphi(H)$ is an element H_0 of $\mathcal{R}^+$. Thus φ is a conditional expectation from $\mathcal{S}$ onto $\mathcal{R}$.

Suppose that $S \in \mathcal{S}^+$ and $\varphi(S) = 0$. Since τ is faithful and $\tau(S) = \tau(\varphi(S)) = 0$, $S = 0$. Hence φ is faithful.

In order to prove that φ is ultraweakly continuous, we have to show that $\omega\circ\varphi \in \mathcal{S}_\sharp$ whenever $\omega \in \mathcal{R}_\sharp$. Since $\mathcal{R}_\sharp$ is the linear span of its positive elements (see Remark 7.4.4), we may assume that ω is a *positive* normal linear functional on $\mathcal{R}$. Then, $\omega\circ\varphi$ is positive, and we have to prove that $\omega\circ\varphi$ is normal. Suppose that $\{H_a\}$ is a monotone increasing net of self-adjoint elements of $\mathcal{S}$, with least upper bound H in $\mathcal{S}$. Since φ is a positive linear mapping, the net $\{\varphi(H_a)\}$ in $\mathcal{R}$ is monotone increasing, has $\varphi(H)$ as an upper bound in $\mathcal{R}$, and therefore has a least upper bound $K(\le \varphi(H))$ in $\mathcal{R}$. Since τ is normal, $\tau(K) = \lim_a \tau(\varphi(H_a)) = \lim_a \tau(H_a) = \tau(H) = \tau(\varphi(H))$; so $\varphi(H) - K \ge 0$ and $\tau(\varphi(H) - K) = 0$. Since τ is faithful, $\varphi(H) = K$. Since ω is normal, it now follows that $\lim_a \omega(\varphi(H_a)) = \omega(K) = \omega(\varphi(H))$. Thus $\omega\circ\varphi$ is normal (and φ is ultraweakly continuous). □

To better understand what Theorem 7 is telling us, it is helpful to examine the case where $\mathcal{S}$ (and hence, $\mathcal{R}$) is commutative. For the purposes of this illustration, we assume that $\mathcal{R}$ and $\mathcal{S}$ have no atoms and that $\mathcal{S}$ acts on a separable Hilbert space $\mathcal{H}$. From Corollary 5.5.17, there is a separating unit vector u for $\mathcal{S}$. The vector state $\omega_u|\mathcal{S}$ is a faithful, normal, tracial state τ of $\mathcal{S}$ and its restriction τ_0 to $\mathcal{R}$ is such a state of $\mathcal{R}$. Let $\mathcal{H}_0$ be $[\mathcal{R}u]$. Then $\mathcal{R}|\mathcal{H}_0$ is maximal abelian in $\mathcal{B}(\mathcal{H}_0)$ (from Corollary 7.2.16). From Theorem 9.4.1 (compare [**vN31**]), we may identify $\mathcal{H}_0$ with $L_2([0,1],\mu)$, where μ is Lebesgue measure on $[0,1]$, u with the constant function 1 on $[0,1]$, and $\mathcal{R}|\mathcal{H}_0$ with the "multiplication algebra" of $([0,1],\mu)$. If X_0 is a measurable subset of $[0,1]$, then multiplication by the characteristic ("indicator") function of X on $L_2([0,1],\mu)$ corresponds to a projection E_0 in $\mathcal{R}$, and $\tau(E_0) = \tau_0(E_0) = \langle E_0u, u\rangle = \mu(X_0)$. With S in $\mathcal{S}$, the functional τ_S on $\mathcal{R}$ defined by

$$\tau_S(R) = \tau(SR) = \langle SRu, u\rangle \qquad (R \in \mathcal{R}),$$

gives rise to a measure μ_0 on $[0,1]$, where

$$\mu_0(X_0) = \tau_S(E_0) = \tau(SE_0) = \langle SE_0u, u\rangle = \langle E_0SE_0u, u\rangle = \langle SE_0u, E_0u\rangle.$$

If $S \ge 0$, then

$$\begin{aligned} 0 \le \langle SE_0u, E_0u\rangle &= \mu_0(X_0) \le \|S\|\|E_0u\|^2 \\ &= \|S\|\langle E_0u, E_0u\rangle = \|S\|\langle E_0u, u\rangle = \|S\|\mu(X_0). \end{aligned}$$

Thus when $S \ge 0$, μ_0 is absolutely continuous with respect to μ. From the Radon-Nikodým theorem, there is a (positive) function f_0 in $L_1([0,1],\mu)$ such that $\int_{[0,1]} g\,d\mu_0 = \int_{[0,1]} gf_0\,d\mu$, for each essentially bounded, μ_0-measurable function g on $[0,1]$. In a formal sense, $d\mu_0 = f_0\,d\mu$, or $d\mu_0/d\mu = f_0$: f_0 is the Radon-Nikodým derivative of μ_0 with respect to μ.

Note that $\tau(SR_g) = \int_{[0,1]} g\,d\mu_0 = \int_{[0,1]} gf_0\,d\mu$, where R_g is the operator in $\mathcal{R}$ corresponding to multiplication by g. Since

$$|\tau(SR_g)| = |\langle SR_gu, u\rangle| = |\langle R_gu, S^*u\rangle| \le \|R_g\|\|S\|,$$

for each μ-essentially bounded g on $[0,1]$, $\|R_g\| = \|g\|_\infty$, and

$$\left|\int_{[0,1]} g f_0 \, d\mu\right| \le \|S\|\|g\|_\infty,$$

we have that f_0 is μ-essentially bounded with $\|f_0\|_\infty \le \|S\|$. Thus there is an R_{f_0} in $\mathcal{R}$ such that $\|R_{f_0}\| = \|f_0\|_\infty \le \|S\|$. Moreover,

$$\tau(R_{f_0}R_g) = \tau(R_{f_0 g}) = \int_{[0,1]} f_0 g \, d\mu = \tau(SR_g).$$

From Theorem 7, there is a *unique* $\varphi(S)$ in $\mathcal{R}$ such that $\tau(SR) = \tau(\varphi(S)R)$ for each R in $\mathcal{R}$. Since each R in $\mathcal{R}$ is of the form R_g for some μ-essentially bounded g on $[0,1]$, $R_{f_0} = \varphi(S)$.

The foregoing discussion shows us that, when we have constructed $\varphi(S)$, we have constructed the Radon-Nikodým derivative of μ_0 with respect to μ, or referring to the associated integration processes rather than the measures, $\varphi(S)$ is the Radon-Nikodým derivative of τ_0 with respect to τ. Theorem 7 applies to general (non-commutative) $\mathcal{S}$ and $\mathcal{R}$; there is every reason to regard $\varphi(S)$ as the general (*non-commutative*) Radon-Nikodým derivative of τ_0 with respect to τ in this case as well. If $\mathcal{S}$ is not finite (hence, not abelian) and $\mathcal{R}$ is abelian, from Proposition 6 there is still a conditional expectation of $\mathcal{S}$ onto $\mathcal{R}$, though it may not be ultraweakly continuous [**Ta72**], [**To59**], [**K-S59**].

Recall that the von Neumann algebra $\mathcal{L}_G$ has a faithful tracial state τ defined by $\tau(A) = \langle Ax_e, x_e\rangle$. When $z \in l_\infty(G)$, let M_z (in $\mathcal{B}(l_2(G))$) be multiplication by the function z.

THEOREM 8. *Let G be a discrete group with unit element e, Φ be a conditional expectation from $\mathcal{B}(l_2(G))$ onto $\mathcal{L}_G$, and ρ be the linear functional on $l_\infty(G)$ defined by*

$$\rho(z) = \tau(\Phi(M_z)) \qquad (z \in l_\infty(G)).$$

Then ρ is an invariant mean on G.

The following three conditions are equivalent:

(1) *There is a conditional expectation from $\mathcal{B}(l_2(G))$ onto $\mathcal{L}_G$ $(= \mathcal{R}'_G)$;*
(2) *There is an invariant mean on G;*
(3) *For each T in $\mathcal{B}(l_2(G))$, $\mathrm{co}_{\mathcal{R}_G}(T)^-$ meets $\mathcal{R}'_G$.*

PROOF. As defined in the statement, ρ is a bounded linear functional on $l_\infty(G)$, and $\|\rho\| \le 1$, since $\|\tau\| = 1$, $\|\Phi\| \le 1$ by Proposition 1, and $\|M_z\| = \|z\|_\infty$ when $z \in l_\infty(G)$. If u is the element of $l_\infty(G)$ that takes the value 1 throughout G, then

$$\rho(u) = \tau(\Phi(M_u)) = \tau(\Phi(I)) = \tau(I) = 1.$$

When $z \in l_\infty(G)$, $y \in l_2(G)$, and $g, h \in G$, we have

$$\begin{aligned}(L_{x_g}M_zL^*_{x_g}y)(h) &= (M_zL^*_{x_g}y)(g^{-1}h)\\ &= z(g^{-1}h)(L^*_{x_g}y)(g^{-1}h) = z(g^{-1}h)y(h) = (M_{z_g}y)(h)\end{aligned}$$

and $L_{x_g}M_zL^*_{x_g} = M_{z_g}$, where z_g (in $l_\infty(G)$) is defined by $z_g(h) = z(g^{-1}h)$. In the notation of Exercise 3.5.7 (dealing with invariant means on groups), z_g is $T_g z$.

Since L_{x_g} and $L_{x_g}^*$ lie in the range $\mathcal{L}_G$ of the conditional expectation Φ, and τ is a tracial state of $\mathcal{L}_G$, we have

$$\begin{aligned}(T_g^\sharp\rho)(z) &= \rho(T_g z) = \rho(z_g) = \tau(\Phi(M_{z_g}))\\ &= \tau(\Phi(L_{x_g}M_z L_{x_g}^*)) = \tau(L_{x_g}\Phi(M_z)L_{x_g}^*) = \tau(\Phi(M_z)) = \rho(z).\end{aligned}$$

Hence $T_g^\sharp\rho = \rho$ for each g in G, and ρ is an invariant mean on G. Thus (1) implies (2).

Suppose, now, that ρ is an invariant mean on G. Since $l_\infty(G)$ is a C^*-algebra with unit u (as defined before), and ρ is a bounded linear functional on $l_\infty(G)$ satisfying $\|\rho\| \le 1 = \rho(u)$, it follows (Theorem 4.3.2) that ρ is a state of $l_\infty(G)$. Given T in $\mathcal{B}(l_2(G))$ and x, y in $l_2(G)$, the complex-valued function $z_{x,y}$, defined at g in G to be $\langle R_{x_g}^* T R_{x_g} x, y\rangle$, is an element of $l_\infty(G)$, the mapping $x \to z_{x,y}$ is linear for each fixed y, the mapping $y \to z_{x,y}$ is conjugate-linear for each fixed x, and $\|z_{x,y}\|_\infty \le \|T\|\|x\|\|y\|$. It follows that the equation $b(x,y) = \rho(z_{x,y})$, for x and y in $l_2(G)$, defines a bounded conjugate-bilinear functional b on $l_2(G)$; corresponding to b, there is an element A_T of $\mathcal{B}(l_2(G))$ such that

$$(*) \qquad \langle A_T x, y\rangle = \rho(z_{x,y}) \qquad (x, y \in l_2(G)).$$

If $h \in G$, and $x, y \in l_2(G)$, we have $\langle R_{x_h}^* A_T R_{x_h} x, y\rangle = \langle A_T u, v\rangle = \rho(z_{u,v})$, where $u = R_{x_h}x$ and $v = R_{x_h}y$. Also, for each g in G,

$$z_{u,v}(g) = \langle R_{x_g}^* T R_{x_g} u, v\rangle = \langle R_{x_g}^* T R_{x_g} R_{x_h} x, R_{x_h} y\rangle = \langle R_{x_{hg}}^* T R_{x_{hg}} x, y\rangle = z_{x,y}(hg).$$

Since ρ is an invariant mean, $\rho(z_{u,v}) = \rho(z_{x,y})$; that is $\langle R_{x_h}^* A_T R_{x_h} x, y\rangle = \langle A_T x, y\rangle$. Thus $R_{x_h}^* A_T R_{x_h} = A_T$, for each h in G, and $A_T \in \{R_{x_h} : h \in G\}' = \mathcal{R}_G'$.

If $A_T \notin \mathrm{co}_{\mathcal{R}_G}(T)^-$, there is a weak-operator continuous linear functional ω on $\mathcal{B}(l_2(G))$ and a real number c such that $\mathrm{Re}\,\omega(A_T) > c \ge \mathrm{Re}\,\omega(S)$ when $S \in \mathrm{co}_{\mathcal{R}_G}(T))$. In particular,

$$(**) \qquad \mathrm{Re}\,\omega(A_T) > c \ge \mathrm{Re}\,\omega(R_{x_g}^* T R_{x_g}) \qquad (g \in G).$$

By expressing ω as a finite sum of vector functionals $\omega_{x,y}$, and using $(*)$ and the definition of $z_{x,y}$, it follows that $\omega(A_T) = \rho(z_\omega)$, where z_ω (in $l_\infty(G)$) is defined by $z_\omega(g) = \omega(R_{x_g}^* T R_{x_g})$ for g in G. Since $\mathrm{Re}\,z_\omega(g) \le c$ for all g in G, by $(**)$, and ρ is a state of $l_\infty(G)$, we have $c < \mathrm{Re}\,\omega(A_T) = \mathrm{Re}\,\rho(z_\omega) \le c$, a contradiction. Thus $A_T \in \mathrm{co}_{\mathcal{R}_G}(T)^-$, and (2) implies (3).

From Proposition 2, (3) implies (1). □

THEOREM 9. *No group containing $\mathcal{F}_2$ has an invariant mean. In particular, $\mathcal{F}_2$ has no invariant mean.*

PROOF. Let a and b be the two generators of $\mathcal{F}_2$, and let S be the set of reduced words in $\mathcal{F}_2$ that begin with a non-zero power of b. We note that $\mathcal{F}_2 = S \cup bS$ and that S, aS, a^2S are disjoint. When $X \subseteq \mathcal{F}_2$, the characteristic function f_X is in $l_\infty(\mathcal{F}_2)$, and

$$f_{gX} = T_g f_X \qquad (g \in \mathcal{F}_2),$$

where $T_g y(h)$ is defined as $y(g^{-1}h)$ $(h \in G,\ y \in l_2(\mathcal{F}_2))$. The inequalities

$$f_S + f_{bS} \ge f_{\mathcal{F}_2} \ge f_S + f_{aS} + f_{a^2S}$$

can be written in the form $f_S + T_b f_S \ge f_{\mathcal{F}_2} \ge f_S + T_a f_S + T_{a^2} f_S$. If ρ is an invariant mean on $\mathcal{F}_2$, these inequalities imply that $2\rho(f_S) \ge 1 \ge 3\rho(f_S)$ (recall that ρ is a state of $l_\infty(G)$), which is impossible. Thus $\mathcal{F}_2$ has no invariant mean.

Let G be a (discrete) group containing $\mathcal{F}_2$ and $\{g_k : k \in \mathbb{K}\}$ be elements of G such that the cosets $\mathcal{F}_2 g_k$ and $\mathcal{F}_2 g_{k'}$ are disjoint unless $k = k'$ and such that $G = \bigcup_{k\in\mathbb{K}} \mathcal{F}_2 g_k$. Let S_0 be $\bigcup_{k\in\mathbb{K}} S g_k$. Then

$$S_0 \cup bS_0 = \left(\bigcup_{k\in\mathbb{K}} Sg_k\right) \cup b\left(\bigcup_{k\in\mathbb{K}} Sg_k\right) = \bigcup_{k\in\mathbb{K}} (S\cup bS)g_k = \bigcup_{k\in\mathbb{K}} \mathcal{F}_2 g_k = G$$

and S_0, aS_0, a^2S_0 are disjoint. The argument given before applies now, with S_0 in place of S, to show that G does not have an invariant mean. □

LEMMA 10. *Let $\mathcal{B}$ be a C^*-subalgebra of the C^*-algebra $\mathfrak{A}$, φ_0 an idempotent linear mapping of $\mathfrak{A}$ onto $\mathcal{B}$ such that $\|\varphi_0\| = 1$, $\mathfrak{A}$ acting on $\mathcal{H}$ the universal representation of $\mathfrak{A}$ and, in this representation, E a projection in $\mathcal{B}^-$. Then*

(i) *φ_0 is a positive linear mapping of $\mathfrak{A}$ onto $\mathcal{B}$ such that $\varphi_0(I) = I$;*
(ii) *φ_0 extends uniquely to an ultraweakly continuous idempotent linear mapping φ of $\mathfrak{A}^-$ onto $\mathcal{B}^-$ such that $\|\varphi\| = 1$, and φ is positive;*
(iii) *$\omega_x\circ\varphi$ is a state of $\mathfrak{A}^-$ definite on E if $\|x\| = 1$ and $x \in E(\mathcal{H})\cup(I-E)(\mathcal{H})$;*
(iv) *the equations $E\varphi(EA)E = E\varphi(AE)E = E\varphi(A)E$, $E\varphi(EAE)E = E\varphi(A)E$, and $(I-E)\varphi(EA)(I-E) = (I-E)\varphi(AE)(I-E) = 0$ hold for each A in $\mathfrak{A}^-$, and $\varphi(EAE) = E\varphi(A)E$ for each A in $\mathfrak{A}^-$;*
(v) *$\varphi(EA(I-E)) = (I-E)\varphi(EA(I-E))E + E\varphi(EA(I-E))(I-E)$ if $A \in \mathfrak{A}^-$.*

PROOF. (i) Since $I \in \mathcal{B}$, and φ_0 is idempotent with range $\mathcal{B}$, $\varphi_0(I) = I$. If ρ is a state of $\mathcal{B}$, then $(\rho\circ\varphi_0)(I) = 1$. Since $\|\rho\circ\varphi_0\| \le \|\rho\|\|\varphi_0\| = 1$, $\rho\circ\varphi_0$ is a state of $\mathfrak{A}$ by Theorem 4.3.2. If H is in $\mathfrak{A}^+$, $\rho(\varphi_0(H)) \ge 0$ for each state ρ of $\mathcal{B}$. Since $\varphi_0(H) \in \mathcal{B}$, $\varphi_0(H) \in \mathcal{B}^+$ by Theorem 4.3.4(iii). Thus φ_0 is a positive linear mapping of $\mathfrak{A}$ onto $\mathcal{B}$.

(ii) By assumption, φ_0 is a bounded linear mapping of $\mathfrak{A}$ onto $\mathcal{B}$. If ω is an ultraweakly continuous linear functional on $\mathcal{B}$, then $\omega\circ\varphi_0$ is a bounded linear functional on $\mathfrak{A}$ and hence is ultraweakly continuous from Proposition 10.1.1. Thus φ_0 is ultraweakly continuous and extends uniquely to an ultraweakly continuous linear mapping φ of $\mathfrak{A}^-$ into $\mathcal{B}^-$ such that $\|\varphi\| = \|\varphi_0\| = 1$. Since $\mathcal{B}^- \subseteq \mathfrak{A}^-$, $\varphi\circ\varphi$ is defined, ultraweakly continuous, and coincides on $\mathfrak{A}$ with $\varphi_0\circ\varphi_0$ $(= \varphi_0 = \varphi|\mathfrak{A})$. The ultraweakly continuous mappings $\varphi\circ\varphi$ and φ agree on the ultraweakly dense subset $\mathfrak{A}$ of $\mathfrak{A}^-$ so that they agree on $\mathfrak{A}^-$. Hence φ is idempotent.

Since the unit ball of $\mathcal{B}$ is contained in the unit ball of $\mathfrak{A}$ and $\|\varphi\| = 1$, φ maps the ultraweakly compact unit ball of $\mathfrak{A}^-$ onto an ultraweakly compact (hence closed) subset of $\mathcal{B}^-$ that contains $(\mathcal{B})_1$. From the Kaplansky density theorem $(\mathcal{B})_1^- = (\mathcal{B}^-)_1$. Hence, $\varphi(\mathfrak{A}^-) = \mathcal{B}^-$. From (i), φ is positive.

(iii) Since $\varphi(I) = \varphi_0(I) = I$, $(\omega_x\circ\varphi)(I) = 1$. From (ii), φ is a positive linear mapping of $\mathfrak{A}^-$ onto $\mathcal{B}^-$ so that $\omega_x\circ\varphi$ is a state of $\mathfrak{A}^-$. As $E^2 = E$, the states ρ of $\mathfrak{A}^-$ that are definite on E are those such that $\rho(E) = \rho(E^2) = \rho(E)^2$; that is, the states definite on E are precisely those that take the value 1 or 0 at E. Since $E \in \mathcal{B}^-$ and φ is idempotent with range $\mathcal{B}^-$, $(\omega_x\circ\varphi)(E) = \omega_x(E)$. When $x \in (I-E)(\mathcal{H})$, $(\omega_x\circ\varphi)(E) = 0$, and when $x \in E(\mathcal{H})$, $(\omega_x\circ\varphi)(E) = 1$. Thus $\omega_x\circ\varphi$ is definite on E when x is a unit vector in either $E(\mathcal{H})$ or $(I-E)(\mathcal{H})$.

(iv) From (iii) and Exercise 4.6.16, when x in $E(\mathcal{H})$ or in $(I-E)(\mathcal{H})$ has norm 1, for all A in $\mathfrak{A}^-$, then $(\omega_x\circ\varphi)(EA) = (\omega_x\circ\varphi)(E)(\omega_x\circ\varphi)(A) = \omega_x(E)(\omega_x\circ\varphi)(A)$.

Thus, with x a unit vector in $E(\mathcal{H})$, $\langle\varphi(EA)x, x\rangle = \langle\varphi(A)x, x\rangle$. This same equality holds for all x in $E(\mathcal{H})$, so that $E\varphi(EA)E = E\varphi(A)E$.

With x a unit vector in $(I - E)(\mathcal{H})$, we have $\langle\varphi(EA)x, x\rangle = 0$. This same equality holds for all x in $(I - E)(\mathcal{H})$. It follows that $(I - E)\varphi(EA)(I - E) = 0$.

In the same way, $E\varphi(AE)E = E\varphi(A)E$ and $(I - E)\varphi(AE)(I - E) = 0$ for all A in $\mathfrak{A}^-$. Thus $E\varphi(EAE)E = E\varphi(AE)E = E\varphi(A)E$.

Since φ is a positive linear mapping (by (ii)) and, with A self-adjoint, $-\|A\|E \leq EAE \leq \|A\|E$, we have that $-\|A\|E = -\|A\|\varphi(E) \leq \varphi(EAE) \leq \|A\|\varphi(E) = \|A\|E$. Hence $\varphi(EAE) = E\varphi(EAE)E = E\varphi(A)E$.

(v) From (iv),

$$\begin{aligned}
\varphi(EA(I-E)) &= E\varphi(EA(I-E))E + (I-E)\varphi(EA(I-E))E \\
&+ E\varphi(EA(I-E))(I-E) + (I-E)\varphi(EA(I-E))(I-E) \\
&= (I-E)\varphi(EA(I-E))E + E\varphi(EA(I-E))(I-E),
\end{aligned}$$

for each A in $\mathfrak{A}^-$. □

The theorem that follows is, in essence, the result first proved by Tomiyama in [**To57**].

THEOREM 11. *With the notation and assumptions of Lemma 10, φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathcal{B}^-$ and φ_0 is a conditional expectation from $\mathfrak{A}$ onto $\mathcal{B}$.*

PROOF. With x a unit vector in $\mathcal{H}$,

$$\begin{aligned}
&\|ET(I-E)x + (I-E)SEx\|^2 = \|ET(I-E)x\|^2 + \|(I-E)SEx\|^2 \\
&\leq \ \|ET(I-E)\|^2\|(I-E)x\|^2 + \|(I-E)SE\|^2\|Ex\|^2 \\
&\leq \ \max\left\{\|ET(I-E)\|^2, \|(I-E)SE\|^2\right\}.
\end{aligned}$$

The last inequality follows from the fact that its left side is a convex combination of $\|ET(I-E)\|^2$ and $\|(I-E)SE\|^2$. On the other hand,

$$\begin{aligned}
\|ET(I-E)\| &= \sup\{\|ET(I-E)y\| : \|y\| \leq 1\} \\
&= \sup\{\|ET(I-E)z\| : z = (I-E)y,\ \|y\| \leq 1\} \\
&= \sup\{\|ET(I-E)z\| : z \in (I-E)(\mathcal{H}),\ \|z\| \leq 1\} \\
&= \sup\{\|[ET(I-E) + (I-E)SE]z\| : z \in (I-E)(\mathcal{H}),\ \|z\| \leq 1\} \\
&\leq \|ET(I-E) + (I-E)SE\|.
\end{aligned}$$

Similarly, $\|(I-E)SE\| \leq \|ET(I-E) + (I-E)SE\|$, so that

$$\|ET(I-E) + (I-E)SE\| = \max\{\|ET(I-E)\|, \|(I-E)SE\|\}.$$

Suppose $(I-E)\varphi(EA(I-E))E \neq 0$. Then for all large positive integers n,

$$\|E\varphi(EA(I-E))(I-E)\| \leq \|n(I-E)\varphi(EA(I-E))E\|,$$

so that from what we have proved and Lemma 10(v), and since φ is an idempotent with range $\mathcal{B}^-$ and norm not exceeding 1,

$$\begin{aligned} n\|(I-E)\varphi(EA(I-E))E\| \\ &= \max\{\|n(I-E)\varphi(EA(I-E))E\|, \|E\varphi(EA(I-E))(I-E)\|\} \\ &= \|E\varphi(EA(I-E))(I-E) + n(I-E)\varphi(EA(I-E))E\| \\ &= \|E\varphi(EA(I-E))(I-E) + (I-E)\varphi(EA(I-E))E \\ &\quad + (n-1)(I-E)\varphi(EA(I-E))E\| \\ &= \|\varphi(EA(I-E) + (n-1)(I-E)\varphi(EA(I-E))E)\| \\ &\leq \|EA(I-E) + (n-1)(I-E)\varphi(EA(I-E))E\| \\ &= (n-1)\|(I-E)\varphi(EA(I-E))E\|, \end{aligned}$$

a contradiction. Thus $(I-E)\varphi(EA(I-E))E = 0$.

From this and Lemma 10(v),

$$(*) \qquad \varphi(EA(I-E)) = E\varphi(EA(I-E))(I-E).$$

Thus for each A in $\mathfrak{A}^-$, from $(*)$ and Lemma 10(iv),

$$\begin{aligned} \varphi(A) &= \varphi(EAE) + \varphi(EA(I-E)) \\ &\quad + \varphi((I-E)AE) + \varphi((I-E)A(I-E)) \\ &= E\varphi(EAE)E + E\varphi(EA(I-E))(I-E) \\ &\quad + (I-E)\varphi((I-E)AE)E + (I-E)\varphi((I-E)A(I-E))(I-E); \end{aligned}$$

so that

$$\begin{aligned} E\varphi(A) &= E\varphi(EAE)E + E\varphi(EA(I-E))(I-E) \\ &= \varphi(EAE) + \varphi(EA(I-E)) \\ &= \varphi(EA). \end{aligned}$$

Similarly, $\varphi(AE) = \varphi(A)E$ for each A in $\mathfrak{A}^-$.

Let B be a self-adjoint element in $\mathcal{B}^-$ and A be in $\mathfrak{A}^-$. From Theorem 5.2.2(v), given a positive ε, there is a (finite) orthogonal family $\{E_1, \ldots, E_n\}$ of projections in $\mathcal{B}^-$ and (real) scalars $a_1, \ldots, a_n$ such that $\|B - \sum_{j=1}^n a_j E_j\| < \varepsilon/(2\|A\|)$. From the preceding, we have

$$\begin{aligned} \|\varphi(BA) - B\varphi(A)\| \\ &\leq \left\|\varphi(BA) - \varphi\left(\left(\sum_{j=1}^n a_j E_j\right) A\right)\right\| + \left\|\varphi\left(\left(\sum_{j=1}^n a_j E_j\right) A\right) - B\varphi(A)\right\| \\ &\leq \left\|BA - \left(\sum_{j=1}^n a_j E_j\right) A\right\| + \left\|\left(\sum_{j=1}^n a_j E_j\right) \varphi(A) - B\varphi(A)\right\| < \varepsilon. \end{aligned}$$

Thus $\varphi(BA) = B\varphi(A)$ and similarly, $\varphi(AB) = \varphi(A)B$ for each B in $\mathcal{B}^-$ and each A in $\mathfrak{A}^-$.

From Lemma 10(ii), φ is a positive linear mapping of $\mathfrak{A}^-$ onto $\mathcal{B}^-$ and $\varphi(I) = I$. From the preceding, $\varphi(BAC) = B\varphi(A)C$ for each A in $\mathfrak{A}^-$ and B, C in $\mathcal{B}^-$. Thus φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathcal{B}^-$.

Since φ_0 maps $\mathfrak{A}$ onto $\mathcal{B}$ and is the restriction of φ to $\mathfrak{A}$, from the preceding, it is immediate that φ_0 is a conditional expectation from $\mathfrak{A}$ onto $\mathcal{B}$. $\square$

Sakai's characterization [**Sa56**] of those C^*-algebras that are $*$-isomorphic to von Neumann algebras can be proved elegantly with the aid of Tomiyama's theorem.

THEOREM 12. *Suppose the C^*-algebra $\mathfrak{A}$ is (linearly isomorphic and isometric to) the norm dual of a Banach space $\mathfrak{A}_\sharp$. Then $\mathfrak{A}$ is $*$ isomorphic to a von Neumann algebra ($\mathfrak{A}$ is a W^*-algebra).*

PROOF. Let η be the natural injection of $\mathfrak{A}_\sharp$ into $\mathfrak{A}^\sharp$. Suppose $\xi \in (\mathfrak{A}_\sharp)_1$. Then, with ν an element of $\mathfrak{A}^{\sharp\sharp}$, since η is an isometry, $\|(\nu\circ\eta)(\xi)\| \le \|\nu\|\|\eta(\xi)\| \le \|\nu\|$, and $\nu\circ\eta$ is a bounded linear functional on $\mathfrak{A}_\sharp$. By assumption, $\mathfrak{A}$ is the norm dual of $\mathfrak{A}_\sharp$. Thus there is an A in $\mathfrak{A}$ such that $\nu\circ\eta = A$, and A is unique. Let $\mathfrak{A}$ acting on $\mathcal{H}$ be the universal representation of $\mathfrak{A}$, and let $A \to \hat{A}$ be the (isometric linear) isomorphism (of Proposition 10.1.21) between $\mathfrak{A}^-$ and $\mathfrak{A}^{\sharp\sharp}$. Let $\varphi(B)$ be the unique element of $\mathfrak{A}$, just obtained, such that $\hat{B}\circ\eta = \varphi(B)$, where $B \in \mathfrak{A}^-$.

Let A be an element of $\mathfrak{A}$ (in $\mathfrak{A}^-$). We show that $\varphi(A) = A$. Since φ is a linear mapping of $\mathfrak{A}^-$ onto $\mathfrak{A}$, this will show that φ is an idempotent mapping of $\mathfrak{A}^-$ onto $\mathfrak{A}$. With ξ in $\mathfrak{A}_\sharp$,

$$\varphi(A)(\xi) = (\hat{A}\circ\eta)(\xi) = \eta(\xi)(A) = A(\xi).$$

Thus $\varphi(A) = A$. At the same time, if $B \in (\mathfrak{A}^-)_1$, then $\widehat{B} \in (\mathfrak{A}^{\sharp\sharp})_1$ and $\|\varphi(B)(\xi)\| = \|(\widehat{B}\circ\eta)(\xi)\| \le \|\eta(\xi)\| = \|\xi\|$. Thus $\|\varphi(B)\| \le 1$. It follows that $\|\varphi\| \le 1$, and from Theorem 11, φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathfrak{A}$.

Let $\mathcal{K}$ be $\varphi^{-1}(0)$. We show that $\mathcal{K}$ is a weak-operator closed two-sided ideal in $\mathfrak{A}^-$. Note first that $\mathcal{K}$ is weak-operator closed. We have that $A \in \mathcal{K}$ if and only if $(\hat{A}\circ\eta)(\xi) = 0$ for all ξ in $\mathfrak{A}_\sharp$. Now $\eta(\xi) \in \mathfrak{A}^\sharp$. Then there are vectors $x(\xi)$ and $y(\xi)$ in $\mathcal{H}$ such that $\eta(\xi) = \omega_{x(\xi),y(\xi)}|\mathfrak{A}$. Thus $A \in \mathcal{K}$ if and only if $\omega_{x(\xi),y(\xi)}(A) = 0$ for all ξ in $\mathfrak{A}_\sharp$. It follows that $\mathcal{K}$ is weak-operator closed.

Since φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathfrak{A}$, $\varphi(BAC) = B\varphi(A)C$ for each A in $\mathfrak{A}^-$ and B, C in $\mathfrak{A}$. Thus, if $A \in \mathcal{K}$, $0 = B\varphi(A)C = \varphi(BAC)$, and $BAC \in \mathcal{K}$. By weak-operator continuity of left (and then right) multiplication, $BAC \in \mathcal{K}$ when $A \in \mathcal{K}$ and $B, C \in \mathfrak{A}^-$. Hence $\mathcal{K}$ is a weak-operator-closed two-sided ideal in $\mathfrak{A}^-$.

Let P be the central projection in $\mathfrak{A}^-$ such that $\mathcal{K} = \mathfrak{A}^-P$. (See Theorem 6.8.8.) Since φ is idempotent, $A - \varphi(A) \in \mathcal{K}$ for each A in $\mathfrak{A}^-$. Thus $A - \varphi(A) \in \mathfrak{A}^-P$ and $A - \varphi(A) = [A - \varphi(A)]P$. It follows that

$$A(I-P) = \varphi(A)(I-P) \in \mathfrak{A}(I-P).$$

Hence $\mathfrak{A}^-(I-P) = \mathfrak{A}(I-P)$.

If $A \in \mathfrak{A}$ and $0 \ne A$ $(= \varphi(A))$, then $A \notin \mathcal{K}$ so that $A \notin \mathfrak{A}^-P$. Thus $A \ne AP$ and $A(I-P) \ne 0$. Since P commutes with $\mathfrak{A}$, the mapping $A \to A(I-P)$ of $\mathfrak{A}$ onto $\mathfrak{A}(I-P)$ is a $*$-homomorphism and from the foregoing, this mapping is a $*$-isomorphism. As we have just proved, $\mathfrak{A}(I-P) = \mathfrak{A}^-(I-P)$, so that $\mathfrak{A}$ is $*$-isomorphic to the von Neumann algebra $\mathfrak{A}^-(I-P)$ (acting on $(I-P)(\mathcal{H})$). Hence $\mathfrak{A}$ is a W^*-algebra. $\square$

The theorem that follows is a formulation of Tomiyama's "slice-mapping" techniques [**To70**].

THEOREM 13. *Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras and ρ and σ be non-zero elements of $\mathcal{R}_\sharp$ and $\mathcal{S}_\sharp$, respectively.*

(i) *There is a unique element $\rho\otimes\sigma$ of $(\mathcal{R}\bar{\otimes}\mathcal{S})_\sharp$ such that $(\rho\otimes\sigma)(R\otimes S) = \rho(R)\sigma(S)$ $(R\in\mathcal{R},\ S\in\mathcal{S})$, $\|\rho\otimes\sigma\| = \|\rho\|\|\sigma\|$.*

(ii) *There are unique operators $\Phi_\sigma(\tilde{T})$ and $\Psi_\rho(\tilde{T})$ in $\mathcal{R}$ and $\mathcal{S}$, respectively, corresponding to each $\tilde{T}$ in $\mathcal{R}\bar{\otimes}\mathcal{S}$, satisfying*

$$\rho'(\Phi_\sigma(\tilde{T})) = (\rho'\otimes\sigma)(\tilde{T}), \qquad \sigma'(\Psi_\rho(\tilde{T})) = (\rho\otimes\sigma')(\tilde{T})$$

for each ρ' in $\mathcal{R}_\sharp$ and each σ' in $\mathcal{S}_\sharp$.

(iii) *Φ_σ and Ψ_ρ are ultraweakly continuous linear mappings of $\mathcal{R}\bar{\otimes}\mathcal{S}$ onto $\mathcal{R}$ and $\mathcal{S}$, respectively, for which*

$$\Phi_\sigma((A\otimes I)\tilde{T}(B\otimes I)) = A\Phi_\sigma(\tilde{T})B$$

$$\Psi_\rho((I\otimes C)\tilde{T}(I\otimes D)) = C\Psi_\rho(\tilde{T})D$$

for each $\tilde{T}$ in $\mathcal{R}\bar{\otimes}\mathcal{S}$, A,B in $\mathcal{R}$, and C,D in $\mathcal{S}$, and

$$\Phi_\sigma(R\otimes S) = \sigma(S)R, \quad \Psi_\rho(R\otimes S) = \rho(R)S \quad (R\in\mathcal{R},\ S\in\mathcal{S}).$$

(iv) *$\Phi_\sigma(\tilde{T})\in\mathcal{R}_0$ and $\Psi_\rho(\tilde{T})\in\mathcal{S}_0$ if $\tilde{T}\in\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$, with $\mathcal{R}_0$ and $\mathcal{S}_0$ von Neumann subalgebras of $\mathcal{R}$ and $\mathcal{S}$, respectively.*

(v) *$\tilde{T}\in\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$ if $\Phi_{\sigma'}(\tilde{T})\in\mathcal{R}_0$ and $\Psi_{\rho'}(\tilde{T})\in\mathcal{S}_0$ for each σ' in $\mathcal{S}_\sharp$ and each ρ' in $\mathcal{R}_\sharp$.*

PROOF. (i) Theorem 11.2.10 assures us that we may consider $\mathcal{R}$ and $\mathcal{S}$ in their universal normal representations on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, without loss of generality. In this case, there are vectors x and y in $\mathcal{H}$ of length $\|\rho\|^{1/2}$ and u and v in $\mathcal{K}$ of length $\|\sigma\|^{1/2}$ such that $\rho=\omega_{x,y}|\mathcal{R}$ and $\sigma=\omega_{u,v}|\mathcal{S}$ from Corollary 7.3.3. The equation

$$(\rho\otimes\sigma)(\tilde{T}) = \langle\tilde{T}(x\otimes u), y\otimes v\rangle \qquad (\tilde{T}\in\mathcal{R}\bar{\otimes}\mathcal{S})$$

defines an ultraweakly continuous linear functional $\rho\otimes\sigma$ on $\mathcal{R}\bar{\otimes}\mathcal{S}$. With R in $\mathcal{R}$ and S in $\mathcal{S}$, we have

$$(\rho\otimes\sigma)(R\otimes S) = \langle(R\otimes S)(x\otimes u), y\otimes v\rangle = \langle Rx,y\rangle\langle Su,v\rangle = \rho(R)\sigma(S).$$

If $R\in(\mathcal{R})_1$ and $S\in(\mathcal{S})_1$, then $R\otimes S\in(\mathcal{R}\otimes\mathcal{S})_1$ and

$$|\rho(R)|\,|\sigma(S)| = |(\rho\otimes\sigma)(R\otimes S)| \le \|\rho\otimes\sigma\|.$$

Hence $\|\rho\|\|\sigma\|\le\|\rho\otimes\sigma\|$. On the other hand, with $\tilde{T}$ in $(\mathcal{R}\otimes\mathcal{S})_1$, we have

$$|(\rho\otimes\sigma)(\tilde{T})| = |\langle\tilde{T}(x\otimes u), y\otimes v\rangle| \le \|x\|\|u\|\|y\|\|v\| = \|\rho\|\|\sigma\|.$$

Thus $\|\rho\otimes\sigma\|\le\|\rho\|\|\sigma\|$, and $\|\rho\otimes\sigma\| = \|\rho\|\|\sigma\|$. Since operators of the form $R\otimes S$, with R in $\mathcal{R}$ and S in $\mathcal{S}$, generate an ultraweakly dense linear submanifold of $\mathcal{R}\bar{\otimes}\mathcal{S}$, there is at most one linear functional on $\mathcal{R}\bar{\otimes}\mathcal{S}$ with the properties prescribed for $\rho\otimes\sigma$.

(ii) From the uniqueness clause of (i),

$$(a\rho+\rho')\otimes\sigma = a(\rho\otimes\sigma)+\rho'\otimes\sigma.$$

Thus the mapping

$$\Phi_\sigma(\tilde{T}) : \rho' \to (\rho'\otimes\sigma)(\tilde{T}) \qquad (\rho'\in\mathcal{R}_\sharp)$$

is a linear functional on $\mathcal{R}_\sharp$. Since

$$|(\Phi_\sigma(\tilde{T}))(\rho')| = |(\rho' \otimes \sigma)(\tilde{T})| \leq \|\rho' \otimes \sigma\| \|\tilde{T}\| = \|\rho'\| \|\sigma\| \|\tilde{T}\|,$$

$\Phi_\sigma(\tilde{T})$, as defined, is an element of $(\mathcal{R}_\sharp)^\sharp$. Thus $\Phi_\sigma(\tilde{T}) \in \mathcal{R}$ from Theorem 7.4.2. Symmetrically, $\Psi_\rho(\tilde{T}) \in \mathcal{S}$.

(iii) Since

$$\begin{aligned} \rho'(\Phi_\sigma(a\tilde{T} + \tilde{T}')) &= (\rho' \otimes \sigma)(a\tilde{T} + \tilde{T}') = a(\rho' \otimes \sigma)(\tilde{T}) + (\rho' \otimes \sigma)(\tilde{T}') \\ &= a\rho'(\Phi_\sigma(\tilde{T})) + \rho'(\Phi_\sigma(\tilde{T}')) \\ &= \rho'(a\Phi_\sigma(\tilde{T}) + \Phi_\sigma(\tilde{T}')) \end{aligned}$$

for each ρ' in $\mathcal{R}_\sharp$, Φ_σ is a linear mapping. Moreover,

$$\tilde{T} \to \rho'(\Phi_\sigma(\tilde{T})) = (\rho' \otimes \sigma)(\tilde{T}) \qquad (\tilde{T} \in \mathcal{R}\bar{\otimes}\mathcal{S})$$

is continuous from $\mathcal{R}\bar{\otimes}\mathcal{S}$ with its ultraweak topology to $\mathbb{C}$ for each ρ' in $\mathcal{R}_\sharp$. Hence Φ_σ is an ultraweakly continuous linear mapping from $\mathcal{R}\bar{\otimes}\mathcal{S}$ into $\mathcal{R}$.

With A, B, and R in $\mathcal{R}$, S in $\mathcal{S}$, ρ' in $\mathcal{R}_\sharp$, and ρ'' the element of $\mathcal{R}_\sharp$ whose value at R_0 in $\mathcal{R}$ is $\rho'(AR_0B)$, we have

$$\begin{aligned} \rho'(\Phi_\sigma((A \otimes I)(R \otimes S)(B \otimes I))) &= (\rho' \otimes \sigma)((ARB) \otimes S) \\ &= \rho'(ARB)\sigma(S) \\ &= \rho''(R)\sigma(S) \\ &= (\rho'' \otimes \sigma)(R \otimes S) \\ &= \rho''(\Phi_\sigma(R \otimes S)) \\ &= \rho'(A\Phi_\sigma(R \otimes S)B). \end{aligned}$$

Thus

$$\Phi_\sigma((A \otimes I)(R \otimes S)(B \otimes I)) = A\Phi_\sigma(R \otimes S)B.$$

Now the mappings

$$\tilde{T} \to \Phi_\sigma((A \otimes I)\tilde{T}(B \otimes I)), \qquad \tilde{T} \to A\Phi_\sigma(\tilde{T})B$$

are ultraweakly continuous linear mappings of $\mathcal{R}\bar{\otimes}\mathcal{S}$ into $\mathcal{R}$ that agree on generators of an ultraweakly dense linear submanifold of $\mathcal{R}\bar{\otimes}\mathcal{S}$. Hence they agree on $\mathcal{R}\bar{\otimes}\mathcal{S}$. The symmetric argument applies to Ψ_ρ, and the first relations set out in (iii) are established. With R in $\mathcal{R}$, S in $\mathcal{S}$, and ρ' in $\mathcal{R}_\sharp$, we have

$$\rho'(\Phi_\sigma(R \otimes S)) = (\rho' \otimes \sigma)(R \otimes S) = \rho'(\sigma(S)R).$$

Thus $\Phi_\sigma(R \otimes S) = \sigma(S)R$ and $\Psi_\rho(R \otimes S) = \rho(R)S$. It follows that Φ_σ maps *onto* $\mathcal{R}$ and Ψ_ρ maps *onto* $\mathcal{S}$.

(iv) From the last relations established in the proof of (iii), $\Phi_\sigma(\tilde{T}) \in \mathcal{R}_0$ when $\tilde{T} = R_0 \otimes S$ with R_0 in $\mathcal{R}_0$ and S in $\mathcal{S}$. Since operators of the form $R_0 \otimes S_0$ $(R_0 \in \mathcal{R}_0, S_0 \in \mathcal{S}_0)$ generate an ultraweakly dense linear submanifold of $\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$ and Φ_σ is an ultraweakly continuous linear mapping, Φ_σ maps $\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$ into $\mathcal{R}_0$. Symmetrically, Ψ_ρ maps $\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$ into $\mathcal{S}_0$.

(v) Suppose $\tilde{T}$ in $\mathcal{R}\bar{\otimes}\mathcal{S}$ is such that $\Phi_{\sigma'}(\tilde{T}) \in \mathcal{R}_0$ and $\Psi_{\rho'}(\tilde{T}) \in \mathcal{S}_0$ for each σ' in $\mathcal{S}_\sharp$ and ρ' in $\mathcal{R}_\sharp$.

With A' in $\mathcal{R}_0'$, x and u in $\mathcal{H}$, and y and v in $\mathcal{K}$, let σ' be $\omega_{y,v}|\mathcal{S}$, ρ' be $\omega_{x,A'^*u}|\mathcal{R}$, and ρ'' be $\omega_{A'x,u}|\mathcal{R}$. Then

$$\begin{aligned}\langle (A'\otimes I)\tilde{T}(x\otimes y), u\otimes v\rangle &= (\rho'\otimes\sigma')(\tilde{T}) = \rho'(\Phi_{\sigma'}(\tilde{T}))\\ &= \langle A'\Phi_{\sigma'}(\tilde{T})x, u\rangle = \langle \Phi_{\sigma'}(\tilde{T})A'x, u\rangle\\ &= \rho''(\Phi_{\sigma'}(\tilde{T})) = (\rho''\otimes\sigma')(\tilde{T})\\ &= \langle \tilde{T}(A'x\otimes y), u\otimes v\rangle\\ &= \langle \tilde{T}(A'\otimes I)(x\otimes y), u\otimes v\rangle.\end{aligned}$$

Thus $\tilde{T}$ commutes with $\mathcal{R}_0'\otimes\mathbb{C}I$. Symmetrically, $\tilde{T}$ commutes with $\mathbb{C}I\otimes\mathcal{S}_0'$. Thus $\tilde{T}\in(\mathcal{R}_0'\bar{\otimes}\mathcal{S}_0')'$. From Theorem 11.2.16 and the double commutant theorem, $(\mathcal{R}_0'\bar{\otimes}\mathcal{S}_0')' = \mathcal{R}_0\bar{\otimes}\mathcal{S}_0$. Hence $\tilde{T}\in\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$. □

THEOREM 14. *Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Suppose $\mathcal{R}_0$ and $\mathcal{S}_0$ are von Neumann subalgebras of $\mathcal{R}$ and $\mathcal{S}$, respectively.*

(i) $(\mathcal{R}_0'\cap\mathcal{R})\bar{\otimes}(\mathcal{S}_0'\cap\mathcal{S}) = (\mathcal{R}_0\bar{\otimes}\mathcal{S}_0)'\cap(\mathcal{R}\bar{\otimes}\mathcal{S})$.

(ii) *$\mathcal{A}\bar{\otimes}\mathcal{B}$ is a maximal abelian subalgebra of $\mathcal{R}\bar{\otimes}\mathcal{S}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are maximal abelian subalgebras of $\mathcal{R}$ and $\mathcal{S}$, respectively.*

(iii) *$\mathcal{C}\bar{\otimes}\mathcal{D}$ is the center of $\mathcal{R}\bar{\otimes}\mathcal{S}$ when $\mathcal{C}$ is the center of $\mathcal{R}$ and $\mathcal{D}$ is the center of $\mathcal{S}$.*

(iv) *$\mathcal{R}_0'\cap\mathcal{R}=\mathbb{C}I$ and $\mathcal{S}_0'\cap\mathcal{S}=\mathbb{C}I$ if and only if we have that the intersection $(\mathcal{R}_0\bar{\otimes}\mathcal{S}_0)'\cap(\mathcal{R}\bar{\otimes}\mathcal{S})$ is $\mathbb{C}I$.*

PROOF. (i) If $R\in\mathcal{R}_0'\cap\mathcal{R}$ and $S\in\mathcal{S}_0'\cap\mathcal{S}$, then $R\otimes S$ commutes with $\mathcal{R}_0\bar{\otimes}\mathcal{S}_0$. Thus

$$(\mathcal{R}_0'\cap\mathcal{R})\bar{\otimes}(\mathcal{S}_0'\cap\mathcal{S}) \subseteq (\mathcal{R}_0\bar{\otimes}\mathcal{S}_0)'\cap(\mathcal{R}\bar{\otimes}\mathcal{S}).$$

Suppose $\tilde{T}\in(\mathcal{R}_0\bar{\otimes}\mathcal{S}_0)'\cap(\mathcal{R}\bar{\otimes}\mathcal{S})$, $A\in\mathcal{R}_0$, $\rho\in\mathcal{R}_\sharp$, and $\sigma\in\mathcal{S}_\sharp$. Then, from Theorem 13(iii),

$$A\Phi_\sigma(\tilde{T}) = \Phi_\sigma((A\otimes I)\tilde{T}) = \Phi_\sigma(\tilde{T}(A\otimes I)) = \Phi_\sigma(\tilde{T})A.$$

Hence $A\Phi_\sigma(\tilde{T}) = \Phi_\sigma(\tilde{T})A$ and $\Phi_\sigma(\tilde{T})\in\mathcal{R}_0'\cap\mathcal{R}$ for each σ in $\mathcal{S}_\sharp$. Symmetrically, $\Psi_\rho(\tilde{T})\in\mathcal{S}_0'\cap\mathcal{S}$ for each ρ in $\mathcal{R}_\sharp$. From Theorem 13(v),

$$\tilde{T}\in(\mathcal{R}_0'\cap\mathcal{R})\bar{\otimes}(\mathcal{S}_0'\cap\mathcal{S}).$$

Hence

$$(\mathcal{R}_0\bar{\otimes}\mathcal{S}_0)'\cap(\mathcal{R}\bar{\otimes}\mathcal{S}) \subseteq (\mathcal{R}_0'\cap\mathcal{R})\bar{\otimes}(\mathcal{S}_0'\cap\mathcal{S}).$$

Combining this with the reverse inclusion, noted earlier, we have the formula of (i).

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are maximal abelian in $\mathcal{R}$ and $\mathcal{S}$, respectively, then $\mathcal{A}'\cap\mathcal{R}=\mathcal{A}$ and $\mathcal{B}'\cap\mathcal{S}=\mathcal{B}$. Thus

$$(\mathcal{A}\bar{\otimes}\mathcal{B})'\cap(\mathcal{R}\bar{\otimes}\mathcal{S}) = (\mathcal{A}'\cap\mathcal{R})\bar{\otimes}(\mathcal{B}'\cap\mathcal{S}) = \mathcal{A}\bar{\otimes}\mathcal{B},$$

from (i), and $\mathcal{A}\bar{\otimes}\mathcal{B}$ is maximal abelian in $\mathcal{R}\bar{\otimes}\mathcal{S}$. If $\mathcal{A}\bar{\otimes}\mathcal{B}$ is maximal abelian in $\mathcal{R}\bar{\otimes}\mathcal{S}$ and T in $\mathcal{R}$ commutes with $\mathcal{A}$, then

$$T\otimes I\in(\mathcal{A}'\cap\mathcal{R})\bar{\otimes}(\mathcal{B}'\cap\mathcal{S}) = (\mathcal{A}\bar{\otimes}\mathcal{B})'\cap(\mathcal{R}\bar{\otimes}\mathcal{S}) = \mathcal{A}\bar{\otimes}\mathcal{B}.$$

From Theorem 13(iii) and (iv), $T=\Phi_\sigma(T\otimes I)\in\mathcal{A}$. Thus $\mathcal{A}$ is maximal abelian in $\mathcal{R}$. Symmetrically, $\mathcal{B}$ is maximal abelian in $\mathcal{S}$.

(iii) From (i),

$$(\mathcal{R}\bar{\otimes}\mathcal{S})'\cap(\mathcal{R}\bar{\otimes}\mathcal{S}) = (\mathcal{R}'\cap\mathcal{R})\bar{\otimes}(\mathcal{S}'\cap\mathcal{S}) = \mathcal{C}\bar{\otimes}\mathcal{D}.$$

But $(\mathcal{R}\bar{\otimes}\mathcal{S})' \cap (\mathcal{R}\bar{\otimes}\mathcal{S})$ is the center of $\mathcal{R}\bar{\otimes}\mathcal{S}$.

(iv) If the tensor product of two von Neumann algebras is the algebra of scalar multiples of I, then each of the von Neumann algebras is the algebra of scalar multiples of I. From (i), then, each of $\mathcal{R}_0' \cap \mathcal{R}$ and $\mathcal{S}_0' \cap \mathcal{S}$ is $\mathbb{C}I$ if and only if

$$(\mathcal{R}_0\bar{\otimes}\mathcal{S}_0)' \cap (\mathcal{R}\bar{\otimes}\mathcal{S}) = \mathbb{C}I.$$

□

4. Applications to Jones index

Although it is not our intention to carry the discussion of the Jones index to an advanced level, we shall describe the basic role that conditional expectations play in the development of the theory around the Jones index and give a careful presentation of the initial portions of that subject in terms of conditional expectations. The conditional expection described in Theorem 7 is an absolutely crucial element of the Jones index theory. We denote the conditional expectation φ constructed in Theorem 7 by $\Phi^{\mathcal{S}}_{\mathcal{R}}$. When the context makes clear the von Neumann algebra from which we are mapping ($\mathcal{S}$ in the present case), we write $\Phi_{\mathcal{R}}$ in place of $\Phi^{\mathcal{S}}_{\mathcal{R}}$. We begin by gathering some more information about that conditional expectation.

PROPOSITION 15. *If $\mathcal{M}$ is a von Neumann algebra with a faithful tracial state τ and $\mathcal{N}$ is a von Neumann subalgebra then $\Phi_{\mathcal{N}}$ is the unique conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ that lifts the tracial state $\tau_{\mathcal{N}}$ $(= \tau_{\mathcal{M}}|\mathcal{N})$ to the trace $\tau_{\mathcal{M}}$ on $\mathcal{M}$; that is*

$$\tau_{\mathcal{N}}(\Phi_{\mathcal{N}}(A)) = \tau_{\mathcal{M}}(A) \qquad (A \in \mathcal{M}).$$

If $\mathcal{P}$ is a von Neumann subalgebra of $\mathcal{N}$, then

$$\Phi^{\mathcal{M}}_{\mathcal{P}} = \Phi^{\mathcal{N}}_{\mathcal{P}} \circ \Phi^{\mathcal{M}}_{\mathcal{N}}.$$

PROOF. From the defining property (†) of $\Phi_{\mathcal{N}}$

$$\tau_{\mathcal{N}}(\Phi_{\mathcal{N}}(A)) = \tau_{\mathcal{N}}(\Phi_{\mathcal{N}}(A)I) = \tau_{\mathcal{M}}(AI) = \tau_{\mathcal{M}}(A) \qquad (A \in \mathcal{M}),$$

whence $\Phi_{\mathcal{N}}$ lifts $\tau_{\mathcal{N}}$ to $\tau_{\mathcal{M}}$.

If Φ is a conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ that lifts $\tau_{\mathcal{N}}$ to $\tau_{\mathcal{M}}$, then

$$\tau_{\mathcal{N}}(\Phi_{\mathcal{N}}(A)) = \tau_{\mathcal{M}}(A) = \tau_{\mathcal{N}}(\Phi(A)) \qquad (A \in \mathcal{M}).$$

Thus $\tau_{\mathcal{N}}((\Phi_{\mathcal{N}} - \Phi)(A)) = 0$ for each A in $\mathcal{M}$. Since $\Phi_{\mathcal{N}}$ and Φ are conditional expectations onto $\mathcal{N}$,

$$0 = \tau_{\mathcal{N}}((\Phi_{\mathcal{N}} - \Phi)(A(\Phi_{\mathcal{N}} - \Phi)(A)^*)) = \tau_{\mathcal{N}}((\Phi_{\mathcal{N}} - \Phi)(A)(\Phi_{\mathcal{N}} - \Phi)(A)^*).$$

As $\tau_{\mathcal{N}}$ is faithful, $(\Phi_{\mathcal{N}} - \Phi)(A) = 0$ $(A \in \mathcal{M})$, and $\Phi_{\mathcal{N}} = \Phi$.

Of course, $\Phi^{\mathcal{N}}_{\mathcal{P}} \circ \Phi^{\mathcal{M}}_{\mathcal{N}}$ is a linear mapping of $\mathcal{M}$ into $\mathcal{P}$. If $A \in \mathcal{P}$, then since $\mathcal{P} \subseteq \mathcal{N}$, $(\Phi^{\mathcal{N}}_{\mathcal{P}} \circ \Phi^{\mathcal{M}}_{\mathcal{N}})(A) = \Phi^{\mathcal{N}}_{\mathcal{P}}(A) = A$. In addition, $\Phi^{\mathcal{N}}_{\mathcal{P}} \circ \Phi^{\mathcal{M}}_{\mathcal{N}}$ is positive since each of $\Phi^{\mathcal{N}}_{\mathcal{P}}$ and $\Phi^{\mathcal{M}}_{\mathcal{N}}$ is positive. At the same time,

$$\tau_{\mathcal{M}}(T) = \tau_{\mathcal{N}}(\Phi^{\mathcal{M}}_{\mathcal{N}}(T)) = \tau_{\mathcal{P}}(\Phi^{\mathcal{N}}_{\mathcal{P}}(\Phi^{\mathcal{M}}_{\mathcal{N}}(T)) \qquad (T \in \mathcal{M}).$$

Thus $\Phi^{\mathcal{N}}_{\mathcal{P}} \circ \Phi^{\mathcal{M}}_{\mathcal{N}}$ lifts $\tau_{\mathcal{P}}$ to $\tau_{\mathcal{M}}$, and $\Phi^{\mathcal{M}}_{\mathcal{P}} = \Phi^{\mathcal{N}}_{\mathcal{P}} \circ \Phi^{\mathcal{M}}_{\mathcal{N}}$. □

PROPOSITION 16. *If $\mathcal{M}$ is a factor acting on a Hilbert space $\mathcal{H}$ with a unit, trace vector u, $\mathcal{N}$ is a von Neumann subalgebra, and $E_{\mathcal{N}}$ is the orthogonal projection on $\mathcal{H}$ with range $[\mathcal{N}u]$ (in $\mathcal{N}'$), then $E_{\mathcal{N}}Tu = \Phi_{\mathcal{N}}(T)u$ for each T in $\mathcal{M}$.*

PROOF. With T in $\mathcal{M}$, note that

$$\langle Tu - \Phi_{\mathcal{N}}(T)u, Su\rangle = \tau_{\mathcal{M}}(S^*T) - \tau_{\mathcal{N}}(S^*\Phi_{\mathcal{N}}(T)) = 0 \qquad (S \in \mathcal{N}),$$

from the defining condition for $\Phi_{\mathcal{N}}(T)$. Thus $Tu - \Phi_{\mathcal{N}}(T)u$ is orthogonal to $[\mathcal{N}u]$ and, since $\Phi_{\mathcal{N}}(T)u \in [\mathcal{N}u]$,

$$0 = E_{\mathcal{N}}(Tu - \Phi_{\mathcal{N}}(T)u) = E_{\mathcal{N}}Tu - \Phi_{\mathcal{N}}(T)u \qquad (T \in \mathcal{M}).$$

□

In the discussion of Jones index, that follows, we refer to Jones [**Jo83**], Pimsner–Popa [**Pi-Po86**], and to the masterful account of this and many related matters in [**E-K98**]. Let $\mathcal{M}$ be a factor of type II_1 acting on a Hilbert space $\mathcal{H}$ with a cyclic, unit, trace vector u. If $\mathcal{N}$ is a subfactor of $\mathcal{M}$ such that $\mathcal{N}'$ is of finite type, then $\mathcal{N}'$ is a finite factor with a tracial state τ'. We call $\tau'(E_u^{\mathcal{N}})^{-1}$ the *index of* $\mathcal{N}$ *in* $\mathcal{M}$ and denote it by '$[\mathcal{M} : \mathcal{N}]$.' In this case, we say that $\mathcal{N}$ *has finite index in* $\mathcal{M}$. Note that, since $\mathcal{H} = [\mathcal{M}'u] \subseteq [\mathcal{N}'u]$, $\tau(E_u^{\mathcal{N}'}) = 1$, and $\tau(E_u^{\mathcal{N}'})/\tau'(E_u^{\mathcal{N}})$, the coupling constant $\mathrm{d}_{\mathcal{N}}(\mathcal{H})$ of $\mathcal{N}$ and $\mathcal{N}'$, is $\tau'(E_u^{\mathcal{N}})^{-1}$, the index $[\mathcal{M} : \mathcal{N}]$ of the subfactor $\mathcal{N}$ in $\mathcal{M}$.

As defined, $[\mathcal{M} : \mathcal{N}]$ is an (isomorphism) invariant of the (ordered) pair $\langle \mathcal{M}, \mathcal{N}\rangle$. To see this, let $\mathcal{P}$ be a factor of type II_1 with a cyclic, unit, trace vector v, acting on a Hilbert space $\mathcal{K}$, $\mathcal{Q}$ a subfactor, and ψ a * isomorphism of $\mathcal{M}$ onto $\mathcal{P}$ that carries $\mathcal{N}$ onto $\mathcal{Q}$. From Theorem 7.2.9, there is a unitary transformation of $\mathcal{H}$ onto $\mathcal{K}$ that implements ψ. Thus $\mathrm{d}_{\mathcal{N}}(\mathcal{H}) = \mathrm{d}_{\mathcal{Q}}(\mathcal{K})$, and $[\mathcal{M} : \mathcal{N}] = [\mathcal{P} : \mathcal{Q}]$. Since $\mathrm{d}_{\mathcal{M}}(\mathcal{H}) = 1$ and $\mathrm{d}_{\mathcal{N}}(\mathcal{H}) = [\mathcal{M} : \mathcal{N}]$, we have that

$$[\mathcal{M} : \mathcal{N}] = \mathrm{d}_{\mathcal{N}}(\mathcal{H}) / \mathrm{d}_{\mathcal{M}}(\mathcal{H}). \tag{‡}$$

This formula is valid no matter what the coupling constant is for $\mathcal{M}$ and $\mathcal{M}'$. (See Proposition A5 of the appendix to this section.)

PROPOSITION 17. *If $\mathcal{M}$ is a finite factor and $\mathcal{S}$ and $\mathcal{N}$ are subfactors such that $\mathcal{S} \subseteq \mathcal{N}$ and $\mathcal{S}'$ is finite, then*

(i) $[\mathcal{M} : \mathcal{M}] = 1$;
(ii) $[\mathcal{M} : \mathcal{S}] \geq 1$;
(iii) $[\mathcal{M} : \mathcal{S}] = [\mathcal{S}' : \mathcal{M}']$;
(iv) $[\mathcal{M} : \mathcal{S}] = [\mathcal{M} : \mathcal{N}][\mathcal{N} : \mathcal{S}]$;
(v) *if* $[\mathcal{M} : \mathcal{N}] = [\mathcal{M} : \mathcal{S}]$, *then* $\mathcal{N} = \mathcal{S}$.

PROOF. We may assume that M acts on the Hilbert space H with a unit, cyclic, trace vector u.

(i) Since $\tau'(E_u^{\mathcal{M}}) = 1$, $[\mathcal{M} : \mathcal{M}] = 1$.
(ii) Since $\tau'(E_u^{\mathcal{S}}) \leq 1$, $[\mathcal{M} : \mathcal{S}] \geq 1$.
(iii) Since $\mathrm{d}_{\mathcal{S}}(\mathcal{H}) = \tau(E_x^{\mathcal{S}'})/\tau'(E_x^{\mathcal{S}})$ (for each unit vector x in $\mathcal{H}$), $\mathrm{d}_{\mathcal{S}'}(\mathcal{H}) = \mathrm{d}_{\mathcal{S}}(\mathcal{H})^{-1}$ and

$$[\mathcal{M} : \mathcal{S}] = \mathrm{d}_{\mathcal{S}}(\mathcal{H}) / \mathrm{d}_{\mathcal{M}}(\mathcal{H}) = \mathrm{d}_{\mathcal{M}'}(\mathcal{H}) / \mathrm{d}_{\mathcal{S}'}(\mathcal{H}) = [\mathcal{S}' : \mathcal{M}'].$$

(iv) Note that

$$[\mathcal{M} : \mathcal{S}] = \frac{\mathrm{d}_{\mathcal{S}}(\mathcal{H})}{\mathrm{d}_{\mathcal{M}}(\mathcal{H})} = \left(\frac{\mathrm{d}_{\mathcal{S}}(\mathcal{H})}{\mathrm{d}_{\mathcal{N}}(\mathcal{H})}\right)\left(\frac{\mathrm{d}_{\mathcal{N}}(\mathcal{H})}{\mathrm{d}_{\mathcal{M}}(\mathcal{H})}\right) = [\mathcal{M} : \mathcal{N}][\mathcal{N} : \mathcal{S}].$$

(v) If $[\mathcal{M} : \mathcal{N}] = [\mathcal{M} : \mathcal{S}]$, then $\tau''(E_u^{\mathcal{S}}) = \tau''(E_u^{\mathcal{N}})$, where τ'' is the tracial state on $\mathcal{S}'$. As $E_u^{\mathcal{S}} \le E_u^{\mathcal{N}}$ and τ'' is faithful, $E_u^{\mathcal{S}} = E_u^{\mathcal{N}}$. Thus if H is a self-adjoint operator in $\mathcal{N}$, there is a sequence $\{T_1, T_2, \ldots\}$ in $\mathcal{S}$ such that $T_n u \to Hu$ in $\mathcal{H}$. As u is a trace vector for $\mathcal{M}$, $T_n^* u \to Hu$. Thus $H_n u \to Hu$, where $H_n = \frac{1}{2}(T_n + T_n^*)$. Let h be the continuous, real-valued function on $\mathbb{R}$ that takes the value t for each t in $[-\|H\|, \|H\|]$, vanishes on $[-\infty, -\|H\| - 1] \cup [\|H\| + 1, \infty]$, and is linear on $[-\|H\| - 1, -\|H\|]$ and $[\|H\|, \|H\| + 1]$. Then $h(H) = H$, and $\|h(H_n)\| \le \|H\|$ for each n. Moreover, from Exercise 12.4.32(ii), $h(H_n)u \to h(H)u = Hu$. Of course, each $h(H_n) \in \mathcal{S}$. Since u is cyclic for $\mathcal{N}'$ and $\{h(H_n)\}$ is a bounded set, $h(H_n)$ is strong-operator convergent to H. Thus $H \in \mathcal{S}$ and $\mathcal{S} = \mathcal{N}$.

□

PROPOSITION 18. *If $\mathcal{M}$ is a finite factor acting on a Hilbert space $\mathcal{H}$, u is a cyclic trace vector for $\mathcal{M}$ and $\mathcal{N}$ is a von Neumann subalgebra, then*

(i) $E_{\mathcal{N}} A E_{\mathcal{N}} = \Phi_{\mathcal{N}}(A) E_{\mathcal{N}} \qquad (A \in \mathcal{M})$;

(ii) *A in $\mathcal{M}$, is in $\mathcal{N}$ if and only if $AE_{\mathcal{N}} = E_{\mathcal{N}} A E_{\mathcal{N}}$, if and only if $E_{\mathcal{N}} A = AE_{\mathcal{N}}$, and $E_{\mathcal{N}}$ is separating for $\mathcal{M}$;*

(iii) *$\mathcal{N}'$ is the von Neumann algebra generated by $\mathcal{M}'$ and $E_{\mathcal{N}}$;*

(iv) *the von Neumann algebra $\mathcal{M}_1$ generated by $\mathcal{M}$ and $E_{\mathcal{N}}$ is the strong-operator closure $\mathcal{F}$ of $\{A_0 + \sum_{j=1}^n A_j E_{\mathcal{N}} B_j : A_j, B_j \in \mathcal{M}\}$*

(v) $\mathcal{N} E_{\mathcal{N}} = E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}}$;

(vi) *the central carrier of $E_{\mathcal{N}}$ relative to $\mathcal{N}'$ and to $\mathcal{M}_1$ is I;*

(vii) *$\mathcal{M}_1$ is a factor if and only if $\mathcal{N}$ is a factor;*

(viii) *$\mathcal{M}_1$ is finite if and only if $\mathcal{N}'$ is.*

PROOF. (i) With B in $\mathcal{N}$ and A in $\mathcal{M}$, from Proposition 16

$$E_{\mathcal{N}} ABu = \Phi_{\mathcal{N}}(AB)u = \Phi_{\mathcal{N}}(A)Bu.$$

Thus $E_{\mathcal{N}} A E_{\mathcal{N}} = \Phi_{\mathcal{N}}(A) E_{\mathcal{N}}$ since $\{Bu : B \in \mathcal{N}\}$ is dense in $E_{\mathcal{N}}(\mathcal{H})$.

(ii) Suppose $A \in \mathcal{M}$ and $AE_{\mathcal{N}} = E_{\mathcal{N}} A E_{\mathcal{N}}$. Then, from (i),

$$AE_{\mathcal{N}} = E_{\mathcal{N}} A E_{\mathcal{N}} = \Phi_{\mathcal{N}}(A) E_{\mathcal{N}}.$$

Hence $Au = \Phi_{\mathcal{N}}(A)u$. As u is separating for $\mathcal{M}$, $E_{\mathcal{N}}$ is separating for $\mathcal{M}$, and $A = \Phi_{\mathcal{N}}(A) \in \mathcal{N}$. From Proposition 16, $E_{\mathcal{N}} \in \mathcal{N}'$, whence $AE_{\mathcal{N}} = E_{\mathcal{N}} A$ in this case. Of course, $AE_{\mathcal{N}} = E_{\mathcal{N}} A E_{\mathcal{N}}$ when $AE_{\mathcal{N}} = E_{\mathcal{N}} A$.

(iii) From (ii),

$$\mathcal{N} = \mathcal{M} \cap \{E_{\mathcal{N}}\}' = (\mathcal{M}' \cup \{E_{\mathcal{N}}\})'.$$

Thus $\mathcal{N}' = (\mathcal{M}' \cup \{E_{\mathcal{N}}\})''$, the von Neumann algebra generated by $\mathcal{M}'$ and $E_{\mathcal{N}}$.

(iv) Since $\mathcal{F}$ contains $\mathcal{M}$ and $E_{\mathcal{N}}$, and $\mathcal{F}$ is a self-adjoint family, it remains to note that $\mathcal{F}$ is an algebra. For this, note that, with A, B, C, and D in $\mathcal{M}$,

$$(AE_{\mathcal{N}}B)(CE_{\mathcal{N}}D) = AE_{\mathcal{N}}BCE_{\mathcal{N}}D = A\Phi_{\mathcal{N}}(BC)E_{\mathcal{N}}D,$$

from (i). As $\Phi_{\mathcal{N}}(BC) \in \mathcal{N}$, $A\Phi_{\mathcal{N}}(BC) \in \mathcal{M}$ and $(AE_{\mathcal{N}}B)(CE_{\mathcal{N}}D) \in \mathcal{F}$.

(v) Since $E_{\mathcal{N}} \in \mathcal{N}'$, $AE_{\mathcal{N}} = E_{\mathcal{N}} A E_{\mathcal{N}} \in E_{\mathcal{N}} \mathcal{F} E_{\mathcal{N}} \subseteq E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}}$ $(A \in \mathcal{N})$, and $\mathcal{N} E_{\mathcal{N}} \subseteq E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}}$. To establish the reverse inclusion, note that if $A =$

$A_0 + \sum_{j=1}^n A_j E_{\mathcal{N}} B_j$, with A_j and B_j in $\mathcal{M}$, then

$$E_{\mathcal{N}} A E_{\mathcal{N}} = \left[\Phi_{\mathcal{N}}(A_0) + \sum_{j=1}^n \Phi_{\mathcal{N}}(A_j)\Phi_{\mathcal{N}}(B_j) \right] E_{\mathcal{N}} \in \mathcal{N} E_{\mathcal{N}},$$

from (i). Thus $E_{\mathcal{N}} \mathcal{F} E_{\mathcal{N}} \subseteq \mathcal{N} E_{\mathcal{N}}$. Since $T \to E_{\mathcal{N}} T E_{\mathcal{N}}$ is weak-operator continuous on $\mathcal{B}(\mathcal{H})$ and $\mathcal{M}_1$ is the strong, hence, weak, -operator closure of $\mathcal{F}$, from (iv), we have that $E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}}$ is the weak-operator closure of $E_{\mathcal{N}} \mathcal{F} E_{\mathcal{N}}$. It follows that $E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}} \subseteq \mathcal{N} E_{\mathcal{N}}$. Thus $\mathcal{N} E_{\mathcal{N}} = E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}}$.

(vi) From Proposition 5.5.2, the central carrier of $E_{\mathcal{N}}$ relative to $\mathcal{N}'$ has range $[\mathcal{N}' E_{\mathcal{N}}(\mathcal{H})]$, which contains $[\mathcal{N}'u]$ $(\supseteq [\mathcal{M}'u] = \mathcal{H})$. Thus $E_{\mathcal{N}}$ has central carrier I relative to $\mathcal{N}'$. The range of the central carrier of $E_{\mathcal{N}}$ relative to $\mathcal{M}_1$ is $[\mathcal{M}_1 E_{\mathcal{N}}(\mathcal{H})]$ which contains $[\mathcal{M} E_{\mathcal{N}}(\mathcal{H})]$. But $[\mathcal{M} E_{\mathcal{N}}(\mathcal{H})]$ contains $[\mathcal{M}u]$, which is $\mathcal{H}$, by assumption. This proves (vi).

(vii) From Proposition 5.5.6, $E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}}$ has center $\mathcal{C} E_{\mathcal{N}}$, where $\mathcal{C}$ is the center of $\mathcal{M}_1$. From (v), $E_{\mathcal{N}} \mathcal{M}_1 E_{\mathcal{N}} = \mathcal{N} E_{\mathcal{N}}$, whence $\mathcal{C} E_{\mathcal{N}}$ is the center of $\mathcal{N} E_{\mathcal{N}}$. From (vi), $E_{\mathcal{N}}$ has central carrier I relative to $\mathcal{M}_1$. As $\mathcal{C} \subseteq \mathcal{M}_1'$, $\mathcal{C}$ and $\mathcal{C} E_{\mathcal{N}}$ are isomorphic. At the same time, $E_{\mathcal{N}}$ has central carrier I relative to $\mathcal{N}'$ from (vi), whence $\mathcal{N}$ and $\mathcal{N} E_{\mathcal{N}}$ are isomorphic. Hence $\mathcal{M}_1$ is a factor if and only if $\mathcal{C} E_{\mathcal{N}}$ is one dimensional, which is the case if and only if $\mathcal{N}$ is a factor.

(viii) With A in $\mathcal{M}$, let $J_0 A u$ be $A^* u$. then J_0 is a conjugate-linear, involutive, isometric mapping of the dense linear manifold $\{\mathcal{M}u\}$ of $\mathcal{H}$ onto itself. Let J be its extension to such a mapping of $\mathcal{H}$ onto itself. Note that, from (i) and Proposition 16,

$$\begin{aligned} J E_{\mathcal{N}} J A u &= J E_{\mathcal{N}} A^* u = J E_{\mathcal{N}} A^* E_{\mathcal{N}} u = J \Phi_{\mathcal{N}}(A^*) E_{\mathcal{N}} u \\ &= J \Phi_{\mathcal{N}}(A)^* u = \Phi_{\mathcal{N}}(A) u = E_{\mathcal{N}} A u \end{aligned}$$

for each A in $\mathcal{M}$. Thus $J E_{\mathcal{N}} J = E_{\mathcal{N}}$.

Note, next, that $J \mathcal{M} J = \mathcal{M}'$, whence $\mathcal{M} = J \mathcal{M}' J$. To see this, choose A, B, and C, in $\mathcal{M}$. Then $Ju = JIu = I^* u = u$, and

$$JAJBu = JAB^* u = BA^* u = BJAu = BJAJu.$$

Thus

$$JAJBCu = BCJAJu = BJAJCu,$$

whence $JAJB = BJAJ$ (since $[\mathcal{M}u] = \mathcal{H}$), and $JAJ \in \mathcal{M}'$. It follows that $J \mathcal{M} J \subseteq \mathcal{M}'$.

From Theorem 7.2.15, for each A in $\mathcal{M}$, there is a unique A' in $\mathcal{M}'$ such that $Au = A'u$ and $A^* u = A'^* u$. Thus J' defined, for $\mathcal{M}'$ and u, as J was for $\mathcal{M}$ and u, coincides with J. From what we have proved, then,

$$J \mathcal{M}' J = J' \mathcal{M}' J' \subseteq \mathcal{M}'' = \mathcal{M},$$

and $\mathcal{M}' \subseteq J \mathcal{M} J$. Hence $J \mathcal{M} J = \mathcal{M}'$.

It follows, from (iii), that

$$J \mathcal{N}' J = J \{\mathcal{M}' \cup \{E_{\mathcal{N}}\}\}'' J = \{\mathcal{M} \cup \{E_{\mathcal{N}}\}\}'' = \mathcal{M}_1,$$

whence $\mathcal{M}_1$ is finite if and only if $\mathcal{N}'$ is. □

We refer to the process by which we arrived at $\mathcal{M}_1$ and the projection $E_{\mathcal{N}}$, with the properties described as the *basic construction* for the factor $\mathcal{M}$, the subfactor $\mathcal{N}$, and the trace vector u.

PROPOSITION 19. *Let $\mathcal{M}$ be a factor of type II_1 acting on a Hilbert space $\mathcal{H}$ with unit, cyclic, trace vector u and $\mathcal{N}$ be a subfactor such that $\mathcal{N}'$ is finite.*

(i) *The von Neumann algebra $\mathcal{M}_1$ generated by $\mathcal{M}$ and the projection $E_{\mathcal{N}}$ is a factor of type II_1, $\tau_{\mathcal{M}_1}(E_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1}$, $[\mathcal{M}_1 : \mathcal{M}] = [\mathcal{M} : \mathcal{N}]$, and $\Phi_{\mathcal{M}}(E_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1}I$, where $\Phi_{\mathcal{M}}$ is the conditional expectation of $\mathcal{M}_1$ onto $\mathcal{M}$ that lifts $\tau_{\mathcal{M}}$ to $\tau_{\mathcal{M}_1}$.*

(ii) *There is a subfactor $\mathcal{P}$ of $\mathcal{N}$ and a projection E in $\mathcal{M} \cap \mathcal{P}'$ such that $\tau_{\mathcal{M}}(E) = [\mathcal{M} : \mathcal{N}]^{-1}$, $\Phi_{\mathcal{N}}(E) = [\mathcal{M} : \mathcal{N}]^{-1}I$, $ETE = \Phi_{\mathcal{P}}(T)E$ for each T in $\mathcal{N}$, where $\Phi_{\mathcal{P}}$ is the conditional expectation of $\mathcal{N}$ onto $\mathcal{P}$ that lifts $\tau_{\mathcal{P}}$ to $\tau_{\mathcal{N}}$, and $\mathcal{M}$ is generated by $\mathcal{N}$ and E.*

PROOF. (i) From Proposition 18 (vii), (viii), $\mathcal{M}_1$ is a finite factor since $\mathcal{N}'$ is a finite factor. Since $\mathcal{N}'$ is a finite factor and contains the factor $\mathcal{M}'$ of type II_1, $\mathcal{N}'$ must be a factor of type II_1. From the proof of Proposition 18 (viii), $\mathcal{M}_1 = J\mathcal{N}'J$. Hence M_1 is a factor of type II_1.

The mapping $A \to JA^*J$ $(= \varphi(A))$ is bijective on $\mathcal{B}(\mathcal{H})$, since $J^2 = I$ and $\varphi^2(A) = A$ $(A \in \mathcal{B}(\mathcal{H}))$. In addition, since J is conjugate linear and u is a trace vector for $\mathcal{M}$,

$$\begin{aligned}\langle J^*Au, Bu\rangle &= \langle JBu, Au\rangle = \langle B^*u, Au\rangle = \langle A^*B^*u, u\rangle \\ &= \langle B^*A^*u, u\rangle = \langle JAu, Bu\rangle \qquad (A, B \in \mathcal{M}).\end{aligned}$$

Thus $J = J^*$, and $\varphi(A^*) = JAJ = (JA^*J)^* = \varphi(A)^*$ for each A in $\mathcal{B}(\mathcal{H})$. Moreover, we have that

$$\varphi(AB) = JB^*A^*J = JB^*JJA^*J = \varphi(B)\varphi(A) \qquad (A, B \in \mathcal{B}(\mathcal{H})).$$

Hence $\varphi|\mathcal{M}$ and $\varphi|\mathcal{M}_1$ are * anti-isomorphisms of $\mathcal{M}$ onto $\mathcal{M}'$ and $\mathcal{M}_1$ onto $\mathcal{N}'$, respectively. Thus

$$(\dagger\dagger) \qquad \tau_{\mathcal{M}_1}(E_{\mathcal{N}}) = \tau_{\mathcal{N}'}(E_{\mathcal{N}}) = \tau_{\mathcal{N}'}(E_u^{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1},$$

by definition of the index $[\mathcal{M} : \mathcal{N}]$. At the same time,

$$\begin{aligned}[\mathcal{M}_1 : \mathcal{M}] &= \mathrm{d}_{\mathcal{M}}(\mathcal{H})/\,\mathrm{d}_{\mathcal{M}_1}(\mathcal{H}) = \mathrm{d}_{\mathcal{M}_1}(\mathcal{H})^{-1} = \mathrm{d}_{\mathcal{N}'}(\mathcal{H})^{-1} \\ &= [\tau_{\mathcal{N}'}(E_u^{\mathcal{N}})/\tau_{\mathcal{N}}(E_u^{\mathcal{N}'})]^{-1} = \tau_{\mathcal{N}'}(E_u^{\mathcal{N}})^{-1} = [\mathcal{M} : \mathcal{N}].\end{aligned}$$

To see that $\Phi_{\mathcal{M}}(E_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1}I$, we note, first, that $\tau_{\mathcal{N}} = \tilde{\tau}$, where

$$\tilde{\tau}(T) = [\mathcal{M} : \mathcal{N}]\tau_{\mathcal{M}_1}(TE_{\mathcal{N}}) \qquad (T \in \mathcal{N}).$$

Since $\mathcal{N}$ is a factor, it suffices to show that $\tilde{\tau}$ is a tracial state on $\mathcal{N}$. Of course, $\tilde{\tau}$ is linear. In addition, $\tilde{\tau}$ is positive since $\tau_{\mathcal{M}_1}$ is and $E_{\mathcal{N}}$ is a positive operator (a *projection*) in $\mathcal{N}'$. Also, $\tilde{\tau}(I) = [\mathcal{M} : \mathcal{N}]\tau_{\mathcal{M}_1}(E_{\mathcal{N}}) = 1$, from ($\dagger\dagger$). Finally,

$$\begin{aligned}\tilde{\tau}(TS) &= [\mathcal{M} : \mathcal{N}]\tau_{\mathcal{M}_1}(TSE_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]\tau_{\mathcal{M}_1}(TE_{\mathcal{N}}S) \\ &= [\mathcal{M} : \mathcal{N}]\tau_{\mathcal{M}_1}(STE_{\mathcal{N}}) = \tilde{\tau}(ST) \qquad (S, T \in \mathcal{N}).\end{aligned}$$

It follows that, with A in $\mathcal{M}$,

$$\begin{aligned}\tau_{\mathcal{M}_1}(E_{\mathcal{N}}A) &= \tau_{\mathcal{M}_1}(E_{\mathcal{N}}AE_{\mathcal{N}}) = \tau_{\mathcal{M}_1}(\Phi_{\mathcal{N}}(A)E_{\mathcal{N}}) \\ &= [\mathcal{M} : \mathcal{N}]^{-1}\tau_{\mathcal{N}}(\Phi_{\mathcal{N}}(A)) = [\mathcal{M} : \mathcal{N}]^{-1}\tau_{\mathcal{M}}(A) \\ &= \tau_{\mathcal{M}}([\mathcal{M} : \mathcal{N}]^{-1}IA),\end{aligned}$$

whence $\Phi_{\mathcal{M}}(E_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1}I$, by definition of $\Phi_{\mathcal{M}}$.

(ii) We note, first, that there is a faithful, normal representation of $\mathcal{M}$ on a Hilbert space such that $\mathcal{N}$ has coupling 1. To see this, we start with $\mathcal{M}$ in its trace representation on a Hilbert space $\mathcal{H}$ with u a unit, cyclic, trace vector for $\mathcal{M}$. Choose a projection F' in $\mathcal{M}'$ such that $(\tau_{\mathcal{N}'}(F') =)\tau_{\mathcal{M}'}(F') = [M : \mathcal{N}]^{-1} = \tau_{\mathcal{N}'}(E_u^{\mathcal{N}})$. Since F' and $E_u^{\mathcal{N}}$ are equivalent in $\mathcal{N}'$, F' is a cyclic projection in $\mathcal{N}'$ (from Proposition 6.2.9). Let x be a unit vector in $\mathcal{H}$ such that $F'(\mathcal{H}) = [\mathcal{N}x]$. From Theorem 7.2.12, $E_x^{\mathcal{N}'}$ and $E_u^{\mathcal{N}'}$ $(= I)$ are equivalent in $\mathcal{N}$. Since $\mathcal{N}$ is finite, $E_x^{\mathcal{N}'} = I$. Thus $\mathcal{N}F'$ and its commutant $F'\mathcal{N}'F'$ acting on $F'(\mathcal{H})$ $(= \mathcal{K})$ have x as a joint generating vector. As $\mathcal{M}$ and $\mathcal{N}$ are factors and F' is a non-zero projection in $\mathcal{M}'$ $(\subseteq \mathcal{N}')$, the mapping $A \to AF'$ of $\mathcal{M}$ onto $\mathcal{M}F'$ is a * isomorphism of $\mathcal{M}$ onto $\mathcal{M}F'$ and $\mathcal{N}$ onto $\mathcal{N}F'$, with commutants $F'\mathcal{M}'F'$ and $F'\mathcal{N}'F'$, respectively, from Proposition 5.5.5.

Henceforth, we assume that $\mathcal{M}$ and $\mathcal{N}$ act on $\mathcal{K}$. Since $\mathcal{N}$ has coupling 1, from Lemma 7.2.8, there is a unit, cyclic, trace vector v for $\mathcal{N}$. We now form the basic construction for the factor $\mathcal{N}'$ of type II_1, the type II_1 subfactor $\mathcal{M}'$, and the trace vector v. This gives us a projection E in $\mathcal{M}$ $(= \mathcal{M}'')$ such that $EA'E = \Phi_{\mathcal{M}'}(A')E$ for each A' in $\mathcal{N}'$, $\mathcal{N}$ and E generate $\mathcal{M}$, $\tau_{\mathcal{M}}(E) = [\mathcal{N}' : \mathcal{M}']^{-1} = [\mathcal{M} : \mathcal{N}]^{-1}$, and $\Phi_{\mathcal{N}'}(E) = [\mathcal{M} : \mathcal{N}]^{-1}I$, where $\Phi_{\mathcal{N}'}$ is the conditional expectation of $\mathcal{N}_1'$, the von Neumann algebra generated by $\mathcal{N}'$ and E, onto $\mathcal{N}'$ that lifts $\tau_{\mathcal{N}'}$ to $\tau_{\mathcal{N}_1'}$.

The mapping $Av \to A^*v$ $(A \in \mathcal{N})$ extends, as in the proof of (i), to a conjugate-linear, isometric, involutive, self-adjoint mapping J' of $\mathcal{K}$ onto itself such that $J'v = v$ and $J'EJ' = E$. If $\varphi'(T) = J'T^*J'$, then φ' is an involutory, * anti-automorphism of $\mathcal{B}(\mathcal{K})$ that maps $\mathcal{M}$ onto $\mathcal{N}_1'$, $\mathcal{N}$ onto $\mathcal{N}'$, and $\mathcal{M}'$ onto a subfactor $\mathcal{P}$ of $\mathcal{N}$. Then $[\mathcal{N} : \mathcal{P}] = [\mathcal{N}' : \mathcal{M}'] = [\mathcal{M} : \mathcal{N}]$. Moreover, $\varphi' \circ \Phi_{\mathcal{N}'} \circ \varphi' = \Phi_{\mathcal{N}}$, $\varphi' \circ \Phi_{\mathcal{M}'} \circ \varphi' = \Phi_{\mathcal{P}}$, and $\varphi'(E) = E$. Thus $E \in \mathcal{M} \cap \mathcal{P}'$, $\tau_{\mathcal{M}}(E) = [\mathcal{M} : \mathcal{N}]^{-1}$, $\Phi_{\mathcal{N}}(E) = [\mathcal{M} : \mathcal{N}]^{-1}I$, $ETE = \Phi_{\mathcal{P}}(T)E$ $(T \in \mathcal{N})$. □

PROPOSITION 20. *Let $\mathcal{M}$ be a factor of type II_1 acting on a Hilbert space $\mathcal{H}$ with u a unit, cyclic trace vector and let $\mathcal{N}$ be a subfactor. Then with $\mathcal{M}_1$ as in Proposition 18 (viii), for each A in $\mathcal{M}_1$, there is a unique element B in $\mathcal{M}$ for which $AE_{\mathcal{N}} = BE_{\mathcal{N}}$, $B = [\mathcal{M} : \mathcal{N}]\Phi_{\mathcal{M}}(AE_{\mathcal{N}})$, and $\|B\| \le [\mathcal{M} : \mathcal{N}]\|A\|$.*

PROOF. We argue as in the proof of Proposition 18(v). If

$$A = A_0 + \sum_{j=1}^{n} A_j E_{\mathcal{N}} B_j$$

with A_j and B_j in $\mathcal{M}$, then $A \in \mathcal{F}$. In this case, $AE_{\mathcal{N}} = [A_0 + \sum_{j=1}^{n} A_j \Phi_{\mathcal{N}}(B_j)]E_{\mathcal{N}}$, and $A_0 + \sum_{j=1}^{n} A_j \Phi_{\mathcal{N}}(B_j)$ $(= B)$ is in $\mathcal{M}$. From Proposition 19(i),

$$\begin{aligned}[\mathcal{M} : \mathcal{N}]\Phi_{\mathcal{M}}(AE_{\mathcal{N}}) &= \left[A_0 + \sum_{j=1}^{n} A_j \Phi_{\mathcal{N}}(B_j)\right][\mathcal{M} : \mathcal{N}]\Phi_{\mathcal{M}}(E_{\mathcal{N}}) \\ &= A_0 + \sum_{j=1}^{n} A_j \Phi_{\mathcal{N}}(B_j) = B.\end{aligned}$$

Thus $A \to (A - [\mathcal{M} : \mathcal{N}]\Phi_{\mathcal{M}}(AE_{\mathcal{N}}))E_{\mathcal{N}}$ is an ultraweakly continuous mapping that vanishes on $\mathcal{F}$ and, hence, on $\mathcal{M}_1$, the ultraweak closure of $\mathcal{F}$. It follows that, with A in $\mathcal{M}_1$,

$$\|B\| = [\mathcal{M} : \mathcal{N}]\|\Phi_{\mathcal{M}}(AE_{\mathcal{N}})E_{\mathcal{N}}\| \le [\mathcal{M} : \mathcal{N}]\|A\|.$$

□

In the theorem that follows, we construct a system of elements, known as a *Pimsner–Popa basis* in a factor of type II_1 with special properties relative to a subfactor of finite index.

THEOREM 21. *If $\mathcal{N}$ is a subfactor of finite index $n+a$, with n a positive integer and a in $[0,1)$, of a factor $\mathcal{M}$ of type II_1, then there are elements $B_1, \dots, B_{n+1}$ in $\mathcal{M}$ such that*

(i) $\Phi_{\mathcal{N}}(B_j^* B_k) = 0$ *when* $j \neq k$;
(ii) $\Phi_{\mathcal{N}}(B_j^* B_j) = I$ *if* $j \in \{1, \dots, n\}$, *and* $\Phi_{\mathcal{N}}(B_{n+1}^* B_{n+1}) = F$, *where F is a projection in $\mathcal{N}$ such that* $\tau_{\mathcal{N}}(F) = a$;
(iii) $\sum_{j=1}^{n+1} B_j E_{\mathcal{N}} B_j^* = I$;
(iv) $\sum_{j=1}^{n+1} B_j B_j^* = [\mathcal{M} : \mathcal{N}]I$;
(v) $T = \sum_{j=1}^{n+1} B_j \Phi_{\mathcal{N}}(B_j^* T)$, *for each T in $\mathcal{M}$.*

PROOF. Let $\mathcal{M}_1$ be the von Neumann algebra generated by $\mathcal{M}$ and $E_{\mathcal{N}}$. From Proposition 19, $\mathcal{M}_1$ is a factor of type II_1 and $\Phi_{\mathcal{M}}(E_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1} I = (n+a)^{-1} I$. There is a projection E_1 in $\mathcal{M}$ such that $\tau_{\mathcal{M}_1}(E_1) = \tau_{\mathcal{M}_1}(E_{\mathcal{N}}) = [\mathcal{M} : \mathcal{N}]^{-1}$. There is a subprojection E_2 of $I - E_1$ in $\mathcal{M}$, such that $\tau_{\mathcal{M}_1}(E_2) = \tau_{\mathcal{M}_1}(E_{\mathcal{N}})$. Continuing in this way, we produce a set $E_1, \dots, E_{n+1}$ of mutually orthogonal projections in $\mathcal{M}$ such that $\tau_{\mathcal{M}_1}(E_j) = [\mathcal{M} : \mathcal{N}]^{-1}$ when $j \in \{1, \dots, n\}$, $E_{n+1} = I - \sum_{j=1}^{n} E_j$, and $\tau_{\mathcal{M}_1}(E_{n+1}) = a[\mathcal{M} : \mathcal{N}]^{-1}$. Since each E_j is equivalent to $E_{\mathcal{N}}$ in $\mathcal{M}_1$, for j in $\{1, \dots, n\}$, there are partial isometries $V_1, \dots, V_n$ in $\mathcal{M}_1$ such that $V_j^* V_j = E_{\mathcal{N}}$ and $V_j V_j^* = E_j$. As $0 \le a < 1$, there is a partial isometry V_{n+1} in $\mathcal{M}_1$ such that $V_{n+1}^* V_{n+1} = E_0 < E_{\mathcal{N}}$ and $V_{n+1} V_{n+1}^* = E_{n+1}$. Thus $V_j E_{\mathcal{N}} = V_j$ for all j. From Proposition 20, there is a unique B_j in $\mathcal{M}$ such that $V_j = V_j E_{\mathcal{N}} = B_j E_{\mathcal{N}}$. Hence, when j and k are distinct,

$$0 = V_j^* E_k V_k = V_j^* V_k = E_{\mathcal{N}} B_j^* B_k E_{\mathcal{N}} = \Phi_{\mathcal{N}}(B_j^* B_k) E_{\mathcal{N}}$$

and with j in $\{1 \dots, n\}$,

$$E_{\mathcal{N}} = V_j^* V_j = E_{\mathcal{N}} B_j^* B_j E_{\mathcal{N}} = \Phi_{\mathcal{N}}(B_j^* B_j) E_{\mathcal{N}}.$$

Since $E_{\mathcal{N}}$ has central carrier I in $\mathcal{N}'$, from (vi) of Proposition 18, we conclude that $\Phi_{\mathcal{N}}(B_j^* B_k) = 0$ when $j \neq k$, and $\Phi_{\mathcal{N}}(B_j^* B_j) = I$ when $j \in \{1, \dots, n\}$. In addition,

$$E_0 = V_{n+1}^* V_{n+1} = E_{\mathcal{N}} B_{n+1}^* B_{n+1} E_{\mathcal{N}} = \Phi_{\mathcal{N}}(B_{n+1}^* B_{n+1}) E_{\mathcal{N}}.$$

Now, $T \to T E_{\mathcal{N}}$ is a $*$ isomorphism of $\mathcal{N}$ onto $\mathcal{N} E_{\mathcal{N}}$. As $\Phi_{\mathcal{N}}(B_{n+1}^* B_{n+1}) E_{\mathcal{N}}$ is a projection, $\Phi_{\mathcal{N}}(B_{n+1}^* B_{n+1})$ is a projection in $\mathcal{N}$. From the proof of (i) of Proposition 19,

$$\begin{aligned}
\tau_{\mathcal{M}}(B_{n+1}^* B_{n+1}) &= \tau_{\mathcal{N}}(\Phi_{\mathcal{N}}(B_{n+1}^* B_{n+1})) \\
&= [\mathcal{M} : \mathcal{N}] \tau_{\mathcal{M}_1}(\Phi_{\mathcal{N}}(B_{n+1}^* B_{n+1}) E_{\mathcal{N}}) \\
&= [\mathcal{M} : \mathcal{N}] \tau_{\mathcal{M}_1}(E_0) = [\mathcal{M} : \mathcal{N}] \tau_{\mathcal{M}_1}(E_{n+1}) \\
&= [\mathcal{M} : \mathcal{N}] a [\mathcal{M} : \mathcal{N}]^{-1} = a.
\end{aligned}$$

This proves (i) and (ii).

At the same time,

$$I = \sum_{j=1}^{n+1} E_j = \sum_{j=1}^{n+1} V_j V_j^* = \sum_{j=1}^{n+1} B_j E_{\mathcal{N}} B_j^*,$$

which proves (iii). From this, (iv) follows since

$$\begin{aligned} I &= \Phi_{\mathcal{M}}(I) = \sum_{j=1}^{n+1} \Phi_{\mathcal{M}}(B_j E_{\mathcal{N}} B_j^*) \\ &= \sum_{n=1}^{n+1} B_j \Phi_{\mathcal{M}}(E_{\mathcal{N}}) B_j^* = [\mathcal{M} : \mathcal{N}]^{-1} \sum_{j=1}^{n+1} B_j B_j^*. \end{aligned}$$

To prove (v), we note, from (iii), that

$$TE_{\mathcal{N}} = \left(\sum_{j=1}^{n+1} B_j E_{\mathcal{N}} B_j^* \right) TE_{\mathcal{N}} = \sum_{j=1}^{n+1} B_j \Phi_{\mathcal{N}}(B_j^* T) E_{\mathcal{N}},$$

from which (v) follows, since $E_{\mathcal{N}}$ is separating for $\mathcal{M}$ ((ii) of Proposition 18). □

COROLLARY 22. *In the notation of Proposition 18, if* $[\mathcal{M} : \mathcal{N}] = n + a$, *where* n *is a positive integer and* $a \in [0, 1)$, *then*

$$\mathcal{M}_1 = \left\{ \sum_{j=1}^{n+1} T_j E_{\mathcal{N}} S_j : T_j, S_j \in \mathcal{M} \right\} = (\mathcal{F}_0).$$

PROOF. The preceding theorem implies that there is a Pimsner–Popa basis $\{B_1, \dots, B_{n+1}\}$ in $\mathcal{M}$. With S in $\mathcal{M}_1$, for each j in $\{1, \dots, n\}$, there is a (unique) A_j in $\mathcal{M}$ such that $SB_j E_{\mathcal{N}} = A_j E_{\mathcal{N}}$ from Proposition 20. It follows from (iii) of Theorem 21 that

$$S = \sum_{j=1}^{n+1} SB_j E_{\mathcal{N}} B_j^* = \sum_{j=1}^{n+1} A_j E_{\mathcal{N}} B_j^* \in \mathcal{F}_0.$$

Thus $\mathcal{M}_1 = \mathcal{F}_0$. □

PROPOSITION 23. *If* $\mathcal{M}$ *is a factor of type* II_1 *acting on a Hilbert space* $\mathcal{H}$ *and* $\mathcal{N}$ *is a subfactor such that* $\mathcal{N}'$ *is finite, then* $\Phi_{\mathcal{N}}(A) \geq [\mathcal{M} : \mathcal{N}]^{-1} A$ $(A \in \mathcal{M}^+)$.

PROOF. From Proposition 19, there is a subfactor $\mathcal{P}$ of $\mathcal{N}$ and a projection E in $\mathcal{M}$ commuting with $\mathcal{P}$ such that $\Phi_{\mathcal{N}}(E) = [\mathcal{M} : \mathcal{N}]^{-1} I$, $ETE = \Phi_{\mathcal{P}}(T)E$ for each T in $\mathcal{N}$, and $\mathcal{M}$ is generated by E and $\mathcal{N}$. If A is a positive element in $\mathcal{M}$, then $A = BB^*$ for some B in $\mathcal{M}$. From Theorem 21, there is a Pimsner–Popa basis $B_1, \dots, B_n$ for $\mathcal{N}$ relative to $\mathcal{P}$. From (the proof of) Corollary 22, there are operators $A_1, \dots, A_n$ in $\mathcal{N}$ such that $B = \sum_{j=1}^{n} A_j E B_j^*$. Thus

$$\begin{aligned} A = BB^* &= \left(\sum_{j=1}^{n} A_j E B_j^* \right) \left(\sum_{k=1}^{n} A_k E B_k^* \right)^* = \sum_{j,k=1}^{n} A_j E B_j^* B_k E A_k^* \\ &= \sum_{j,k=1}^{n} A_j \Phi_{\mathcal{P}}(B_j^* B_k) E A_k^* = \sum_{j=1}^{n} A_j \Phi_{\mathcal{P}}(B_j^* B_j) E A_j^* \end{aligned}$$

and since $E \in \mathcal{P}'$,

$$\begin{aligned}
\Phi_{\mathcal{N}}(A) &= \sum_{j=1}^{n} A_j \Phi_{\mathcal{P}}(B_j^* B_j) \Phi_{\mathcal{N}}(E) A_j^* \\
&= [\mathcal{M} : \mathcal{N}]^{-1} \sum_{j=1}^{n} A_j \Phi_{\mathcal{P}}(B_j^* B_j) A_j^* \\
&\geq [\mathcal{M} : \mathcal{N}]^{-1} \sum_{j=1}^{n} A_j \Phi_{\mathcal{P}}(B_j^* B_j) E A_j^* \\
&= [\mathcal{M} : \mathcal{N}]^{-1} A.
\end{aligned}$$

□

Jones index – Appendix. In this appendix, we prove the assertion (‡) (Proposition A5) as well as other basic identities involving $d_{\mathcal{N}}(\mathcal{H})$ and $d_{\mathcal{M}}(\mathcal{H})$.

LEMMA A1. *If $\mathcal{M}$ is a finite factor acting on a Hilbert space $\mathcal{H}$ and E' is a non-zero projection in $\mathcal{M}'$, then*

$$\mathrm{d}_{\mathcal{M}E'}(E'(\mathcal{H})) = \tau_{\mathcal{M}'}(E')\, \mathrm{d}_{\mathcal{M}}(\mathcal{H}).$$

PROOF. Note that the mapping $A \to AE'$ is a * isomorphism of $\mathcal{M}$ onto $\mathcal{M}E'$ (acting on $E'(\mathcal{H})$ with commutant $E'\mathcal{M}'E'$) from Proposition 5.5.5. Thus $\tau_{\mathcal{M}E'}(AE') = \tau_{\mathcal{M}}(A)$ when $A \in \mathcal{M}$. At the same time, $\tau_{E'\mathcal{M}'E'}(E'A'E') = \tau_{\mathcal{M}'}(E')^{-1}\tau_{\mathcal{M}'}(E'A'E')$ for each A' in $\mathcal{M}'$.

Let x be a unit vector in $E'(\mathcal{H})$. Note that $E_x^{E'\mathcal{M}'E'} = E'E_x^{\mathcal{M}'}$ and $E_x^{\mathcal{M}E'} = E_x^{\mathcal{M}}$. Thus

$$\begin{aligned}
\mathrm{d}_{\mathcal{M}E'}(E'(\mathcal{H})) &= \tau_{\mathcal{M}E'}(E_x^{E'\mathcal{M}'E'})/\tau_{E'\mathcal{M}'E'}(E_x^{\mathcal{M}E'}) \\
&= \tau_{\mathcal{M}E'}(E'E_x^{\mathcal{M}'})/\tau_{\mathcal{M}'}(E')^{-1}\tau_{\mathcal{M}'}(E_x^{\mathcal{M}}) \\
&= \tau_{\mathcal{M}'}(E')\tau_{\mathcal{M}}(E_x^{\mathcal{M}'})/\tau_{\mathcal{M}'}(E_x^{\mathcal{M}}) \\
&= \tau_{\mathcal{M}'}(E')\, \mathrm{d}_{\mathcal{M}}(\mathcal{H}).
\end{aligned}$$

□

COROLLARY A2. *If $\mathcal{M}$ is a finite factor acting on a Hilbert space $\mathcal{H}$ and $\mathcal{N}$ is a subfactor, with $\mathcal{N}'$ finite, then*

$$\mathrm{d}_{\mathcal{N}}(\mathcal{H})/\mathrm{d}_{\mathcal{M}}(\mathcal{H}) = \mathrm{d}_{\mathcal{N}E'}(E'(\mathcal{H}))/\,\mathrm{d}_{\mathcal{M}E'}(E'(\mathcal{H}))$$

where E' is a non-zero projection in $\mathcal{M}'$.

LEMMA A3. *If $\mathcal{M}$ is a finite factor acting on a Hilbert space $\mathcal{H}$ and $\mathcal{K}$ is an n-dimensional Hilbert space, with n finite, and $\tilde{\mathcal{M}} = \mathcal{M} \otimes \mathbb{C}I_{\mathcal{K}}$, then $\mathrm{d}_{\tilde{\mathcal{M}}}(\mathcal{H} \otimes \mathcal{K}) = n\, \mathrm{d}_{\mathcal{M}}(\mathcal{H})$.*

PROOF. Choosing an orthonormal basis for $\mathcal{K}$, we represent $\mathcal{H} \otimes \mathcal{K}$ as $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, the n-fold direct sum of $\mathcal{H}$ with itself. In this representation, $\tilde{\mathcal{M}}$ appears as the algebra of diagonal matrices with the same element of $\mathcal{M}$ at each diagonal entry. (We write $\tilde{A}$ for this matrix, corresponding to the element A in $\mathcal{M}$ and note that the mapping $A \to \tilde{A}$ is a * isomorphism of $\mathcal{M}$ onto $\tilde{\mathcal{M}}$.) The commutant $\tilde{\mathcal{M}}'$ appears as the algebra of $n \times n$ matrices with arbitrary entries from $\mathcal{M}'$. The matrices act on column vectors $(x_1, \ldots, x_n)$ $(x_j \in \mathcal{H})$.

Let x be a unit vector in $\mathcal{H}$ and $\tilde{x}$ be $(x, 0, \ldots, 0)$ in $\mathcal{H} \otimes \mathcal{K}$. Then $E_{\tilde{x}}^{\tilde{\mathcal{M}}}$ is the matrix whose only non-zero entry is $E_x^{\mathcal{M}}$ at the 1,1 entry and $E_{\tilde{x}}^{\tilde{\mathcal{M}}'}$ is the diagonal matrix with $E_x^{\mathcal{M}'}$ at each diagonal entry. Thus $\tau_{\tilde{\mathcal{M}}'}(E_{\tilde{x}}^{\tilde{\mathcal{M}}}) = n^{-1}\tau_{\mathcal{M}'}(E_x^{\mathcal{M}})$ and $\tau_{\tilde{\mathcal{M}}}(E_{\tilde{x}}^{\tilde{\mathcal{M}}'}) = \tau_{\mathcal{M}}(E_x^{\mathcal{M}'})$. It follows that

$$\begin{aligned} \mathrm{d}_{\tilde{\mathcal{M}}}(\mathcal{H} \otimes \mathcal{K}) &= \tau_{\tilde{\mathcal{M}}}(E_{\tilde{x}}^{\tilde{\mathcal{M}}'})/\tau_{\tilde{\mathcal{M}}'}(E_{\tilde{x}}^{\tilde{\mathcal{M}}}) \\ &= \tau_{\mathcal{M}}(E_x^{\mathcal{M}'})/n^{-1}\tau_{\mathcal{M}'}(E_x^{\mathcal{M}}) = n\,\mathrm{d}_{\mathcal{M}}(\mathcal{H})\,. \end{aligned}$$

□

COROLLARY A4. *If $\mathcal{M}$ is a finite factor acting on a Hilbert space $\mathcal{H}$ and $\mathcal{N}$ is a subfactor, with $\mathcal{N}'$ finite, then*

$$\mathrm{d}_{\tilde{\mathcal{M}}}(\mathcal{H} \otimes \mathcal{K})/\,\mathrm{d}_{\tilde{\mathcal{N}}}(\mathcal{H} \otimes \mathcal{K}) = \mathrm{d}_{\mathcal{M}}(\mathcal{H})/\,\mathrm{d}_{\mathcal{N}}(\mathcal{H}),$$

where $\mathcal{K}$ and $\tilde{\mathcal{M}}$ are as in the preceding lemma.

In the cases described in Lemmas A1 and A3, there are representations of $\mathcal{M}$ involved. In Lemma A1, there is the isomorphism of $\mathcal{M}$ onto $\mathcal{M}E'$, and the coupling changes to $\tau_{\mathcal{M}'}(E')\,\mathrm{d}_{\mathcal{M}}(\mathcal{H})$ (from $\mathrm{d}_{\mathcal{M}}(\mathcal{H})$). In Lemma A3, there is the isomorphism $A \to \tilde{A}$ of $\mathcal{M}$ onto $\tilde{\mathcal{M}}$, and the coupling changes to $n\,\mathrm{d}_{\mathcal{M}}(\mathcal{H})$. If we apply these isomorphisms successively, in the appropriate order (to $\mathcal{M}$ and then the image), we produce a representation of $\mathcal{M}$ with coupling $n\tau_{\mathcal{M}'}(E')\,\mathrm{d}_{\mathcal{M}}(\mathcal{H})$. Suitable choice of E' and n yields any positive real number we please as coupling, when $\mathcal{M}$ is a factor of type II$_1$ with $\mathcal{M}'$ finite. If $\mathcal{M}$ acting on $\mathcal{H}'$ is any of these representations, we have noted that $\mathrm{d}_{\mathcal{N}}(\mathcal{H}')/\,\mathrm{d}_{\mathcal{M}}(\mathcal{H}') = \mathrm{d}_{\mathcal{N}}(\mathcal{H})/\,\mathrm{d}_{\mathcal{M}}(\mathcal{H})$. Since coupling is a unitary invariant for representations of $\mathcal{M}$ with finite commutant (see Exercise 9.6.30), these representations constitute all representations of $\mathcal{M}$ (up to unitary equivalence). On the other hand, when $\mathcal{H}$ is chosen such that $\mathrm{d}_{\mathcal{M}}(\mathcal{H}) = 1$, we have defined the index $[\mathcal{M} : \mathcal{N}]$ of $\mathcal{N}$ in $\mathcal{M}$ to be $\mathrm{d}_{\mathcal{N}}(\mathcal{H})$ $(= \mathrm{d}_{\mathcal{N}}(\mathcal{H})/\,\mathrm{d}_{\mathcal{M}}(\mathcal{H}))$. From this discussion, we conclude the following result.

PROPOSITION A5. *If $\mathcal{M}$ is a factor of type II$_1$ acting on a Hilbert space $\mathcal{H}$ and $\mathcal{N}$ is a subfactor such that $\mathcal{N}'$ is finite, then $[\mathcal{M} : \mathcal{N}] = \mathrm{d}_{\mathcal{N}}(\mathcal{H})/\,\mathrm{d}_{\mathcal{M}}(\mathcal{H})$.*

In the discussion that follows we give an alternate proof of the fact that $\mathcal{N} = \mathcal{M}$ when $\mathcal{M}$ is a factor of type II$_1$ and $\mathcal{N}$ is subfactor such that $[\mathcal{M} : \mathcal{N}] = 1$. According to our definition, we consider $\mathcal{M}$ in its trace representation with cyclic trace vector u and $[\mathcal{M} : \mathcal{N}] = \tau_{\mathcal{N}'}(E_u^{\mathcal{N}})$. Thus when $[\mathcal{M} : \mathcal{N}] = 1$, we have that $\tau_{\mathcal{N}'}(E_u^{\mathcal{N}}) = 1$ and $E_u^{\mathcal{N}} = I$. We have also proved that, with T in $\mathcal{M}$, $E_u^{\mathcal{N}}(Tu) = \Phi_{\mathcal{N}}(T)u$. Under the present assumption, then, $Tu = \Phi_{\mathcal{N}}(T)u$ (see Proposition 2). As u is separating for $\mathcal{M}$, $T = \Phi_{\mathcal{N}}(T) \in \mathcal{N}$, and $\mathcal{M} = \mathcal{N}$.

5. The Schur inequalities

In this section, we prove some tracial, matrix-operator inequalities that include the "Schur Inequalities" [**Sch23**] and extend these inequalities to more general (infinite-dimensional) situations. When we carry these results to the case of factors of type II$_1$, the conditional expectation of Theorem 7 will play a decisive role. The material presented here is joint work with W. B. Arveson. It is part of a project-in-progress and represents a second approach to certain results presented in another way in that project.

We begin with a numerical inequality that underlies our later results.

LEMMA 24. *If $a_1, a_2, \ldots,$ are in $[0,1]$, $\sum_{j=1}^{\infty} a_j \le m$ for some integer m, and $\lambda, \lambda_1, \lambda_2, \ldots,$ are real numbers such that $\lambda_j \ge \lambda \ge 0$ when $1 \le j \le m$ and $\lambda \ge \lambda_k$ when $m+1 \le k$, then when $n \ge m$,*

$$(*) \qquad \sum_{j=1}^{n} \lambda_j a_j \le \sum_{j=1}^{m} \lambda_j.$$

If equality holds and $\lambda_j > \lambda$ when $1 \le j \le m$, then $a_1 = \cdots = a_m = 1$ and $a_j = 0$ when $j > m$.

PROOF. Since $\sum_{j=1}^{n} a_j \le m$, we have that $\sum_{j=m+1}^{n} a_j \le \sum_{j=1}^{m} 1 - a_j$. Thus

$$\begin{aligned}
\sum_{j=1}^{m} \lambda_j - \sum_{j=1}^{n} \lambda_j a_j &= \sum_{j=1}^{m} \lambda_j (1 - a_j) - \sum_{j=m+1}^{n} \lambda_j a_j \\
&\ge \sum_{j=1}^{m} \lambda_j (1 - a_j) - \sum_{j=m+1}^{n} \lambda a_j \\
&\ge \sum_{j=1}^{m} \lambda_j (1 - a_j) - \sum_{j=1}^{m} \lambda (1 - a_j) \\
&= \sum_{j=1}^{m} (\lambda_j - \lambda)(1 - a_j) \ge 0.
\end{aligned}$$

If equality holds in $(*)$, then $\sum_{j=1}^{m} (\lambda_j - \lambda)(1 - a_j) = 0$. By assumption, $\lambda_j - \lambda > 0$ when $1 \le j \le m$. Thus $a_1 = \cdots = a_m = 1$. Since $\sum_{j=1}^{\infty} a_j \le m$ and $a_j \ge 0$, it follows that $a_j = 0$ when $j > m$. $\square$

Note that if the condition '$\lambda_j > \lambda$' is not in force, in the case of equality, and we let $\lambda_1, \ldots, \lambda_{20}$ be 1 and m be 10, then we may choose $\frac{1}{2}$ for $a_1, \ldots, a_{20}$ and equality will hold, or we may choose 1 for $a_1, \ldots, a_{10}$. Thus uniqueness fails when that condition is not in force (and nothing replaces it).

The theorem that follows is a version of the Schur inequalities extended to trace class operators.

THEOREM 25. *If A is a trace-class operator acting on the Hilbert space $\mathcal{H}$ and $\{e_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ such that $Ae_j = \lambda_j e_j$, where $\lambda_j \ge \lambda \ge 0$ when $1 \le j \le m$ and $\lambda \ge \lambda_k$ when $k \ge m+1$, then*

$$(**) \qquad \sup\{\operatorname{tr}(HAH) : H = H^*,\, H^2 \le I,\, \operatorname{tr}(H^2) \le m\} = \sum_{j=1}^{m} \lambda_j = \operatorname{tr}(AE),$$

where E is the projection with range spanned by $\{e_1, \ldots, e_m\}$.

If the supremum in $(**)$ is attained at H_0 and $\lambda_j > \lambda$ when $1 \le j \le m$, then $H_0^2 = E$. In particular, if $\lambda_j > \lambda_{j+1} > 0$ for all j, then $(**)$ holds for each positive integer m and the projection E is the unique positive H at which the supremum is attained.

PROOF. Note that, with H as in $(**)$,

$$\operatorname{tr}(HAH) = \operatorname{tr}(AH^2) = \sum_{j=1}^{\infty} \langle AH^2 e_j, e_j \rangle = \sum_{j=1}^{\infty} \lambda_j \langle H^2 e_j, e_j \rangle = \sum_{j=1}^{\infty} \lambda_j \|He_j\|^2$$

and that $\sum_{j=1}^{\infty} \|He_j\|^2 = \operatorname{tr}(H^2) \leq m$. By applying the preceding lemma with $\|He_j\|^2$ in place of a_j, we conclude that

$$\operatorname{tr}(HAH) \leq \sum_{j=1}^{m} \lambda_j = \operatorname{tr}(AE) = \operatorname{tr}(EAE)$$

from which (**) follows.

For the last assertions, note that, by assumption,

$$\sum_{j=1}^{\infty} \lambda_j \|H_0 e_j\|^2 = \operatorname{tr}(H_0 A H_0) = \sum_{j=1}^{m} \lambda_j,$$

and $\lambda_j > \lambda > 0$. From the equality condition of the preceding lemma, $\|H_0 e_1\|^2 = \cdots = \|H_0 e_m\|^2 = 1$ and $H_0 e_j = 0$ when $j > m$. From the equality condition of the Cauchy-Schwarz inequality, $H_0^2 e_j = e_j$ for $j = 1, \ldots, m$. It follows that $H_0^2 = E$. □

Without the condition $\lambda_j > \lambda$, we can choose for A a projection of, say, dimension 20 and take m to be 10. Then the supremum is attained at each 10 dimensional subprojection of A. Thus uniqueness requires some condition such as $\lambda_j > \lambda$.

COROLLARY 26. *If A is a positive, trace-class operator acting on the Hilbert space $\mathcal{H}$ with eigenvalues $\lambda_1, \lambda_2, \cdots$ listed in decreasing order, then* $\sup\{\operatorname{tr}(FAF) : F = F^* = F^2, \dim F(\mathcal{H}) \leq m\}$ *is attained when F is the projection E with range spanned by an orthonormal set $\{f_1, \ldots, f_m\}$ such that $Af_j = \lambda_j f_j$ when $1 \leq j \leq m$. If $\{e_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then $\sum_{j=1}^{m} \langle Ae_j, e_j \rangle \leq \sum_{j=1}^{m} \lambda_j$. If $\lambda_m > \lambda_{m+1} \geq 0$, then E is the unique projection at which the supremum is attained.*

PROOF. With F a projection, $\dim(F(\mathcal{H})) = \operatorname{tr}(F) = \operatorname{tr}(F^2)$. Hence the supremum in the statement of this corollary is taken over a smaller subset than the supremum in (**). Nevertheless, for the projection E of the statement, $\operatorname{tr}(EAE) = \sum_{j=1}^{m} \lambda_j$, the maximum in (**).

The last assertion of this corollary follows from the last assertion in the statement of Theorem 25. □

The Schur inequalities are described in the next corollary.

COROLLARY 27. *If A is an hermitian $n \times n$ matrix over $\mathbb{C}$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ and diagonal $p_1, \ldots, p_n$, both listed in decreasing order, then*

$$p_1 + \cdots + p_k \leq \lambda_1 + \cdots + \lambda_k \qquad k \in \{1, \ldots, n\}$$

and $p_1 + \cdots + p_n = \lambda_1 + \cdots + \lambda_n$.

PROOF. Both $p_1 + \cdots + p_n$ and $\lambda_1 + \cdots + \lambda_n$ are $\operatorname{tr}(A)$. If F is the projection with 1 at each of the first k diagonal entries and 0 at all other entries, then

$$p_1 + \cdots + p_k = \operatorname{tr}(FAF) \leq \operatorname{tr}(EAE) = \lambda_1 + \cdots + \lambda_k,$$

where E is the projection matrix whose range is spanned by the orthonormal set $e_1, \ldots, e_k$ such that $Ae_1 = \lambda_1 e_1, \ldots, Ae_k = \lambda_k e_k$. □

LEMMA 28. *Let (X, μ) be a measure space, X_0 a measurable subset of X with indicator function χ_0, f and g integrable functions such that $0 \leq g \leq 1$ a.e. and*

$\int g\,d\mu \le \mu(X_0)$, and λ a non-negative real number such that both $f\chi_0 \ge \lambda\chi_0$ and $f(1-\chi_0) \le \lambda(1-\chi_0)$ a.e. Then

$$\int fg\,d\mu \le \int f\chi_0\,d\mu.$$

If equality holds and $f\chi_0 > \lambda\chi_0$ a.e.on X_0, then $g = \chi_0$ a.e. This same inequality holds for all real λ if $\mu(X_0) = \int g\,d\mu$.

PROOF. By assumption, $\int \chi_0\,d\mu \ge \int \chi_0 g\,d\mu + \int(1-\chi_0)g\,d\mu$. Consequently $\int \chi_0(1-g)\,d\mu \ge \int(1-\chi_0)g\,d\mu$. It follows that

$$\begin{aligned}\int f\chi_0\,d\mu - \int fg\,d\mu &= \int f\chi_0\,d\mu - \int f\chi_0 g\,d\mu - \int f(1-\chi_0)g\,d\mu \\ &= \int f\chi_0(1-g)\,d\mu - \int f(1-\chi_0)g\,d\mu \\ &\ge \int \lambda\chi_0(1-g)\,d\mu - \int \lambda(1-\chi_0)g\,d\mu \ge 0.\end{aligned}$$

If equality holds and $f\chi_0 > \lambda\chi_0$ a.e. on X_0, then

$$\begin{aligned}0 = \int f\chi_0\,d\mu - \int fg\,d\mu &= \int f\chi_0(1-g)\,d\mu - \int f(1-\chi_0)g\,d\mu \\ &\ge \int f\chi_0(1-g)\,d\mu - \int \lambda(1-\chi_0)g\,d\mu \\ &\ge \int f\chi_0(1-g)\,d\mu - \int \lambda\chi_0(1-g)\,d\mu \\ &= \int (f-\lambda)\chi_0(1-g)\,d\mu \ge 0.\end{aligned}$$

(For the last inequality, recall that $\int(1-\chi_0)g\,d\mu \le \int \chi_0(1-g)\,d\mu$ and that $\lambda \ge 0$ by assumption.) Thus $\int(f-\lambda)\chi_0(1-g)\,d\mu = 0$. Since $(f-\lambda)\chi_0 > 0$ a.e. on X_0 and $0 \le g \le 1$ a.e., we conclude that $g = 1$ a.e. on X_0. Thus $\int g\chi_0\,d\mu = \mu(X_0)$. Since $\mu(X_0) \ge \int g\chi_0\,d\mu + \int g(1-\chi_0)\,d\mu$ by assumption, $\int g(1-\chi_0)\,d\mu = 0$. Now, $g \ge 0$ a.e., whence $g = 0$ a.e. on $X \setminus X_0$. Hence $g = \chi_0$ a.e.

If $\mu(X_0) = \int g\,d\mu$, then $\int \lambda\chi_0(1-g)\,d\mu = \int \lambda(1-\chi_0)g\,d\mu = 0$ for all λ. Again, $\int f\chi_0\,d\mu \ge \int fg\,d\mu$. □

We note that Lemma 28 is an extension of Lemma 24. For this, choose X to be $\mathbb{N}$, the natural numbers, μ the measure on $\mathbb{N}$ that counts the number of elements in the intersection of a set with $\{1,\dots,n\}$, g the function that assigns to j (in $\mathbb{N}$) the real number a_j (in [0,1]), X_0 the set $\{1,\dots,m\}$, and f the function that assigns λ_j to j. With these choices, $\int fg\,d\mu$ is $\sum_{j=1}^{n} \lambda_j a_j$ and $\int f\chi_0\,d\mu$ is $\sum_{j=1}^{m} \lambda_j$.

Lemma 28 allows us to prove the appropriate version of the Schur Inequalities for factors of type II_1.

THEOREM 29. *Let $\mathcal{M}$ be a factor of type II_1, τ the (unique) normalized tracial state on $\mathcal{M}$, $\mathcal{A}$ a maximal abelian self-adjoint subalgebra (masa) in $\mathcal{M}$, A a self-adjoint operator in $\mathcal{A}$, and a a number in $[0,1]$. There is a projection E in $\mathcal{A}$ and a real number λ such that $\tau(E) = a$, $AE \ge \lambda E$, and $A(I-E) \le \lambda(I-E)$. If $B \in \mathcal{M}$, $0 \le B \le I$, and $\tau(B) = a$, then $\tau(AB) \le \tau(AE)$. If $\tau(B) \le a$ and $\lambda \ge 0$, then again, $\tau(AB) \le \tau(AE)$. If $\tau(AB) = \tau(AE)$ and $AE \ge \lambda' E$, where $\lambda' > \lambda \ge 0$, then $B = E$.*

PROOF. We note that $\tau(AB)$ is real since A and B are self-adjoint. To see this, we may assume that $B \geq 0$ (otherwise, replace B by $B + \|B\|I$, and observe that $\|B\|\tau(A)$ is real). Since τ is a tracial state, $\tau(AB) = \tau(B^{\frac{1}{2}}AB^{\frac{1}{2}})$, which is real as $B^{\frac{1}{2}}AB^{\frac{1}{2}}$ is self-adjoint.

Assume that we have found E and λ as described. We give two proofs of the stated inequality. Our first proof is a modification of the proof of Lemma 28 in the von Neumann algebra setting that uses the properties of the trace to bypass the commutativity missing in $\mathcal{M}$ and present in the measure-theoretic setting of Lemma 28. The second proof uses the conditional expectation of $\mathcal{M}$ onto $\mathcal{A}$ to reduce the problem to the commutative case and then, uses von Neumann's results to identify $\mathcal{A}$ with the algebra of bounded measurable functions on the appropriate measure space. Lemma 28 then applies as it is stated.

For our first proof, we have that $a = \tau(E) = \tau(B) = \tau(EB) + \tau((I-E)B)$ by assumption. Thus $\tau(E(I-B)) = \tau((I-E)B)$. It follows that

$$\begin{aligned} \tau(AE) - \tau(AB) &= \tau(AE) - \tau(AEB) - \tau(A(I-E)B) \\ &= \tau((I-B)^{\frac{1}{2}}AE(I-B)^{\frac{1}{2}}) - \tau(B^{\frac{1}{2}}A(I-E)B^{\frac{1}{2}}) \\ &\geq \tau(\lambda(I-B)^{\frac{1}{2}}E(I-B)^{\frac{1}{2}}) - \tau(\lambda B^{\frac{1}{2}}(I-E)B^{\frac{1}{2}}) \\ &= \lambda\tau(E(I-B)) - \lambda\tau((I-E)B) = 0. \end{aligned}$$

If we assume only that $\tau(B) \leq a$, we must require that $\lambda \geq 0$ in order to conclude that $\lambda(\tau(E(I-B)) - \tau((I-E)B)) \geq 0$.

Since $\tau(E) = a \geq \tau(B) = \tau(BE) + \tau(B(I-E))$, we have that $\tau(E(I-B)) \geq \tau(B(I-E))$. Suppose $\tau(AB) = \tau(AE)$ and $AE \geq \lambda' E$ where $\lambda' > \lambda \geq 0$. Then $\tau(AE) = \tau(AB) = \tau(AEB) + \tau(A(I-E)B)$, whence $\tau(AE(I-B)) = \tau(A(I-E)B)$. Thus

$$\begin{aligned} 0 &= \tau(AE(I-B)) - \tau(A(I-E)B) \\ &= \tau((I-B)^{\frac{1}{2}}AE(I-B)^{\frac{1}{2}}) - \tau(B^{\frac{1}{2}}A(I-E)B^{\frac{1}{2}}) \\ &\geq \tau((I-B)^{\frac{1}{2}}AE(I-B)^{\frac{1}{2}}) - \tau(\lambda B^{\frac{1}{2}}(I-E)B^{\frac{1}{2}}) \\ &\geq \tau((I-B)^{\frac{1}{2}}AE(I-B)^{\frac{1}{2}}) - \tau(\lambda(I-B)^{\frac{1}{2}}E(I-B)^{\frac{1}{2}}) \\ &= \tau((I-B)^{\frac{1}{2}}(AE-\lambda E)(I-B)^{\frac{1}{2}}) \geq 0. \end{aligned}$$

Hence $\tau((I-B)^{\frac{1}{2}}(A-\lambda I)E(I-B)^{\frac{1}{2}}) = 0$, and $(I-B)^{\frac{1}{2}}(A-\lambda I)E(I-B)^{\frac{1}{2}} = 0$. It follows that

$$[(A-\lambda I)E]^{\frac{1}{2}}(I-B)^{\frac{1}{2}} = [(A-\lambda I)E]^{\frac{1}{2}}E(I-B)^{\frac{1}{2}} = 0.$$

Now, $(A-\lambda I)E \geq (\lambda'-\lambda)E$ and $\lambda'-\lambda > 0$. Thus

$$0 = (I-B)^{\frac{1}{2}}(A-\lambda I)E(I-B)^{\frac{1}{2}} \geq (I-B)^{\frac{1}{2}}(\lambda'-\lambda I)E(I-B)^{\frac{1}{2}} \geq 0,$$

and $E(I-B)^{\frac{1}{2}} = 0$. Hence $E(I-B) = 0$, and $a = \tau(E) = \tau(B^{\frac{1}{2}}EB^{\frac{1}{2}}) \leq \tau(B) \leq a$. Therefore, $\tau(B) = \tau(E) = \tau(EB)$, and $0 = \tau((I-E)B) = \tau((I-E)B(I-E))$. As $B \geq 0$, $(I-E)B(I-E) \geq 0$, whence $(I-E)B^{\frac{1}{2}} = 0$. Of course, $(I-E)B = 0$, from which $B = EB = E$.

For our second proof, we use the (unique) trace-lifting conditional expectation Φ of $\mathcal{M}$ onto $\mathcal{A}$. (The mapping Φ is an idempotent, positive, linear, $\mathcal{A}$-bimodule mapping of $\mathcal{M}$ onto $\mathcal{A}$, and $\tau(\Phi(T)) = \tau(T)$ for each T in $\mathcal{M}$. See Exercises 8.7.23–8.7.30, 10.5.85–10.5.87.) To show that $\tau(AB) \leq \tau(AE)$ under the given conditions on A, B, and E, we note that

$$\tau(A\Phi(B)) = \tau(\Phi(AB)) = \tau(AB).$$

Thus it suffices to show that $\tau(A\Phi(B)) \le \tau(AE)$. Now, A, E, and $\Phi(B)$, are in $\mathcal{A}$. Moreover, $0 \le \Phi(B) \le I$ since $0 \le B \le I$ and Φ is a positive, linear, idempotent mapping with I in its range (so that $\Phi(I) = \Phi(\Phi(T)) = \Phi(T) = I$). In addition, $\tau(\Phi(B)) = \tau(B) \le \tau(E)$. Thus A, E, and $\Phi(B)$, satisfy the various conditions we assumed for A, E, and B. To complete this second proof, we shall see that $\mathcal{A}$ with the restriction of τ to it is (equivalent to) the measure-theoretic situation of Lemma 28. After seeing that, we conclude that Lemma 28 applies to yield our inequality.

By applying the GNS construction to τ, we may assume that $\mathcal{M}$ acts on the Hilbert space $\mathcal{H}$ and u is a separating and generating unit vector for $\mathcal{M}$ such that $\tau(T) = \langle Tu, u\rangle$ for each T in $\mathcal{M}$. (We call u a *trace vector* for M. See the discussion preceding and the proof of Proposition 12.1.4.) Let G be the projection with range $[\mathcal{A}u]$ (the closure of $\{Tu : T \in \mathcal{A}\}$). Since u is a separating vector for $\mathcal{M}$ (and hence, for $\mathcal{A}$), the mapping $T \to TG$ of $\mathcal{A}$ into $\mathcal{B}(G(\mathcal{H}))$, is a normal *isomorphism of $\mathcal{A}$ onto its range $\mathcal{A}_0$. Let $\mathcal{H}_0$ be the Hilbert space $G(\mathcal{H})$. As $I \in \mathcal{A}$, $u \in G(\mathcal{H})$. By construction u is generating for the abelian von Neumann algebra $\mathcal{A}_0$. From Corollary 7.2.16, $\mathcal{A}_0$ is a masa in $\mathcal{B}(\mathcal{H}_0)$. Since $\mathcal{A}_0$ is a masa in the II_1 factor $\mathcal{M}$, it follows from Exercise 6.9.17 that $\mathcal{A}_0$ has no minimal projections. From Theorem 9.4.1, there is a unitary transformation U of $\mathcal{H}_0$ onto $L_2([0,1],\mu)$, where μ is Lebesgue measure, such that Uu is the constant function 1 on $[0,1]$, and $U\mathcal{A}_0U^{-1}$ is the algebra of all bounded measurable functions on $[0,1]$. Note that, with T in $\mathcal{M}$,

$$\tau(T) = \langle Tu, u\rangle = \langle UTu, Uu\rangle = \langle UTU^{-1}Uu, Uu\rangle = \int UTU^{-1}\, d\mu.$$

Lemma 28 now applies to yield the desired inequality and the uniqueness resulting when equality holds.

It remains to establish the existence of the projection E and the real number λ with the properties posited in the statement of this theorem. Since A and E are to be in $\mathcal{A}$ and τ is a normal, faithful state of $\mathcal{A}$, we may carry out the construction of E and λ in the von Neumann algebra – Hilbert space framework or in the measure algebra – measure space framework. Let $\{E_\lambda\}$ be the spectral resolution of A. (We follow the construction of $\{E_\lambda\}$ described in Theorem 5.2.2 and make use of the properties proved there for $\{E_\lambda\}$. Let a' be $1-a$, F be $\vee\{E_\lambda : \tau(E_\lambda) \le a'\}$, λ_0 be $\sup\{\lambda : \tau(E_\lambda) \le a'\}$, G be $\wedge\{E_\lambda : \tau(E_\lambda) \ge a'\}$, and λ_1 be $\inf\{\lambda : \tau(E_\lambda) \ge a'\}$. Since $E_\lambda \le E_{\lambda'}$ when $\lambda \le \lambda'$ and τ is normal, $E_\lambda \to F$, and $\tau(E_\lambda) \to \tau(F)$ as $\lambda \uparrow \lambda_0$. Similarly, $E_\lambda \to G$ and $\tau(E_\lambda) \to \tau(G)$ as $\lambda \downarrow \lambda_1$. Hence $\tau(F) \le a' \le \tau(G)$ and $\lambda_0 \le \lambda_1$. If $\lambda_0 < \lambda_1$ and we choose λ' in (λ_0, λ_1), then either $\tau(E_{\lambda'}) \ge a'$, contradicting the choice of λ_1, or $\tau(E_{\lambda'}) \le a'$, contradicting the choice of λ_0. Thus $\lambda_0 = \lambda_1$ $(= \lambda)$. From Theorem 5.2.2(iii), $E_{\lambda_1} = E_\lambda = G$, and $F \le E_{\lambda_0} = E_{\lambda_1} = E_\lambda = G$. It is possible that $F < E_{\lambda_0} = G$. However, if $F = G$ $(= I - E)$, then $\tau(I-E) = a'$, $I - E = E_\lambda$, $A(I-E) = AE_\lambda \le \lambda E_\lambda = \lambda(I-E)$, $AE = A(I-E_\lambda) \ge \lambda(I-E_\lambda) = \lambda E$ from Theorem 5.22(iv), and $\tau(E) = a$.

Suppose, now, that $F < G$ and that N is a projection in $\mathcal{A}$ such that $F \le N \le G$. Since $AG = AE_\lambda \le \lambda E_\lambda = \lambda G$ and N is a positive operator commuting with A and G, $AN = AGN \le \lambda GN = \lambda N$. We note, next, that $A(G-F) = \lambda(G-F)$, whence $A(G-N) = \lambda(G-N)$ and $A(N-F) = \lambda(N-F)$. Since $AG \le \lambda G$, we see that $A(G-F) = AG(G-F) \le \lambda G(G-F) = \lambda(G-F)$. We show that $A(G-F) \ge \lambda(G-F)$. By construction, there are spectral projections $E_{\lambda'}$, with $\lambda' < \lambda$ and $\lambda - \lambda'$ as small as we wish, that are subprojections of F. Now, $G - F \le I - E_{\lambda'}$ for all such λ', whence for each such λ',

$$A(G-F) = A(I-E_{\lambda'})(G-F) \ge \lambda'(I-E_{\lambda'})(G-F) = \lambda'(G-F).$$

As $\lambda' \uparrow \lambda$, we conclude that $A(G-F) \ge \lambda(G-F)$ and $A(G-F) = \lambda(G-F)$. Hence

$$\begin{aligned} A(I-N) = A(I-G) + A(G-N) &= A(I-E_\lambda) + \lambda(G-N) \\ &\ge \lambda(I-G) + \lambda(G-N) = \lambda(I-N). \end{aligned}$$

From the proposition that follows, we have that there is a choice of N as above for which $\tau(N) = a'$. (Recall that $\tau(F) \le a' \le \tau(G)$ and $\mathcal{M}$ is a factor of type II_1.) Letting E be $I - N$, we have found a projection E and a real λ with the desired properties. □

PROPOSITION 30. *Let $\mathcal{M}$ be a factor of type II_1, $\mathcal{A}$ a masa in $\mathcal{M}$, G a projection in $\mathcal{A}$ such that $\tau(G) = s$, where τ is the unique tracial state on $\mathcal{M}$, then for t in $[0, s]$ there is a subprojection F of G such that $F \in \mathcal{A}$ and $\tau(F) = t$.*

PROOF. Represent t in dyadic form as $(.a_1a_2\ldots)s$, where each a_j is either 0 or 1. From Exercise 6.9.15, $G\mathcal{M}G$ is a factor of type II_1. Since $G \in \mathcal{A}$, $G\mathcal{A}G$ is an abelian von Neumann subalgebra or $\mathcal{GMG}$. If $T \in G\mathcal{M}\,G$ and T commutes with $G\mathcal{A}G$, then T commutes with each A $(= AG + A(I-G)$ in $\mathcal{A}$, from which, $T \in \mathcal{A}$. As $T = GTG$, we have that $T \in G\mathcal{A}G$, whence $G\mathcal{A}G$ is maximal abelian in $G\mathcal{M}G$. From Exercise 6.9.29, we have that G is the sum of n orthogonal equivalent projections in $G\mathcal{A}G$ $(\subseteq \mathcal{A})$. Using this observation, we can find an orthogonal family $\{G_1, G_2, \ldots\}$ of subprojections of G in $\mathcal{A}$ such that $\tau(G_j) = s2^{-j}$, for j in $\{1, 2, \ldots\}$, by "bisecting" G, then bisecting one of the resulting subprojections, and so forth. Now, follow the "instructions" coded in the sequence $a_1, a_2, \ldots$; let F_1 be $G_{j(1)}$, where $j(1)$ is the first a_j that is 1, F_2 be $G_{j(2)}$, where $j(2)$ is the next a_j after $a_{j(1)}$ that is 1, and so forth. Let F be $\sum_{k=1}^\infty F_k$. Since τ is normal,

$$\tau(F) = \tau\left(\sum_{k=1}^\infty F_k\right) = \sum_{k=1}^\infty \tau(F_k) = (.a_1a_2\ldots)s = t.$$

□

Without the condition $\lambda' > \lambda$, one cannot assert that $B = E$ (that is, "uniqueness of the maximum") even in the commutative case, because the projection E with the given properties is not, itself, unique. For example, the 4×4 diagonal matrix A with diagonal 3,2,1,1 has the two projections E_1 and E_2 of trace 3 that are diagonal matrices with diagonals 1,1,1,0 and 1,1,0,1 as maxima of $\{\mathrm{tr}(AB) : 0 \le B \le I, \mathrm{tr}\, B = 3\}$. This same example is easily transferred to a factor of type II_1 by working in a type I_4 subfactor. The fact that "higher multiplicity" is the basis of this example suggests that, with the condition of simple multiplicity imposed, the unique maximum of $\mathrm{tr}(AB)$, when $0 \le B \le I$, $\mathrm{tr}\, B \le 0$, and $\lambda \ge 0$, is a projection E in $\mathcal{A}$ such that $AE \ge \lambda E$ and $A(I-E) \le \lambda(I-E)$. Here, "simple multiplicity"is relative to the factor $\mathcal{M}$ of type II_1, that is, A generates a masa A in $\mathcal{M}$ (as a von Neumann subalgebra). The argument of Theorem 29, when we assume that $\tau(AB) = \tau(AE)$ and $\lambda \ge 0$, allows us to conclude that

$[(A-\lambda I)E]^{\frac{1}{2}}E(I-B)^{\frac{1}{2}} = 0$ without the introduction of a λ' such that $AE \geq \lambda' E$ and $\lambda' > \lambda$. If $[(A-\lambda I)E]^{\frac{1}{2}}$ has $(I-E)(\mathcal{H})$ as its null space (that is, has null space (0) in $E(\mathcal{H})$), then $E(I-B)^{\frac{1}{2}} = 0$ and the rest of that argument applies to yield that $B = E$. With the "simple spectrum" hypothesis in force, we shall show that the null space of $[(A-\lambda I)E]^{\frac{1}{2}}$ is, indeed, $(I-E)(\mathcal{H})$. If G is the projection on this null space, then $G \in \mathcal{A}$ and $(F =)\ G-(I-E) \in \mathcal{A}$. Now, $F \leq E$ and $[(A-\lambda I)E]^{\frac{1}{2}}F = 0$. Hence $(A-\lambda I)F = [(A-\lambda I)E]^{\frac{1}{2}}[(A-\lambda I)E]^{\frac{1}{2}}F = 0$. Thus $AF = \lambda F = FA$. If $T \in \mathcal{M}$ and $FTF = T$, then $TA = FTFA = FT\lambda F = \lambda FTF = AT$. Since A is assumed to generate the masa $\mathcal{A}$, we see that $T \in \mathcal{A}$. It follows that $F\mathcal{M}F \subseteq \mathcal{A}$. If $F \neq 0$, then $F\mathcal{M}F$ is a factor of type II_1 on $F(\mathcal{H})$. But $\mathcal{A}$ is abelian. Thus $F = 0$. We have proved the following theorem.

THEOREM 31. *If $\mathcal{M}$ is a factor of type II_1, A is a self-adjoint operator in $\mathcal{M}$ that generates a masa $\mathcal{A}$ in $\mathcal{M}$, E is a projection in $\mathcal{A}$ such that $AE \geq \lambda E$ and $A(I-E) \leq \lambda(I-E)$ for some non-negative, real λ, then $\tau(AB) \leq \tau(AE)$, when $0 \leq B \leq I$ and $\tau(B) \leq \tau(E)$, and $\tau(AB) = \tau(AE)$ if and only if $B = E$.*

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