The Pythagorean Theorem: II. The infinite discrete case

Richard V. Kadison†

Mathematics Department, University of Pennsylvania, Philadelphia, PA 19104-6395

Contributed by Richard V. Kadison, December 17, 2001

The study of the Pythagorean Theorem and variants of it as the basic result of noncommutative, metric, Euclidean Geometry is continued. The emphasis in the present article is the case of infinite discrete dimensionality.

1. Introduction

We continue our study of the Pythagorean Theorem begun in ref. 1. The numbering of results and remarks in ref. 1 will be used in this article in two ways: A reference to *Proposition 3* is a reference to that proposition in ref. 1, and the results in this article will be numbered from 13 on (following *Proposition 12*, the last result in ref. 1).

Our focus in this article is the case of infinite-dimensional Hilbert space and an infinite-dimensional subspace, although we study first the case of a finite-dimensional subspace of infinite-dimensional space. The novelty in that case is that we are projecting the vectors of an *infinite* orthonormal basis onto the finite-dimensional subspace. *Propositions 1* and *2* apply, as they stand, to prove those variants of the Pythagorean Theorem in that situation. The Carpenter's Theorem for that case is quite another matter. Several "proofs" of it were developed, the first of which was invalid: as argued, the "proof" yielded results that did not respect necessary conditions (discovered later). There is a small warning here: the intricacies of the arguments in the case of an infinite-dimensional subspace with infinite-dimensional orthogonal complement are needed to cause certain infinitely repeated processes to produce convergent sums. That convergence is far less automatic than might sometimes seem natural. The shortest of our arguments in the case of the finite-dimensional subspace was *not* short. Junhao Shen suggested a change in strategy that produces a shortened version, which appears as the proof of *Theorem 13.* I am happy to express my gratitude for his suggestion. The results involving finite-dimensional subspaces appear in the next section.

In the third section, we study the case of an infinite-dimensional subspace with infinite-dimensional complement. Although the precise formula of *Proposition 3* does not apply in that case, the "integrality condition" implicit in that assertion (and mentioned) plays a crucial role in these results.

In the last section, we resume the examination of the relation between the Pythagorean Theorem and doubly stochastic (now, infinite) matrices. We produce such matrices with a block and its complement of finite weight (as promised). Although the formula of *Proposition 12* is not applicable here, the integrality condition is and is proved in the concluding section.

MATHEMATICS

2. Finite and Cofinite-Dimensional Subspaces

We prove versions of the Carpenter's Theorem for subspaces of infinite-dimensional Hilbert space $\mathcal{H}$ that are finite or cofinite-dimensional in $\mathcal{H}$. From *Proposition 2*, if we specify numbers in $[0, 1]$, they must have sum m if they are to be the squares of the lengths of the projections of the elements in an orthonormal basis onto some m-dimensional subspace of $\mathcal{H}$. Subject to this condition, is there such a subspace? There is, as we shall show. The proof uses the finite case (*Theorem 6*) and some additional constructions.

Theorem 13. *If $\{e_b\}_{b\in\mathbb{B}}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$ and numbers t_b in $[0, 1]$ are specified, there is an m-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\|Fe_b\|^2 = t_b$ for each b in $\mathbb{B}$, where F is the projection with range $\mathcal{V}$, if and only if $\Sigma_{b\in\mathbb{B}}t_b = m$.*

Proof: Let $\mathbb{B}_0$ be $\{b \in \mathbb{B} : t_b = 0\}$ and $\mathcal{H}_0$ the closed linear span of $\{e_b\}_{b\in\mathbb{B}_0}$. Since $\Sigma_{b\in\mathbb{B}\setminus\mathbb{B}_0}t_b = m$ and $t_b > 0$ when $b \notin \mathbb{B}_0$, $\mathbb{B}\setminus\mathbb{B}_0$ is a countable set. Note that $m = \Sigma_{b\in\mathbb{B}\setminus\mathbb{B}_0}t_b \le \Sigma_{b\in\mathbb{B}\setminus\mathbb{B}_0} 1 = \dim(\mathcal{H}\ominus\mathcal{H}_0)$. If we find $\mathcal{V}$ in $\mathcal{H}\ominus\mathcal{H}_0$ such that $\|Fe_b\|^2 = t_b$ for each b in $\mathbb{B}\setminus\mathbb{B}_0$, we are done, because $Fe_b = 0$ when $b \in \mathbb{B}_0$. Restricting to $\mathcal{H}\ominus\mathcal{H}_0$, we may assume that $\mathcal{H}$ is separable and that each $t_b > 0$. We have dealt with the case where $\mathcal{H}$ has finite dimension. Henceforth, we assume that $\mathcal{H}$ has dimension $\aleph_0$, that our given orthonormal basis is $e_1, e_2, \dots,$ and that the specified numbers are $a_1, a_2, \dots.$

Some further reductions are useful. By restricting our attention to the orthogonal complement of the subspace of $\mathcal{H}$ spanned by the basis elements e_j for which a_j is 0 or 1 and constructing a projection with matrix having diagonal the remaining a_j with respect to the remaining e_j, we may assume that each $a_j \in (0, 1)$. Since $\Sigma_{j=1}^{\infty}a_j = m$, there are at most a finite number of a_j greater than a given number, and these can be written in decreasing order. Thus, for some permutation π of $\mathbb{N}$, $a_{\pi(1)} \ge a_{\pi(2)} \ge \cdots$. Suppose E_π is a projection with matrix relative to $e_1, e_2, \dots$ having diagonal $a_{\pi(1)}, a_{\pi(2)}, \dots.$ Then $U_\pi^* E_\pi U_\pi$ has matrix with diagonal $a_1, a_2, \dots,$ where U_π is the permutation unitary that maps e_j to $e_{\pi^{-1}(j)}$ for each j in $\mathbb{N}$. It suffices, in general, to construct our matrix with the specified diagonal in any order.

We may assume that $m \ge 2$ (from *Proposition 1*) and that $a_1 \ge a_2 \ge \cdots$ with each a_j in $(0, 1)$. Now, $\Sigma_{j=2}^{\infty}a_j = m - a_1 > m - 1$. Thus there is an s such that $\Sigma_{j=2}^{s}a_j < m - 1$ and $\Sigma_{j=2}^{s+1}a_j \ge m - 1$. If $t = \Sigma_{j=2}^{s+1}a_j - (m - 1)$, then $t \le a_{s+1}$. From *Theorem 6*, there is a projection E of rank $m - 1$ with matrix relative to the basis $e_2, e_3, \dots, e_{s+1}$ having $a_2, a_3, \dots, a_{s+1} - t$ as its diagonal. Moreover, $\Sigma_{j=s+2}^{\infty}a_j + a_1 + t = 1$. Thus, from *Proposition 1*, there is a projection G of rank 1 with matrix relative to the basis $e_1, e_{s+2}, e_{s+3}, \dots$ whose diagonal is $a_1 + t, a_{s+2}, a_{s+3}, \dots.$ Noting that $a_{s+1} - t \le a_{s+1} \le a_1 \le a_1 + t$, we can splice the two projections E and G together and then "permute" the splice, to form a projection F whose matrix relative to the basis $e_1, e_2, \dots$ has diagonal $a_1, a_2, \dots.$ ■

†E-mail: kadison@math.upenn.edu.

Following this fifteenth variation, the situation in which the projections of the elements of our orthonormal basis $\{e_n\}$ on the subspace $\mathcal{V}$ of $\mathcal{H}$ have lengths whose squares sum to ∞ remains to be studied. In this case, from *Proposition 2*, $\mathcal{V}$ must be infinite dimensional. Once again, the question of whether the squares of the lengths can be assigned in $[0, 1]$ arbitrarily, subject only to the condition that their sum diverges is the more involved aspect of this case. Our experience, to this point, makes it tempting to believe that the question has an affirmative answer. Some further consideration makes it clear that there is more to be said.

Again, the question can be formulated in terms of projections, their matrices relative to $\{e_n\}$, and diagonals of those matrices. If $\mathcal{V}$ is the (infinite-dimensional) subspace for the specified lengths $a_1, a_2, \ldots$ and E is the projection with $\mathcal{V}$ as range, then $I - E$ has $\mathcal{H}\ominus\mathcal{V}$, the orthogonal complement of $\mathcal{V}$, as its range. The matrix for the projection $I - E$ has $1 - a_1, 1 - a_2, \ldots$ as its diagonal. If $\Sigma_j 1 - a_j$ converges, it must converge to an integer, since $I - E$ is a projection. If we choose the a_j such that $\Sigma_j 1 - a_j$ converges, but not to an integer (e.g., let a_j be $1 - (1/j^2)$), then $a_j \to 1$, whence $\Sigma a_j = \infty$, and there is no projection with diagonal $a_1, a_2, \ldots$ relative to $\{e_j\}$. (If E were such a projection, $I - E$ would have diagonal $1 - a_1, 1 - a_2, \ldots$, with finite sum other than an integer, contradicting *Proposition 2*). However, if $\Sigma_j 1 - a_j = m$, with m an integer, then there is a projection $I - E$ with diagonal $1 - a_1, 1 - a_2, \ldots$, and hence a projection E with diagonal $a_1, a_2, \ldots$. This discussion provides our sixteenth variation.

Theorem 14. *If $\{e_a\}_{a\in\mathbb{A}}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$ and $\{t_a\}_{a\in\mathbb{A}}$ is a family of numbers in* $[0, 1]$, *there is an infinite-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ with m-dimensional orthogonal complement such that $\|Fe_a\|^2 = t_a$ for each a in $\mathbb{A}$, where F is the projection with range $\mathcal{V}$, if and only if $\Sigma_{a\in\mathbb{A}} 1 - t_a = m$.*

3. Infinite-Dimensional Subspaces with Infinite-Dimensional Complement

In the context of orthonormal bases for Hilbert space, there remains the case where $\Sigma_{a\in\mathbb{A}} t_a$ and $\Sigma_{a\in\mathbb{A}} 1 - t_a$ diverge. This case leads to our seventeenth variation. As we shall see in the course of the proof, and as noted in the statement, there is more to the story than the divergence of the two sums noted. To recognize this in advance, we need only consider the case where the assigned diagonal entries consist of an infinite number of 0s and terms $a_1, a_2, \ldots$ in $[\frac{1}{2}, 1]$ such that $\Sigma 1 - a_j$ converges to a number a not an integer. If E is a projection whose matrix has that diagonal, then the restriction of E to the space generated by the basis elements corresponding to all the a_j is also a projection of the sub-Hilbert space onto some subspace of it. Relative to those basis elements corresponding to the a_j, that projection has a matrix whose diagonal has entries $a_1, a_2, \ldots$. But $\Sigma 1 - a_j$ is a number other than an integer, by assumption. We have seen that such a diagonal is not a possibility for a projection. This restriction appears in more complex form as well. If we replace a finite number of the 0s by numbers $r_1, \ldots, r_k$ in $(0, \frac{1}{2})$ such that $(r_1 + \cdots + r_k) - \Sigma_{j=1}^{\infty}(1 - a_j)$ is not an integer, then the resulting assignment of numbers is not the diagonal of a projection. This reduces to the case just considered by noting that, with the present assumption, $\Sigma_{j=1}^{\infty}(1 - a_j) + (1 - r_1) + \cdots + (1 - r_k)$ is not an integer. Of course, the restriction on the diagonal in this case is foreshadowed by *Proposition 3* (and the remarks following it).

To simplify the discussion and focus on essentials, we deal with the case where $\mathcal{H}$ has dimension $\aleph_0$.

Theorem 15. *Let* $\{e_1, e_2, \ldots\}$ *be an orthonormal basis for the Hilbert space $\mathcal{H}$ and numbers $a_1, a_2, \ldots$ in* $[0, 1]$ *be specified. Let $a'_1, a'_2, \ldots$ be the a_j in* $(\frac{1}{2}, 1]$, *$a''_1, a''_2, \ldots$ those in* $[0, \frac{1}{2}]$, *a the sum of the a''_j, and b the sum $\Sigma_{j=1}^{\infty} 1 - a'_j$. There is an infinite-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ with infinite-dimensional complement such that $\|Fe_j\|^2 = a_j$ for each j, where F is the projection with range $\mathcal{V}$, if and only if $\Sigma_{j=1}^{\infty} a_j$ and $\Sigma_{j=1}^{\infty} 1 - a_j$ diverge and either of a or b is infinite or both are finite and $a - b$ is an integer.*

Proof: As in the proof of *Theorem 13*, by restricting to the orthogonal complement of the subspace of $\mathcal{H}$ generated by the basis elements e_j for which a_j is either 0 or 1, we may assume that each $a_j \in (0, 1)$. Pursuing this same idea, we note that if $\{\mathbb{N}_j\}_{j=1,2,\ldots}$ is a set of mutually disjoint subsets of $\mathbb{N}$ with union $\mathbb{N}$ such that we can find a projection E_j with range contained in the closed subspace generated by all e_n for which $n \in \mathbb{N}_j$, with $\|E_j e_n\|^2 = a_n$, for each such e_n, and $E_j e_m = 0$ for each other e_m, then $\Sigma_{j=1}^{\infty} E_j$ $(= F)$ is a projection such that $\|Fe_j\|^2 = a_j$ for each j.

Let $\mathbb{N}'$ be the set of a_j-indices of a'_j and $\mathbb{N}''$ those of a''_j. We suppose, first, that $\Sigma_{j=1}^{\infty} a''_j = \infty$. Let $n(1)$ be the least integer n such that $a'_1 + a''_1 + \cdots + a''_n \ge 3$. Since $a'_1 < 1$ and each $a''_j \le \frac{1}{2}$, $n(1) \ge 5$. Let $b_2, \ldots, b_{n(1)}$ be $a''_2, \ldots, a''_{n(1)}$ rearranged in decreasing order (so, $b_2 \ge b_3 \ge \cdots \ge b_{n(1)}$). Let $m(1)$ be the least integer m such that $a'_1 + b_1 + b_2 + \cdots + b_m \ge 3$, where $b_1 = a''_1$. Then $5 \le m(1) \le n(1)$ and $a'_1 + \Sigma_{j=1}^{m(1)-1} b_j < 3$. Let $\tilde{a}$ be $3 - a'_1 - \Sigma_{j=1}^{m(1)-1} b_j$, $b'_{m(1)-1}$ be $b_{m(1)-1} + \tilde{a}$, and $b'_{m(1)}$ be $b_{m(1)} - \tilde{a}$. Then $a'_1 + \Sigma_{j=1}^{m(1)-2} b_j + b'_{m(1)-1} = 3$, and

$$(*) \qquad 0 \le b'_{m(1)} < b_{m(1)} \le b_{m(1)-1} < b'_{m(1)-1} \le 1.$$

[For the first inequality of $(*)$, note that $a'_1 + \Sigma_{j=1}^{m(1)} b_j \ge 3$, whence $b_{m(1)} \ge 3 - a'_1 - \Sigma_{j=1}^{m(1)-1} b_j = \tilde{a}$; for the last inequality, note that $b_{m(1)}$ is some a''_j $(\le \frac{1}{2})$ as is $b_{m(1)-1}$. Thus $b'_{m(1)-1} = b_{m(1)-1} + \tilde{a} \le b_{m(1)-1} + b_{m(1)} \le 1$.]

Let $\mathbb{N}_1$ be the set of indices of the a_j in $\{a'_1, a''_1, \ldots, a''_{n(1)}\}$ corresponding to $a'_1, b_1, \ldots, b_{m(1)-1}$. [Recall that $b_1, \ldots, b_{n(1)}$ is a rearrangement of $a''_1, \ldots, a''_{n(1)}$.] Let $j(1), j(2), \ldots$ be the numbers in $\mathbb{N}''\backslash\mathbb{N}_1$ in ascending order, except that $j(1)$ is the index of the a_j in $\{a''_1, \ldots, a''_{n(1)}\}$ that $b_{m(1)}$ represents, and $j(2)$ is the index of the a_j that a'_2 represents. Let $n(2)$ be the least integer n such that $a'_2 + b'_{m(1)} + \Sigma_{k=3}^{n} a_{j(k)} \ge 3$. Let c_1 be $b'_{m(1)}$, c_2 be a'_2, and $c_3, \ldots, c_{n(2)}$ be $a_{j(3)}, \ldots, a_{j(n(2))}$ rearranged in decreasing order except that c_3 is $a_{j(3)}$. [Note that the smallest number in $\mathbb{N}''\backslash\mathbb{N}_1$ is one of $j(1)$ or $j(3)$.] Let $m(2)$ be the least integer m such that $\Sigma_{j=1}^{m} c_j \ge 3$. Then $\Sigma_{j=1}^{m(2)-1} c_j < 3$, $m(2) \le n(2)$, and $m(2) \ge 6$. Let $\tilde{b}$ be $3 - \Sigma_{j=1}^{m(2)-1} c_j$, $c'_{m(2)-1}$ be $c_{m(2)-1} + \tilde{b}$, and $c'_{m(2)}$ be $c_{m(2)} - \tilde{b}$. Note that $0 < \tilde{b} \le c_{m(2)} \le \frac{1}{2}$, whence

$$0 \le c'_{m(2)} < c_{m(2)} \le c_{m(2)-1} < c'_{m(2)-1} \le 1.$$

We repeat this process, letting $\mathbb{N}_2$ be $j(1), j(2)$ and the set of indices of the a_j in $\{a_{j(3)}, \ldots, a_{j(n(2))}\}$ corresponding to $c_3, \ldots, c_{m(2)-1}$. Let $k(1), k(2), \ldots$ be the numbers in $\mathbb{N}''\backslash(\mathbb{N}_1 \cup \mathbb{N}_2)$ in ascending order, except that $k(1)$ is the index of the a_j in $\{a_{j(1)}, \ldots, a_{j(n(2))}\}$ that $c_{m(2)}$ represents, and $k(2)$ is the index of the a_j that a'_3 represents. Let $n(3)$ be the least integer n such that $a'_3 + c'_{m(2)} + \Sigma_{r=3}^{n} a_{k(r)} \ge 3$. Let d_1 be $c'_{m(2)}$, d_2 be a'_3, and $d_3, \ldots, d_{n(3)}$ be $a_{k(3)}, \ldots, a_{k(n(3))}$ rearranged in decreasing order except that d_3 is $a_{k(3)}$. Again, the smallest number in $\mathbb{N}''\backslash(\mathbb{N}_1 \cup \mathbb{N}_2)$ is one of $k(1)$ or $k(3)$. Let $m(3)$ be the least integer m such that $\Sigma_{j=1}^{m} d_j \geq 3$. Then $\Sigma_{j=1}^{m(3)-1} d_j < 3$, $m(3) \leq n(3)$, and $m(3) \geq 6$. Let $\tilde{c}$ be $3 - \Sigma_{j=1}^{m(3)-1} d_j$, $d'_{m(3)-1}$ be $d_{m(3)-1} + \tilde{c}$, and $d'_{m(3)}$ be $d_{m(3)} - \tilde{c}$. Note that $0 < \tilde{c} \leq d_{m(3)} \leq \frac{1}{2}$ so that

$$0 \leq d'_{m(3)} < d_{m(3)} \leq d_{m(3)-1} < d'_{m(3)-1} \leq 1.$$

Continuing in this way, we construct disjoint subsets $\mathbb{N}_1, \mathbb{N}_2, \ldots$ of $\mathbb{N}$ with union $\mathbb{N}$. In addition, if $r(1), \ldots, r(m(j)-1)$ are the elements of $\mathbb{N}_j$, with $r(3), \ldots, r(m(j)-1)$ in ascending order, there are alterations $\tilde{a}_{r(1)}$ and $\tilde{a}_{r(m(j)-1)}$ of $a_{r(1)}$ and $a_{r(m(j)-1)}$, as described, such that $\tilde{a}_{r(1)} + a'_j + \Sigma_{k=3}^{m(j)-2} a_{r(k)} + \tilde{a}_{r(m(j)-1)} = 3$. At the same time, with $s(1), \ldots, s(m(j+1)-1)$, the elements of $\mathbb{N}_{j+1}$ and $\tilde{a}_{s(1)}, a_{s(2)}, a_{s(3)}, \ldots, a_{s(m(j+1)-2)}, \tilde{a}_{s(m(j+1)-1)}$ summing to 3, with $\tilde{a}_{s(1)}$ and $\tilde{a}_{s(m(j+1)-1)}$ the elements $a_{s(1)}$ and $a_{s(m(j+1)-1)}$ altered as described, we have $\tilde{a}_{r(m(j)-1)} + \tilde{a}_{s(1)} = a_{r(m(j)-1)} + a_{s(1)}$ and

$$0 \leq \tilde{a}_{s(1)} < a_{s(1)} \leq a_{r(m(j)-1)} < \tilde{a}_{r(m(j)-1)} \leq 1.$$

We include the possibility that $\mathbb{N}'$ is finite (or null) in the foregoing argument. If $\mathbb{N}'$ is $\{a'_1, \ldots, a'_{j-1}\}$, we eliminate the references to "a'_k" in the construction of $\mathbb{N}_k$ when $k \geq j$.

From *Theorem 6*, there are three-dimensional projections E_j and E_{j+1} whose matrices relative to the bases $e_{r(1)}, \ldots, e_{r(m(j)-1)}$ and $e_{s(1)}, \ldots, e_{s(m(j+1)-1)}$ have diagonals $\tilde{a}_{r(1)}, a_{r(2)}, a_{r(3)}, \ldots, a_{r(m(j)-2)}, \tilde{a}_{r(m(j)-1)}$ and $\tilde{a}_{s(1)}, a_{s(2)}, a_{s(3)}, \ldots, a_{s(m(j+1)-2)}, \tilde{a}_{s(m(j+1)-1)}$, respectively. We extend each of the projections E_j to a projection (denoted, again, by "E_j") defined on all of $\mathcal{H}$ by letting it annihilate all other basis elements. Then, because the sets $\mathbb{N}_j$ are disjoint, $E_jE_k = 0$ when $j \neq k$. Let E be $\Sigma_{j=1}^{\infty} E_j$. The next part of this argument is devoted to describing the splicing used to transform E into the projection with the specified diagonal.

We transform E by means of a sequence of unitary operators $W_n(\theta_n)$ of the form appearing in the proof of *Theorem 7*. In the present case, with $h(1)$ the index of the a_j represented by $b_{m(1)-1}$ [recall that $j(1)$ is the index of the a_j represented by $b_{m(1)}$], $W_1(\theta_1)e_{h(1)} = \sin\theta_1 e_{h(1)} + \cos\theta_1 e_{j(1)}$, $W_1(\theta_1)e_{j(1)} = -\cos\theta_1 e_{h(1)} + \sin\theta_1 e_{j(1)}$, $W_1(\theta_1)e_j = e_j$ for all other e_j, whence

$$\langle W_1(\theta_1)EW_1(\theta_1)^* e_{h(1)}, e_{h(1)}\rangle = b'_{m(1)-1}\sin^2\theta_1 + b'_{m(1)}\cos^2\theta_1 = b_{m(1)-1} = a_{h(1)}$$

and

$$\langle W_1(\theta_1)EW_1(\theta_1)^* e_{j(1)}, e_{j(1)}\rangle = b'_{m(1)-1}\cos^2\theta_1 + b'_{m(1)}\sin^2\theta_1 = b_{m(1)} = a_{j(1)}.$$

Here, θ_1 is chosen [as it may be, by virtue of (*)] so that the convex combination $b'_{m(1)-1}\sin^2\theta_1 + b'_{m(1)}\cos^2\theta_1$ is $b_{m(1)-1}$, from which $b'_{m(1)-1}\cos^2\theta_1 + b'_{m(1)}\sin^2\theta_1 = b_{m(1)}$, since $b'_{m(1)-1} + b'_{m(1)} = b_{m(1)-1} + b_{m(1)}$.

In the same way, we define $W_2(\theta_2)$. If $g(1)$ is the index of the a_j represented by $c_{m(2)-1}$, then $W_2(\theta_2)e_{g(1)} = \sin\theta_2 e_{g(1)} + \cos\theta_2 e_{k(1)}$, $W_2(\theta_2)e_{k(1)} = -\cos\theta_2 e_{g(1)} + \sin\theta_2 e_{k(1)}$. [Recall that $k(1)$ is the index of the a_j represented by $c_{m(2)}$.] Again, $W_2(\theta_2)EW_2(\theta_2)^*$ "splices" $c'_{m(2)-1}$ and $c'_{m(2)}$ to $c_{m(2)-1}$ and $c_{m(2)}$, when θ_2 is suitably chosen, and leaves other "diagonal entries" of E unaltered. Note, too, that $W_1(\theta_1)W_2(\theta_2)e_j = W_2(\theta_2)W_1(\theta_1)e_j$ for each j. Here, we use the fact that $\mathbb{N}_1$ and $\mathbb{N}_2$ each contain four or more elements so that no two of $j(1)$, $h(1)$, $g(1)$, and $k(1)$ are equal. Thus $W_1(\theta_1)$ and $W_2(\theta_2)$ commute.

Note that $W_k(\theta_k)e_j = e_j$ unless $j \in \mathbb{N}_k \cup \mathbb{N}_{k+1}$. Thus $W_k(\theta_k)E_n = E_n$ if n is neither k nor $k+1$, since E_n has range in the space generated by e_j with j in $\mathbb{N}_n$. Suppose $j \in \mathbb{N}_n$. Then

$$\begin{aligned} F_r(e_j) &= W_r(\theta_r) \cdots W_n(\theta_n) \cdots W_1(\theta_1)EW_1(\theta_1)^* \cdots W_n(\theta_n)^* \cdots W_r(\theta_r)^* e_j \\ &= W_r(\theta_r) \cdots W_1(\theta_1)EW_n(\theta_n)^* W_{n-1}(\theta_{n-1})^* e_j \\ &= W_r(\theta_r) \cdots W_1(\theta_1)\left(\sum_{h=n-1}^{n+1} E_h W_n(\theta_n)^* W_{n-1}(\theta_{n-1})^* e_j\right) \\ &= W_{n+1}(\theta_{n+1})W_n(\theta_n) \cdots W_{n-2}(\theta_{n-2})\left(\sum_{h=n-1}^{n+1} E_h W_n(\theta_n)^* W_{n-1}(\theta_{n-1})^* e_j\right). \end{aligned}$$

In any event, $F_r(e_j) = F_s(e_j)$ when r and s exceed $n+1$. Thus $\{F_r(e_j)\}$ converges as $r \to \infty$ for each fixed j. As $\{F_r\}$ is a sequence of projections, it is uniformly bounded by 1. The basis $\{e_j\}$ generates a dense linear manifold on each element of which $\{F_r\}$ acts to produce a convergent sequence of vectors in $\mathcal{H}$. It follows that $\{F_r x\}$ is Cauchy convergent, hence convergent for each x in $\mathcal{H}$. If Fx is its limit, then F is linear and $\|F\| \leq 1$. Thus the sequence of projections $\{F_r\}$ is strong-operator convergent to F, and F is a projection whose matrix relative to $\{e_j\}$ has diagonal $\{a_j\}$ in some order.

The foregoing argument establishes our result when $a = \infty$. If $b = \infty$, the argument shows that there is a projection G with $b_1, b_2, \ldots$ as its diagonal, where $b_j = 1 - a_j$. The diagonal of $I - G$ is $a_1, a_2, \ldots (= 1 - b_1, 1 - b_2, \ldots)$. It remains to treat the case where a and b are finite and $a - b$ is an integer. With this assumption, and the added hypothesis that both the sums of the a_j and of the b_j are infinite, $\mathbb{N}'$ and $\mathbb{N}''$ must be infinite sets. There are at most a finite number of a''_j exceeding $1/n$ for a given positive integer n. By arranging those a''_j in decreasing order and letting n take on the values $1, 2, \ldots$, successively, we may re-label the a''_j so that $a''_1 \geq a''_2 \geq \cdots$. Similarly, we may assume that $b''_1 \geq b''_2 \geq \cdots$, where $b''_j = 1 - a'_j$, whence $a'_1 \leq a'_2 \leq \cdots$. As $\Sigma a''_j$ and $\Sigma b''_j$ are finite, we have that $a''_j \to 0$ and $b''_j \to 0$ as $j \to \infty$. It follows that $a'_j \to 1$ as $j \to \infty$. We may assume that $a \leq b$ (otherwise, we work with $b_1, b_2, \ldots$).

We discuss a procedure for "distributing the a''_j among $a'_1, a'_2, \ldots$." Let n be the smallest integer k such that $a''_1 \leq \Sigma_{j=1}^{k} b''_j$. Then $\Sigma_{j=1}^{n-1} b''_j < a''_1$. Replace $a'_1, \ldots, a'_{n-1}$ by $1, \ldots, 1$ and a'_n by $a'_n + a''_1 - \Sigma_{j=1}^{n-1} b''_j (= \tilde{a}'_n)$. Then $0 < a'_n < \tilde{a}'_n$ and $a''_1 - \Sigma_{j=1}^{n-1} b''_j \leq b''_n$, so $\tilde{a}'_n \leq a'_n + b''_n = 1$. In addition,

MATHEMATICS

$$\sum_{j=1}^{n} a'_j + a''_1 = \sum_{j=1}^{n} a'_j + \sum_{j=1}^{n-1} b''_j + a''_1 - \sum_{j=1}^{n-1} b''_j = n - 1 + a'_n + a''_1 - \sum_{j=1}^{n-1} b''_j = n - 1 + \tilde{a}'_n.$$

It follows from *Lemma 5* that $(a'_n, \ldots, a'_1, a''_1)$ is contained in the permutation polytope generated by $(1, \ldots, 1, \tilde{a}'_n, 0)$. As at the end of the proof of *Theorem 6*, there is a unitary operator U_1 on $\mathcal{H}$ such that $U_1 e_k = e_k$ for each k not in $\mathbb{N}_1$, the set of indices $\{j(1), \ldots, j(n+1)\}$, of the a_j in $\{a'_n, \ldots, a'_1, a''_1\}$, which restricts to $\mathcal{Y}_1$, the subspace of $\mathcal{H}$ generated by $e_{j(1)}, \ldots, e_{j(n+1)}$, as a unitary operator U'_1 such that $U'_1 A U'^*_1$ has matrix with diagonal $a''_1, a'_1, \ldots, a'_n$, when A, on $\mathcal{Y}_1$, has matrix with diagonal $0, 1, \ldots, 1, \tilde{a}'_n$ relative to the basis $e_{j(1)}, \ldots, e_{j(n+1)}$.

We now distribute a''_2 among $a'_{n+1}, a'_{n+2}, \ldots$ by the procedure just described, forming a finite subset $\mathbb{N}_2$ $(= \{k(1), \ldots, k(m+1)\})$ of $\mathbb{N}$, disjoint from $\mathbb{N}_1$, a unitary operator U_2 on $\mathcal{H}$, such that $U_2 e_k = e_k$ for each k not in $\mathbb{N}_2$, whose restriction U'_2 to $\mathcal{Y}_2$, the space generated by $e_{k(1)}, \ldots, e_{k(m+1)}$, transforms each operator A, on $\mathcal{Y}_2$, whose matrix has diagonal $0, 1, \ldots, 1, \tilde{a}'_{n+m}$ into one whose matrix has diagonal $a''_2, a'_{n+1}, \ldots, a'_{n+m}$ relative to the basis $e_{k(1)}, \ldots, e_{k(m+1)}$.

Continuing in this way, we construct disjoint subsets $\mathbb{N}_1, \mathbb{N}_2, \ldots$ of $\mathbb{N}$ with union $\mathbb{N}$, commuting unitaries $U_1, U_2, \ldots$ on $\mathcal{H}$, and (finite-dimensional) subspaces $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ with span dense in $\mathcal{H}$, as described before. Distributing all the a''_j among the a'_j yields an infinite sequence of 0s in place of the a''_j and an infinite sequence $\tilde{a}_1, \tilde{a}_2, \ldots$ of numbers in $(\frac{1}{2}, 1]$ such that $\Sigma_{j=1}^{\infty} 1 - \tilde{a}_j = b - a$, an integer, by assumption. (There is "room" for the distribution of *all* the a''_j among the a'_j from the assumption that $a \le b$.) From *Theorem 14*, there is a projection E_0 with diagonal $\tilde{a}_1, \tilde{a}_2, \ldots$ relative to the basis $\{e_j\}_{j\in\mathbb{N}'}$, and of course, a projection E with diagonal $0, 0, \ldots, \tilde{a}_1, \tilde{a}_2, \ldots$ relative to the basis $\{e_j\}$. We organize the basis $\{e_j\}$ according to the sets $\mathbb{N}_1, \mathbb{N}_2, \ldots$ and the diagonal of the matrix for E such that the entries at the diagonal positions corresponding to numbers in $\mathbb{N}_j$ are the numbers obtained from the matching step of the distribution procedure. If we now form $U_1 E U_1^*$, $U_2 U_1 E U_1^* U_2^*, \ldots$, successively, we construct a sequence of projections that converges, in the strong-operator topology, to a projection F (by the same argument used for the first part of the proof, where we assumed that a is infinite). The diagonal of F relative to $\{e_j\}$ is $a''_1, a''_2, \ldots, a'_1, a'_2, \ldots$, that is, $a_1, a_2, \ldots$, by choice of $U_1, U_2, \ldots$.

Having completed the proof of the "Carpenter's Theorem" in this case, we are left with the task of verifying the curious "integrality" condition imposed on $a - b$. In more detail, we let our orthonormal basis for $\mathcal{H}$ be $\{e_j\}_{j\in\mathbb{Z}_0}$, where $\mathbb{Z}_0$ is the set of nonzero integers, and F be a projection on $\mathcal{H}$ with matrix (a_{jk}) relative to $\{e_j\}$. Let $\mathbb{Z}_-$ and $\mathbb{Z}_+$ be the negative and positive integers in $\mathbb{Z}_0$, respectively. Our assumption now is that $\Sigma_{j=-1}^{-\infty} a_{jj}$ $(= a)$ and $\Sigma_{j=1}^{\infty} 1 - a_{jj}$ $(= b)$ are finite. We wish to prove that $a - b$ is an integer.

Let E be the projection whose range is spanned by $\{e_j\}_{j\in\mathbb{Z}_-}$. In effect, we have assumed that the positive operators EFE and $(I - E)(I - F)(I - E)$ are of trace class (L_1). It follows that FE, EF, $(I - F)(I - E)$, and $(I - E)(I - F)$ are operators of Hilbert–Schmidt Class (L_2). Thus the sum of all $|a_{jk}|^2$ with j or k in $\mathbb{Z}_-$ converges as does the sum of all $|b_{jk}|^2$ with j or k in $\mathbb{Z}_+$, where (b_{jk}) is the matrix of $I - F$. If T is a Hilbert–Schmidt operator on $\mathcal{H}$, we denote by $\|T\|_2$ the Hilbert–Schmidt (L_2-) norm of T. That is, $\|T\|_2^2$ is $\mathrm{tr}(T^*T)$ $(= \mathrm{tr}(TT^*))$, where "$\mathrm{tr}(T^*T)$" denotes the sum of the diagonal entries of the matrix for T^*T relative to an arbitrary orthonormal basis for $\mathcal{H}$, in particular, relative to $\{e_j\}_{j\in\mathbb{Z}_0}$. Thus $\|T\|_2^2$ is the sum of the squares of the absolute values of all the entries of the matrix for T (or of T^*). Since

$$\|T\|_2^2 = \mathrm{tr}(T^*T) = \sum_{j\in\mathbb{Z}_0} \langle T^*Te_j, e_j\rangle = \sum_{j\in\mathbb{Z}_0} \|Te_j\|^2,$$

with B a bounded operator on $\mathcal{H}$,

$$\|BT\|_2^2 = \sum_{j\in\mathbb{Z}_0} \|BTe_j\|^2 \le \|B\|^2 \sum_{j\in\mathbb{Z}_0} \|Te_j\|^2 = \|B\|^2\|T\|_2^2,$$

and $\|BT\|_2 \le \|B\|\|T\|_2$. In this notation, we have $\|EF\|_2^2 = \|FE\|_2^2 = \mathrm{tr}(EFE) = a$ and $\|(I - F)(I - E)\|_2^2 = b$.

Given a positive ε $(<(2 + 2a^{1/2} + 2b^{1/2})^{-1})$, choose n_0 in $\mathbb{Z}_+$ such that $\Sigma_{j \text{ or } k<-n_0}|a_{jk}|^2$ and $\Sigma_{j \text{ or } k>n_0}|b_{jk}|^2$ are each less than ε'^2, where $\varepsilon' = \varepsilon[28(1 + a^{1/2} + b^{1/2})]^{-1}$. Let A be the matrix that has a_{jk} as its j,k entry when $|j|$ and $|k|$ do not exceed n_0, 1 at the j,j entry when $j > n_0$, and 0 at all other entries. Then, by choice of n_0 and A, $\|F - A\|_2 < 2\varepsilon'$. Hence $|\|\mathrm{F}\|_2 - \|\mathrm{A}\|_2| < 2\varepsilon'$.

Let E_0 be the projection with range spanned by $\{e_{-1}, \ldots, e_{-n_0}\}$. Again, by choice of n_0 and A, $\|EF - E_0A\|_2 < 2\varepsilon'$. Hence $|\|EF\|_2 - \|E_0A\|_2| < 2\varepsilon'$. Let E'_0 be the projection with range spanned by $\{e_1, \ldots, e_{n_0}\}$ and I_0 be the projection $E_0 + E'_0$. Then $\|(I - E)A - (I - E)F\|_2 < 2\varepsilon'$. At the same time, $(I - E)(I - A) = E'_0(I - A)$, whence

$$\begin{aligned}|\|E'_0(I - A)\|_2 - \|(I - E)(I - F)\|_2| &\le \|E'_0(I - A) - (I - E)(I - F)\|_2\\ &= \|(I - E)(I - A) - (I - E)(I - F)\|_2\\ &\le \|I - E\|\|I - A - (I - F)\|_2 = \|F - A\|_2 < 2\varepsilon'.\end{aligned}$$

Note, too, that

$$\begin{aligned}\|A - A^2\|_2 &\le \|A - F\|_2 + \|F^2 - FA\|_2 + \|FA - A^2\|_2\\ &\le \|A - F\|_2 + \|F\|\|F - A\|_2 + \|A\|\|F - A\|_2 < 6\varepsilon'.\end{aligned}$$

Let A_0 be I_0FI_0. Then $A_0 - A_0^2 = A - A^2$, whence $\|A_0 - A_0^2\|_2 < 6\varepsilon'$. Of course, we may treat A_0 as the $2n_0 \times 2n_0$ matrix $(a_{jk})_{|j|,|k|\le n_0}$. Since A_0 is I_0FI_0, $\|A_0\| \le 1$ and A_0 is positive. Let $\lambda_{-n_0}, \ldots, \lambda_{n_0}$ be the $2n_0$ eigenvalues of A_0 (with repetitions). Then $\lambda_{-n_0} - \lambda_{-n_0}^2, \ldots, \lambda_{n_0} - \lambda_{n_0}^2$ are the eigenvalues of $A_0 - A_0^2$. Let ε_j be $\lambda_j - \lambda_j^2$ $(= \lambda_j(1 - \lambda_j))$. If $\lambda_j \ge \frac{1}{2}$, then $(1 - \lambda_j)^2 \le$

$4\varepsilon_j^2$. If $\lambda_j \le \frac{1}{2}$, then $\lambda_j^2 \le 4\varepsilon_j^2$. We suppose that m_0 of the eigenvalues λ_j are in $[\frac{1}{2}, 1]$. Let G_0 be the projection whose matrix relative to the basis that diagonalizes A_0 is diagonal with 1 in place of each of the λ_j in $[\frac{1}{2}, 1]$ and 0 in place of the other λ_j. Then

$$\|A_0 - G_0\|_2^2 \le 4 \sum_{j=-n_0}^{n_0} \varepsilon_j^2 = 4\|A_0 - A_0^2\|_2^2,$$

and $\|A_0 - G_0\|_2 \le 2\|A_0 - A_0^2\|_2 < 12\varepsilon'$. Thus $\|E_0A_0 - E_0G_0\|_2 < 12\varepsilon'$. Since $E_0A = E_0A_0$,

$$\|EF - E_0G_0\|_2 \le \|EF - E_0A\|_2 + \|E_0A_0 - E_0G_0\|_2 < 14\varepsilon'.$$

At the same time, $(I - E)(I - A) = E_0'(I_0 - A_0)$, whence

$$\|(I-E)(I-F) - E_0'(I_0 - G_0)\|_2 \le \|(I-E)(I-F) - (I-E)(I-A)\|_2 + \|E_0'(I_0 - A_0) - E_0'(I_0 - G_0)\|_2 < 14\varepsilon'.$$

It follows that $|\|EF\|_2 - \|E_0G_0\|_2| < 14\varepsilon' < 1$, and

$$|\|(I-E)(I-F)\|_2 - \|E_0'(I_0 - G_0)\|_2| < 14\varepsilon' < 1.$$

From the foregoing, we have that

$$\begin{aligned} |a - \|E_0G_0\|_2^2| &= |\|EF\|_2^2 - \|E_0G_0\|_2^2| \\ &= |(\|EF\|_2 - \|E_0G_0\|_2)(\|E_0G_0\|_2 + \|EF\|_2)| \\ &< 14\varepsilon'(1 + 2\|EF\|_2) = 14(1 + 2a^{\frac{1}{2}})\varepsilon'. \end{aligned}$$

For the last inequality, we note that $\|E_0G_0\|_2 \le 1 + \|EF\|_2$. In the same way,

$$\begin{aligned} |b - \|E_0'(I_0 - G_0)\|_2^2| &= |\|(I-E)(I-F)\|_2^2 - \|E_0'(I_0 - G_0\|_2^2| \\ &= |(\|(I-E)(I-F)\|_2 - \|E_0'(I_0 - G_0)\|_2)(\|(I-E)(I-F)\|_2 + \|E_0'(I_0 - G_0)\|_2)| \\ &< 14\varepsilon'(1 + 2\|(I-E)(I-F)\|_2) = 14(1 + 2b^{\frac{1}{2}})\varepsilon'. \end{aligned}$$

Thus, as I_0, E_0, and G_0 have ranks $2n_0$, n_0, and m_0, respectively, from *Proposition 3*,

$$\begin{aligned} |a - b - (m_0 - n_0)| &= |a - b - (m_0 - 2n_0 + n_0)| \\ &= |a - b - [\mathrm{tr}(E_0G_0E_0) - \mathrm{tr}((I_0 - E_0)(I_0 - G_0)(I_0 - E_0))]| \\ &= |a - b - (\|E_0G_0\|_2^2 - \|E_0'(I_0 - G_0)\|_2^2)| < 28(1 + a^{\frac{1}{2}} + b^{\frac{1}{2}})\varepsilon' = \varepsilon. \end{aligned}$$

Now, m_0 and n_0 vary with the choice of ε, but they are always integers. As $a - b$ is arbitrarily close to an integer, $a - b$ is an integer.

We have established that $a - b$ is an integer for an arbitrary subset $\mathcal{S}$ of diagonal elements of F with convergent sum a whose complementary set of diagonal elements $\mathcal{S}'$, subtracted from 1, also has a convergent sum b. To conclude our proof, we note that the existence of any such $\mathcal{S}$ implies that the set $\mathcal{S}_0$ of a_{jj} in $[0, \frac{1}{2}]$ is such a set. As $\mathcal{S}$ has a convergent sum, it contains at most a finite number of a_{jj} exceeding $\frac{1}{2}$. Similarly, $\mathcal{S}'$ contains at most a finite number of a_{jj} less than or equal to $\frac{1}{2}$. Shifting one finite set from $\mathcal{S}$ to $\mathcal{S}'$ and the other from $\mathcal{S}'$ to $\mathcal{S}$ produces the sets $\mathcal{S}_0$ and $\mathcal{S}_0'$ with convergent sums a_0 and b_0, respectively. Moreover, $a_0 - b_0$ and $a - b$ differ by an integer, as described in the comment following the proof of *Proposition 3.* Thus $a_0 - b_0$ is an integer if and only if $a - b$ is. ■

4. Pythagorean Matrices

In *Remark 11*, we discussed doubly stochastic matrices. We used *Proposition 12* to provide another proof of *Proposition 3*, referring to *Proposition 12* as a "Pythagorean Theorem" for finite doubly stochastic matrices. In this section, we consider infinite doubly stochastic matrices. We prove a Pythagorean Theorem for a class of them, the *Pythagorean matrices* (*Proposition 16*).

We say that a doubly stochastic matrix A $(= (a_{jk}))$ is *Pythagorean* when there is a Hilbert space $\mathcal{H}$ and two orthonormal bases $\{e_j\}$ and $\{f_k\}$ for $\mathcal{H}$ such that $a_{jk} = |\langle e_j, f_k\rangle|^2$. We show that a Pythagorean Theorem holds for Pythagorean matrices: The difference of the weights of complementary blocks is an integer when those weights are finite. As promised in *Remark 11*, we establish the existence of doubly stochastic matrices with infinite complementary blocks, each of which has finite weight and no zero diagonal entries. We use the "Carpenter" result in *Theorem 15* and the notation of the proof of "integrality" for this.

When $j \in \mathbb{Z}_-$, let a_j be 2^j. When $j \in \mathbb{Z}_+$, let a_j be $1 - 2^{-j}$. Since $1 = \Sigma_{j\in\mathbb{Z}_-} a_j$ $(= a)$ and $1 = \Sigma_{j\in\mathbb{Z}_+} 1 - a_j$ $(= b)$ and $a - b = 0$, an integer, we may apply *Theorem 15* to conclude that there is an infinite-dimensional subspace $\mathcal{H}_0$ of $\mathcal{H}$ with infinite-dimensional complement $\mathcal{H}\ominus\mathcal{H}_0$ $(= \mathcal{H}_0')$ such that $\|Fe_j\|^2 = a_j$, for each j in $\mathbb{Z}_0$, where F is the projection of $\mathcal{H}$ on $\mathcal{H}_0$. If $j \in \mathbb{Z}_+$, $\|(I - F)e_j\|^2 = 1 - \|Fe_j\|^2 = 1 - a_j = 2^{-j}$. Let $\{f_k\}_{k\in\mathbb{Z}_+}$ and $\{f_k\}_{k\in\mathbb{Z}_-}$ be orthonormal bases for $\mathcal{H}_0$ and $\mathcal{H}_0'$, respectively. Let a_{jk} be $|\langle e_j, f_k\rangle|^2$, so that (a_{jk}) is an infinite doubly stochastic matrix, as noted before. Since

MATHEMATICS

$$\sum_{k \in \mathbb{Z}_+} a_{jk} = \sum_{k \in \mathbb{Z}_+} |\langle e_j, f_k \rangle|^2 = \|Fe_j\|^2 = a_j,$$

for each j in $\mathbb{Z}_0$, and $\Sigma_{k\in\mathbb{Z}_-} a_{jk} = \|(I - F)e_j\|^2 = 1 - a_j$, for each such j, we have that $\Sigma_{j\in\mathbb{Z}_-} \Sigma_{k\in\mathbb{Z}_+} a_{jk} = \Sigma_{j\in\mathbb{Z}_-} 2^j = 1$ and $\Sigma_{j\in\mathbb{Z}_+} \Sigma_{k\in\mathbb{Z}_-} a_{jk} = \Sigma_{j\in\mathbb{Z}_+} 2^{-j} = 1$. Thus the weight of each of the $\mathbb{Z}_+$, $\mathbb{Z}_-$ and $\mathbb{Z}_-$, $\mathbb{Z}_+$ complementary blocks is 1. Of course, we can construct other examples of infinite complementary blocks, each having finite weight, by using *Theorem 15* in the manner just described.

We prove the analogue of *Proposition 12* for Pythagorean matrices.

Proposition 16. *The difference of the weights of a block and its complementary block in an infinite Pythagorean matrix is an integer when those weights are finite.*

Proof: Since the complement of a finite block has infinite weight, we may assume that the block A_0 and its complement A_0' in the matrix A are infinite with finite weights. Assume that A is Pythagorean. We may also assume that $A = (a_{jk})_{j,k\in\mathbb{Z}_0}$ and that $A_0 = (a_{jk})_{j,k\in\mathbb{Z}_-}$ (so that $A_0' = (a_{jk})_{j,k\in\mathbb{Z}_+}$). By assumption, there are orthonormal bases $\{e_j\}_{j\in\mathbb{Z}_0}$ and $\{f_k\}_{k\in\mathbb{Z}_0}$, for a Hilbert space $\mathcal{H}$, such that $a_{jk} = |\langle e_j, f_k\rangle|^2$. Let $\mathcal{H}_0$ be the closure of the space spanned by $\{f_k\}_{k\in\mathbb{Z}_-}$ and F the projection of $\mathcal{H}$ on $\mathcal{H}_0$. As in the proof of *Proposition 12*, $\|Fe_j\|^2$ is the sum of the entries of the jth row of A_0, when $j \in \mathbb{Z}_-$, while $\|(I - F)e_j\|^2$ is the sum of the entries in the jth row of A_0', when $j \in \mathbb{Z}_+$. At the same time, $\|Fe_j\|^2$ is the j diagonal entry of the matrix for F and $\|(I - F)e_j\|^2$ is the j diagonal entry for the matrix of $I - F$ (matrices formed relative to the basis $\{e_j\}_{j\in\mathbb{Z}_0}$). Thus $w(A_0)$ is the sum of the j diagonal entries in the matrix for F with j in $\mathbb{Z}_-$, and $w(A_0')$ is the sum of the j diagonal entries in the matrix for $I - F$ (that is, 1 minus the j diagonal entry for F), with j in $\mathbb{Z}_+$. These sums are finite, by assumption [because $w(A_0)$ and $w(A_0')$ are finite]. But *Theorem 15* assures us that the difference of these sums is an integer. Thus $w(A_0) - w(A_0')$ is an integer when A is Pythagorean. ■

There is a great deal more to be said about doubly stochastic matrices in this context, and there are a number of questions that have not been answered (for example: Are there non-Pythagorean doubly stochastic matrices?) That discussion must await another occasion.

1. Kadison, R. (2002) *Proc. Natl. Acad. Sci. USA* **99,** 4178–4184.