Contemporary Mathematics
Volume **228**, 1998

Dual Cones And Tomita–Takesaki Theory

Richard V. Kadison

In this article, we present the theory of dual cones as introduced by M. Takesaki in [T] and developed by H. Araki [A1, A2], A. Connes [C], and U. Haagerup [H1, H2]. Takesaki, in [T], defines and studies the cones we denote by $\mathcal{V}_u^0$ and $\mathcal{V}_u^{\frac{1}{2}}$ (following Araki's notation). He proves that they are dual to each other and establishes more of their important special properties. In [C] and [H2], Connes and Haagerup introduce the self-dual cone, which becomes $\mathcal{V}_u^{\frac{1}{4}}$ in the Araki notation (and later, $\mathcal{V}_u$ in our notation). In [A1] (see also [A2]), Araki extends the theory to a one-parameter family $\mathcal{V}_u^a$, $a \in [0, \frac{1}{2}]$, of cones. It is, largely, the Araki theory that we present. Most, but not all, of the results in this article appear in exercise form in [KR4; Exercises 9.6.51–65].

This theory is, in essence, a deep and detailed examination of the structure of the space of normal states of a von Neumann algebra and, at a more primitive level, a study of which operators, affiliated with a von Neumann algebra acting on a Hilbert space in standard form, map a given vector onto or near other vectors. As such, it is an important aspect of the theory of non-commutative integration, on the one hand, and a fundamental part of the theory of non-commutative approximation, on the other.

The results presented are spread throughout the mathematical literature. It seems worthwhile to gather them into one article and to present them with complete proofs in a unified and simplified style. All the major results that appear are known. Many of the results on the way to these are new as are most of the arguments. In formulating and proving Theorem 13, some unpublished computations of Uffe Haagerup provided us with crucial help.

The Friedrichs Extension [F] is a vital element in Takesaki's pioneering work with the original dual cones. We have included a complete proof of it as an appendix, with the appropriate additions and statement for use with von Neumann algebras. The proof is different from the earlier proofs and is substantially that appearing in [KR4; Exercises 7.6.52-55]. Our notation and terminology is that of [KR1–4]. The results and exercises of [KR1–4] are referred to with their numbering in [KR1–4] and no further reference.

Theorem 1. *Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$, acting on a Hilbert space $\mathcal{H}$, and let J be a conjugate-linear isometry of $\mathcal{H}$ onto $\mathcal{H}$ such that $J^2 = I$, $J\mathcal{R}J = \mathcal{R}'$, and $JCJ = C^*$ for each C in $\mathcal{C}$.*

(i) $A \to JA^*J$ is a $*$ anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$.

(ii) If ψ is a $*$ anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$, then there is a unitary operator U on $\mathcal{H}$ such that (the conjugate-linear isometry) JU implements ψ:

$$JUA^*(JU)^* = \psi(A) \qquad (A \in \mathcal{R}).$$

Proof. (i) Note, first, that since J^* is the mapping of $\mathcal{H}$ into $\mathcal{H}$ obtained from J by using the adjoint of J when J is viewed as a linear mapping of $\mathcal{H}$ into $\overline{\mathcal{H}}$ and that J so viewed is a unitary transformation of $\mathcal{H}$ onto $\overline{\mathcal{H}}$, we have $J = J^*$ (as mappings of $\mathcal{H}$ into $\mathcal{H}$) for both J and J^* are the mapping inverse to J on $\mathcal{H}$. Note, too, that with A in $\mathcal{B}(\mathcal{H})$, $(JAJ)^* = JA^*J$, for $JAJ \in \mathcal{B}(\mathcal{H})$ and, with x, y in $\mathcal{H}$,

$$\begin{aligned}\langle (JAJ)^*x, y\rangle &= \langle x, JAJy\rangle = \langle AJy, Jx\rangle \\ &= \langle Jy, A^*Jx\rangle = \langle JA^*Jx, y\rangle .\end{aligned}$$

Thus

$$\phi(A^*) = JA^{**}J = JAJ = (JA^*J)^* = \phi(A)^*,$$

where $\phi(B) = JB^*J$ for B in $\mathcal{R}$. In addition, with A, B in $\mathcal{R}$,

$$\begin{aligned}\phi(aA + B) = J(aA + B)^*J &= J\bar{a}A^*J + JB^*J \\ &= aJA^*J + JB^*J \\ &= a\phi(A) + \phi(B),\end{aligned}$$

and

$$\phi(AB) = J(AB)^*J = JB^*JJA^*J = \phi(B)\phi(A).$$

With A' in $\mathcal{R}'$, there is, by assumption, an A^* in $\mathcal{R}$ such that $\phi(A) = JA^*J = A'$, so that ϕ maps $\mathcal{R}$ onto $\mathcal{R}'$. Finally, since J is an isometry of $\mathcal{H}$ onto $\mathcal{H}$, given an x in $\mathcal{H}$ there is a y in $\mathcal{H}$ such that $Jy = x$. If $0 = \phi(A) = JA^*J$ for some A in $\mathcal{R}$, then $0 = JA^*Jy = JA^*x$, and $A^*x = 0$ for each x in $\mathcal{H}$. Thus $A^* = 0$ and $A = 0$. It follows that ϕ is a $*$ anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$.

(ii) With ϕ as in (i), let η be the $*$ *isomorphism* $\phi^{-1} \circ \psi$ of $\mathcal{R}$ onto $\mathcal{R}$. From Exercise 9.6.25, there is a unitary operator U such that $\eta(A) = UAU^*$ for each A in $\mathcal{R}$. Hence

$$\psi(A) = \phi(UAU^*) = J(UAU^*)^*J = JUA^*(JU)^* \qquad (A \in \mathcal{R}).$$

Theorem 2. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, u be a generating and separating unit vector for $\mathcal{R}$, and S, F J, and Δ, be the modular operators for $\{\mathcal{R}, u\}$.*

(i) With A in $\mathcal{R}$, A is in the centralizer of $\omega_u|\mathcal{R}$ if and only if

$$JAu = A^*u.$$

(ii) If $C \in \mathcal{R}$, then C is in the center of $\mathcal{R}$ if and only if

$$JCJ = C^*.$$

Proof. (i) Suppose $JAu = A^*u$. Then $Au = JA^*u$, and

$$\begin{aligned}\langle ABu, u\rangle &= \langle ASB^*u, u\rangle = \langle AJ\Delta^{\frac{1}{2}}B^*u, u\rangle \\ &= \langle J\Delta^{\frac{1}{2}}B^*u, A^*u\rangle = \langle JA^*u, \Delta^{\frac{1}{2}}B^*u\rangle \\ &= \langle Au, \Delta^{\frac{1}{2}}B^*u\rangle = \langle B\Delta^{\frac{1}{2}}Au, u\rangle \\ &= \langle BJJ\Delta^{\frac{1}{2}}Au, u\rangle = \langle BJA^*u, u\rangle \\ &= \langle BAu, u\rangle\end{aligned}$$

for each B in $\mathcal{R}$. Thus A is in the centralizer of $\omega_u|\mathcal{R}$.

Suppose, now, that A is in the centralizer of $\omega_u|\mathcal{R}$. Then $\Delta^{it}A\Delta^{-it} = A$, for each real t, from Proposition 9.2.14(iii). Let $\mathcal{A}$ be the (abelian) von Neumann algebra generated by $\{\Delta^{it} : t \in \mathcal{R}\}$. Then $\Delta\,\eta\,\mathcal{A}$ and $\Delta^{\frac{1}{2}}\,\eta\,\mathcal{A}$. It follows that $A\Delta^{\frac{1}{2}} \subseteq \Delta^{\frac{1}{2}}A$ and that

$$JAu = JA\Delta^{\frac{1}{2}}u = J\Delta^{\frac{1}{2}}Au = SAu = A^*u.$$

(ii) Suppose $JCJ = C^*$. Since $C^* \in \mathcal{R}$, $JCJ \in \mathcal{R} \cap \mathcal{R}'$. Thus JCJ, C^*, and C, are in the center of $\mathcal{R}$.

If C is in the center of $\mathcal{R}$, then C is in the centralizer of $\omega_u|\mathcal{R}$. Hence, from (i),

$$JCJu = JCu = C^*u.$$

As $C \in \mathcal{R}'$, $JCJ \in \mathcal{R}$. Since u is separating for $\mathcal{R}$,

$$JCJ = C^*.$$

Theorem 3. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with generating and separating vector u. Let S_0, S, F_0, F, J, and Δ be the modular operators for u Suppose J' is a conjugate-linear isometry of $\mathcal{H}$ into $\mathcal{H}$ such that*

$$J'u = u, \quad J'^2 = I, \quad J'\mathcal{R}J' = \mathcal{R}', \quad \langle AJ'AJ'u, u\rangle \geq 0 \qquad (A \in \mathcal{R}).$$

Let H_0Au be $J'A^*u$ ($A \in \mathcal{R}$) and U be $J'J$. Then

(i) $0 \leq \langle AJAJu, u\rangle \quad (A \in \mathcal{R})$;

(ii) $H_0 = J'S_0$, *and* H_0 *has closure* $J'S\,(= H)$;

(iii) $\langle Hx, x\rangle \geq 0$ *for each* x *in* $\mathcal{D}(H)\,(= \mathcal{D}(\Delta^{1/2}))$ *and* H *is symmetric;*

(iv) H *is self-adjoint;*

(v) H *is positive and* $H = U\Delta^{1/2}$;

(vi) $J = J'$, *and* J *is the unique operator with the properties assumed for* J'.

Proof. (i) With A in $\mathcal{R}$, we have

$$\langle AJAJu, u\rangle = \langle AJSA^*u, u\rangle = \langle \Delta^{1/2}A^*u, A^*u\rangle \geq 0,$$

since $\Delta^{1/2} \geq 0$.

(ii) By definition $\mathcal{D}(H_0) = \mathcal{D}(J'S_0) = \mathcal{D}(S_0) = \mathcal{R}u$, and

$$H_0 Au = J'A^*u = J'S_0Au \qquad (A \in \mathcal{R}).$$

Hence $H_0 = J'S_0$. Now (x, y) is in the closure of the graph of $J'S_0$ if and only if there is a sequence $\{x_n\}$ in the domain of S_0, tending to x, such that $\{J'S_0x_n\}$ tends to y, which occurs if and only if $\{S_0x_n\}$ tends to $J'y$. Thus, (x, y) is in the closure of the graph of $J'S_0$ if and only if $(x, J'y)$ is in the closure of the graph of S_0, that is, if and only if $x \in \mathcal{D}(S)$ and $Sx = J'y$ (equivalently, $J'Sx = y$). Thus $H = J'S$.

(iii) It follows from (ii) that $\mathcal{D}(H) = \mathcal{D}(S) = \mathcal{D}(\Delta^{1/2})$. By hypothesis, with A in $\mathcal{R}$,

$$\langle HAu, Au\rangle = \langle A^*J'A^*u, u\rangle = \langle A^*J'A^*J'u, u\rangle \geq 0;$$

hence $\langle Hx, x\rangle \geq 0$ for each x in $\mathcal{D}(H)$ since $\mathcal{R}u$ is a core for H. From Exercise 7.6.52(i), with A_0 in place of H, $H \subseteq H^*$.

(iv) From (ii), $H_0 = J'S_0$ and $S_0 = J'H_0$. Thus

$$S_0^*J' \subseteq H_0^* = H^*, \quad H_0^*J' \subseteq S_0^*, \quad H^* = H_0^* \subseteq S_0^*J'.$$

Hence, $H^* = S_0^*J' = FJ'$. Now $\mathcal{R}'u$ is a core for F, so that $J'\mathcal{R}'u$ is a core for FJ'. But $H^* = FJ'$ and $J'\mathcal{R}'u = J'\mathcal{R}'J'u = \mathcal{R}u$. Thus $\mathcal{R}u$ $(= \mathcal{D}(H_0))$ is a core for H and for H^*. From (iii), $H \subseteq H^*$, so that $H = H^*$.

(v) From (iii), $\langle Hx, x\rangle \geq 0$ for each x in $\mathcal{D}(H)$. From (iv), H is self-adjoint. Hence H is positive. From (ii),

$$H = J'S = J'J\Delta^{1/2} = U\Delta^{1/2}.$$

(vi) Since $H \geq 0$ and $U\Delta^{1/2}$ is a polar decomposition for H, from Theorem 6.1.11, we have that $I = U = J'J$. Hence, $J = J'$, and J is the unique operator with the properties assumed for J'.

Theorem 4. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, u be a separating and generating vector for $\mathcal{R}$, and S, F, J and Δ be the modular operators for u. With x a vector in $\mathcal{H}$, let $\phi_x(A)$ be $\langle Au, x\rangle$ for each A in $\mathcal{R}$, and $\phi'_x(A')$ be $\langle A'u, x\rangle$ for each A' in $\mathcal{R}'$. Then*

(i) $x \in \mathcal{D}(S)$ $(= \mathcal{D}(F_0^*) = \mathcal{D}(\Delta^{1/2}))$ and $Sx = x$ for a vector x in $\mathcal{H}$ if and only if the (normal) linear functional ϕ'_x on $\mathcal{R}'$ is hermitian; symmetrically, $y \in \mathcal{D}(F)$ and $Fy = y$ if and only if ϕ_y is hermitian;

(ii) $\phi'_x \geq 0$ if and only if $x = Hu$ for some positive H affiliated with $\mathcal{R}$;

(iii) the set of vectors x in $\mathcal{H}$ such that $\phi'_x \geq 0$ is a (norm-)closed cone $\mathcal{V}_u^0$ in $\mathcal{H}$ and (by symmetry) the same is true of the set $\mathcal{V}_u^{1/2}$ of vectors x in $\mathcal{H}$ such that $\phi_x \geq 0$;

(iv) $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ (of (iii)) are dual cones, that is, $w \in \mathcal{V}_u^0$ if and only if $\langle w, v\rangle \geq 0$ for each v in $\mathcal{V}_u^{1/2}$, and $v \in \mathcal{V}_u^{1/2}$ if and only if $\langle w, v\rangle \geq 0$ for each w in $\mathcal{V}_u^0$;

(v) $\mathcal{V}_u^0$ is the norm closure of $\mathcal{R}^+u$ and $\mathcal{V}_u^{1/2}$ is the norm closure of $\mathcal{R}'^+u$;

(vi) $\Delta^{1/2}\mathcal{R} + u = \mathcal{R}'^+u$, $\Delta^{-1/2}\mathcal{R}'^+u = \mathcal{R}^+u$, and $\mathcal{V}_u^{1/2}$, $\mathcal{V}_u^0$ are the norm closures of $\Delta^{1/2}\mathcal{R}^+u$, $\Delta^{-1/2}\mathcal{R}'^+u$, respectively.

Proof. (i) If $x \in \mathcal{D}(S)$ and $Sx = x$, then for each self-adjoint A' in $\mathcal{R}'$,

$$\langle A'u, x\rangle = \langle FA'u, x\rangle = \langle Sx, A'u\rangle = \langle x, A'u\rangle;$$

whence $\phi'_x(A')$ is real and ϕ'_x is hermitian. (See Corollary 9.2.30.)

Suppose, now, that ϕ'_x is hermitian and $T' \in \mathcal{R}'$. Then

$$\langle FT'u, x\rangle = \langle T'^*u, x\rangle = \phi'_x(T'^*) = \overline{\phi'_x(T')} = \overline{\langle T'u, x\rangle} = \langle x, T'u\rangle,$$

whence $x \in \mathcal{D}(F_0^*)$ and $x = F_0^*x = Sx$.

(ii) Suppose H is a positive operator affiliated with $\mathcal{R}$ such that $u \in \mathcal{D}(H)$ and $x = Hu$. Let $\{E_\lambda\}$ be the resolution of the identity for H, and let H_n be HE_n. Then $H_n \in \mathcal{R}$ and $E_nH \subseteq H_n$. Hence

$$H_nu = E_nHu \to Hu = x \qquad (n \to \infty)$$

since E_n tends to I in the strong-operator topology. Now $\langle A'u, H_nu\rangle = \langle H_nA'u, u\rangle \geq 0$, when $A' \in \mathcal{R}'^+$, since H_n and A' are commuting positive operators. Thus

$$0 \leq \lim\langle A'u, H_nu\rangle = \langle A'u, Hu\rangle = \langle A'u, x\rangle,$$

and $\phi'_x \geq 0$.

Suppose x in $\mathcal{H}$ is such that $\phi'_x \geq 0$. Then, in particular, ϕ'_x is hermitian, and $x \in \mathcal{D}(S)$ $(= \mathcal{D}(F_0^*))$ from (i). From Lemma 9.2.28, $L_x \,\eta\, \mathcal{R}$ (and $L_xu = x$). By definition of L_x,

$L_xT'u = T'x$ for each T' in $\mathcal{R}'$, so that

$$\langle L_xT'u, T'u\rangle = \langle T'x, T'u\rangle = \overline{\langle T'^*T'u, x\rangle} \geq 0$$

since $T'^*T' \in \mathcal{R}'^+$ and $\phi'_x \geq 0$. Thus $\langle L_xy, y\rangle \geq 0$ for each y in $\mathcal{R}'u$, a core for L_x, and $\langle L_xz, z\rangle \geq 0$ for each z in $\mathcal{D}(L_x)$. From Theorems $2'$ and $4'$ of the appendix, L_x has a positive self-adjoint extension (the Friedrichs extension) H affiliated with $\mathcal{R}$. As $L_xu = x$, $Hu = x$.

(iii) If $A' \in \mathcal{R}'^+$, then $\langle A'u, ax + y\rangle \geq 0$ when $a \geq 0$ and $x, y \in \mathcal{V}_u^0$. Thus $ax + y \in \mathcal{V}_u^0$. If v and $-v$ are in $\mathcal{V}_u^0$, then $\langle A'u, v\rangle = 0$ for each A' in $\mathcal{R}'^+$. Since each operator T' in $\mathcal{R}'$ is a linear combination of (four) operators in $\mathcal{R}'^+$, $\langle T'u, v\rangle = 0$. As $[\mathcal{R}'u] = \mathcal{H}$, $v = 0$. Thus $\mathcal{V}_u^0$ and, symmetrically, $\mathcal{V}_u^{1/2}$ are cones in $\mathcal{H}$.

If $\{x_n\}$ is a sequence of vectors in $\mathcal{V}_u^0$ tending to x in norm and $A' \in \mathcal{R}'^+$, then $0 \leq \langle A'u, x_n\rangle \to \langle A'u, x\rangle$. Hence $\mathcal{V}_u^0$ and, symmetrically, $\mathcal{V}_u^{1/2}$ are (norm-)closed cones in $\mathcal{H}$.

(iv) If $v \in \mathcal{V}_u^{1/2}$, then $\langle Au, v\rangle \geq 0$ for each A in $\mathcal{R}^+$. If $w \in \mathcal{V}_u^0$, then $w = Hu$ for some positive H affiliated with $\mathcal{R}$ from (ii). With H_n as in the proof of (ii),

$$0 \leq \langle H_nu, v\rangle \to \langle Hu, v\rangle = \langle w, v\rangle .$$

If $\langle w, v\rangle \geq 0$ for each v in $\mathcal{V}_u^{1/2}$, then $0 \leq \langle w, A'u\rangle = \langle A'u, w\rangle$ for each A' in $\mathcal{R}'^+$, since $A'u \in \mathcal{V}_u^{1/2}$ for such A' (from (ii) applied with $\mathcal{R}'$ in place of $\mathcal{R}$). Hence $\phi'_w \geq 0$ and $w \in \mathcal{V}_u^0$. Thus $w \in \mathcal{V}_u^0$ if and only if $\langle w, v\rangle \geq 0$ for each v in $\mathcal{V}_u^{1/2}$. Symmetrically, $v \in \mathcal{V}_u^{1/2}$ if and only if $\langle w, v\rangle \geq 0$ for each w in $\mathcal{V}_u^0$.

(v) From (ii), $\mathcal{R}^+u \subseteq \mathcal{V}_u^0$. If $x \in \mathcal{V}_u^0$, there is a positive H affiliated with $\mathcal{R}$ such that $x = Hu$. With the notation of the solution to (ii), $H_nu \in \mathcal{R}^+u$ and

$H_n u \to Hu = x$. Thus $\mathcal{V}_u^0$ is contained in the norm closure of $\mathcal{R}^+u$ and, hence, coincides with this norm closure. Symmetrically, $\mathcal{V}_u^{1/2}$ is the norm closure of $\mathcal{R}'^+u$.

(vi) Let $\Phi(A)$ be JA^*J for A in $\mathcal{R}$. From the discussion at the beginning of Section 9.2, Φ is a $*$ anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$. Hence $\Phi(\mathcal{R}^+) = \mathcal{R}'^+$. With A in $\mathcal{R}^+$, $Au \in \mathcal{D}(S) = \mathcal{D}(\Delta^{1/2})$ and

$$\Delta^{1/2}Au = JSAu = JA^*u = JA^*Ju = \Phi(A)u.$$

Thus $\Delta^{1/2}\mathcal{R}^+u = \mathcal{R}'^+u$. Since $\mathcal{R}'^+u$ is norm dense in $\mathcal{V}_u^{1/2}$ from (v), $\mathcal{V}_u^{1/2}$ is the norm closure of $\Delta^{1/2}\mathcal{R}^+u$. Symmetrically, with A' in $\mathcal{R}'^+$, $A'u \in \mathcal{D}(F) = \mathcal{D}(\Delta^{-1/2})$ and

$$\Delta^{-1/2}A'u = JFA'u = JA'^*u = JA'^*Ju = \Phi^{-1}(A')u\,.$$

Hence $\Delta^{-1/2}\mathcal{R}'^+u = \mathcal{R}^+u$, and $\mathcal{V}_u^0$ is the norm closure of the cone $\Delta^{-1/2}\mathcal{R}'^+u$.

Theorem 5. *With the notation and assumptions of Theorem 4 and with ω a normal state of $\mathcal{R}$:*

(i) there is a vector v in $\mathcal{V}_u^0$ such that $\omega_v|\mathcal{R} = \omega$;

(ii) $\|v-u\| = \inf\{\|z-u\| : \omega_z|\mathcal{R} = \omega\}$, where v is as in (i);

(iii) the vector v in (i) is unique.

Proof. (i) Since u is separating for $\mathcal{R}$, there is a unit vector z in $\mathcal{H}$ such that $\omega = \omega_z\,|\,\mathcal{R}$ from Theorem 7.2.3. From Theorem 7.3.2, there is a partial isometry V' in $\mathcal{R}'$ such that ω' is a positive normal linear functional on $\mathcal{R}'$, where $\omega'(A') = \phi'_z(V'A')$ for each A' in $\mathcal{R}'$, and such that $\phi'_z(A') = \omega'(V'^*A')$.

Now $\omega'(A') = \phi'_z(V'A') = \langle V'A'u, z\rangle = \langle A'u, V'^*z\rangle$, so that $(v =)\ V'^*z \in \mathcal{V}_u^0$. In addition,

$$\langle A'u, z\rangle = \phi'_z(A') = \omega'(V'^*A') = \phi'_z(V'V'^*A') = \langle A'u, V'V'^*z\rangle\,.$$

Since u is generating for $\mathcal{R}'$, $z = V'V'^*z$, whence $\omega_v\,|\,\mathcal{R} = \omega_z\,|\,\mathcal{R} = \omega$.

(ii) If H is a positive operator, affiliated with $\mathcal{R}$ and $u \in \mathcal{D}(H)$, then $u \in \mathcal{D}(H^{1/2})$, $H^{1/2}u \in \mathcal{D}(H^{1/2})$, $H^{1/2}H^{1/2}u = Hu$ (from 5.6.(18)), and $A'H^{1/2} \subseteq H^{1/2}A'$ for each A' in $\mathcal{R}'$. Thus, if V' is a partial isometry in $\mathcal{R}'$,

$$\begin{aligned} |\langle V'Hu, u\rangle| &= |\langle V'H^{1/2}u, H^{1/2}u\rangle| \\ &\le \|V'H^{1/2}u\|\|H^{1/2}u\| \\ &\le \|H^{1/2}u\|^2 \\ &= \langle Hu, u\rangle, \end{aligned} \tag{1}$$

and

$$\operatorname{Re}\langle V'Hu, u\rangle \le \langle Hu, u\rangle\,. \tag{2}$$

Suppose z is a unit vector in $\mathcal{H}$ such that $\omega_z\,|\,\mathcal{R} = \omega_{Hu}\,|\,\mathcal{R}$. From Exercise 7.6.23(ii), there is a partial isometry W' in $\mathcal{R}'$, with initial space $[\mathcal{R}Hu]$, such that $W'Hu = z$. From (2),

$$\operatorname{Re}\langle z, u\rangle = \operatorname{Re}\langle W'Hu, u\rangle \le \langle Hu, u\rangle,$$

so that

$$\|Hu-u\|^2 = 2 - 2\operatorname{Re}\langle Hu, u\rangle \le 2 - 2\operatorname{Re}\langle z, u\rangle = \|z-u\|^2\,. \tag{3}$$

From Theorem 4(ii), there is a positive operator H, affiliated with $\mathcal{R}$ such that $v = Hu$. From (3),

$$\|v - u\| = \inf\{\|z - u\| \ : \ \omega_z \,|\, \mathcal{R} = \omega\}\,.$$

(iii) If v' is another vector in $\mathcal{V}_u^0$ such that $\omega_{v'} \,|\, \mathcal{R} = \omega$, then

$$\|v - u\| = \inf\{\|z - u\| \ : \ \omega_z \,|\, \mathcal{R} = \omega\} = \|v' - u\|,$$

and $v' = V'Hu$ for some partial isometry V' in $\mathcal{R}'$ with Hu in its initial space (where H is as in (ii)). Hence

$$\mathrm{Re}\,\langle V'Hu, u\rangle = \mathrm{Re}\,\langle v', u\rangle = \mathrm{Re}\,\langle v, u\rangle = \langle Hu, u\rangle,$$

and the inequality of (2) is equality in the present case. It follows that

$$\langle V'Hu, u\rangle = |\langle V'Hu, u\rangle| = \mathrm{Re}\,\langle V'Hu, u\rangle = \langle Hu, u\rangle,$$

so that

$$\langle V'H^{1/2}u, H^{1/2}u\rangle = \|V'H^{1/2}u\|\|H^{1/2}u\| = \|H^{1/2}u\|^2\,.$$

Thus $V'H^{1/2}u = H^{1/2}u$ and $v' = V'Hu = Hu = v$.

Theorem 6. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, u be a separating and generating vector for $\mathcal{R}$, and S, F, J and Δ be the modular operators for u. With a in $[0, \frac{1}{2}]$, let $\mathcal{V}_u^a$ be the norm closure of $\{\Delta^a Au : A \in \mathcal{R}^+\}$. (The notation $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ of Theorem 4 is in agreement with the definition of $\mathcal{V}_u^a$ by virtue of Theorem 4(v) and (vi).) Let a' be $\frac{1}{2} - a$. Then*

(i) $\mathcal{V}_u^a$ is a (closed) cone and

$$J\Delta^a Au = \Delta^{a'}Au \quad (A \in \mathcal{R}^+), \qquad J\mathcal{V}_u^a = \mathcal{V}_u^{a'};$$

(ii) $\mathcal{W}_u^0$, the real-linear span of $\mathcal{V}_u^0$, is contained in the domain of $\Delta^{1/2}$ and

$$\Delta^{1/2}y = Jy, \qquad \|\Delta^{1/2}y\| = \|y\| \qquad (y \in \mathcal{W}_u^0)$$

(iii) $\|Hx\| \le \|Kx\|$ when $x \in \mathcal{D}(H) \cap \mathcal{D}(K)$ and H and K are self-adjoint operators affiliated with an abelian von Neumann algebra such that $H^2 \le K^2$;

(iv)

$$\|\Delta^a y\| \le 2^{1/2}\|y\| \qquad (y \in \mathcal{W}_u^0);$$

(v) $\Delta^a\mathcal{V}_u^0$ is dense in $\mathcal{V}_u^a$;

(vi) $\mathcal{V}_u^a$ and $\mathcal{V}_u^{a'}$ are dual cones; in particular, $\mathcal{V}_u^{1/4}$ is self-dual.

Proof. (i) Since $\Delta^{1/2} = \Delta^{a'}\Delta^a$ (from 5.6.(18)), $\mathcal{R}u \subseteq \mathcal{D}(\Delta^{1/2}) \subseteq \mathcal{D}(\Delta^a)$. With A and B in $\mathcal{R}^+$ and b a positive number,

$$\Delta^a Au + b\Delta^a Bu = \Delta^a(A + bB)u \in \{\Delta^a Ku \ : \ K \in \mathcal{R}^+\}\,.$$

When H and K are in $\mathcal{R}^+$, there is a K' in $\mathcal{R}'^+$ such that

$$\langle \Delta^a Hu, \Delta^{a'}Ku\rangle = \langle Hu, \Delta^{\frac{1}{2}}Ku\rangle = \langle Hu, K'u\rangle \ge 0,$$

from Theorem 4(vi). Thus, with x in $\mathcal{V}_u^a$ and y in $\mathcal{V}_u^{a'}$, $\langle x, y\rangle \ge 0$. If x and $-x$ are in $\mathcal{V}_u^a$, then $\langle x, \Delta^{a'}Ku\rangle = 0$ when $K \in \mathcal{R}^+$. Since each operator in $\mathcal{R}$ is a linear combination of four operators in $\mathcal{R}^+$, x is orthogonal to $\Delta^{a'}\mathcal{R}u$. We note that $\Delta^{a'}\mathcal{R}u$ is dense in $\mathcal{H}$, from which $x = 0$.

Suppose that $y \in \mathcal{D}(\Delta^{a'})$. Let E_n be the spectral projection for Δ, corresponding to the interval $[0, n]$. Then $E_n y \in \mathcal{D}(\Delta^{\frac{1}{2}}) \subseteq \mathcal{D}(\Delta^{a'})$. Since $\mathcal{R}u$ is a core for $\Delta^{\frac{1}{2}}$, there is a sequence $\{A_m\}$ in $\mathcal{R}$ such that

$$A_m u \to_m E_n y, \qquad \Delta^{\frac{1}{2}} A_m u \to_m \Delta^{\frac{1}{2}} E_n y.$$

$\Delta^{a'}$ is everywhere defined and does not increase norm on $E_1(\mathcal{H})$. Δ^a does not decrease norm on $\mathcal{D}(\Delta^a) \cap (I - E_1)(\mathcal{H})$. Thus, with x_m for $A_m u - E_n y$, we have that

$$\begin{aligned} \|\Delta^{a'} A_m u - \Delta^{a'} E_n y\|^2 &= \|E_1 \Delta^{a'} x_m\|^2 + \|(I - E_1)\Delta^{a'} x_m\|^2 \\ &\le \|\Delta^{a'} E_1 x_m\|^2 + \|\Delta^a (I - E_1)\Delta^{a'} x_m\|^2 \\ &= \|\Delta^{a'} E_1 x_m\|^2 + \|(I - E_1)\Delta^{\frac{1}{2}} x_m\|^2 \\ &\to_m 0. \end{aligned}$$

Thus

$$\langle x, E_n \Delta^{a'} y\rangle = \langle x, \Delta^{a'} E_n y\rangle = \lim_m \langle x, \Delta^{a'} A_m u\rangle = 0.$$

Now $\lim_n E_n \Delta^{a'} y = \Delta^{a'} y$, whence $\langle x, \Delta^{a'} y\rangle = 0$ for all y in $\mathcal{D}(\Delta^{a'})$. Since $\Delta^{a'}$ is self-adjoint and one-to-one, its range is dense in $\mathcal{H}$, so $x = 0$. Thus $\{\Delta^a K u : K \in \mathcal{R}^+\}$ and its closure $\mathcal{V}_u^a$ are cones.

If $A \in \mathcal{R}^+$, then from Exercise 9.6.10,

$$\begin{aligned} J\Delta^a A u &= \Delta^{-a} J A u = \Delta^{-a} J S A u \\ &= \Delta^{-a} J J \Delta^{1/2} A u = \Delta^{a'} A u\,. \end{aligned}$$

Hence J maps a dense subset of $\mathcal{V}_u^a$ onto a dense subset of $\mathcal{V}_u^{a'}$. Since J is an isometry, $J\mathcal{V}_u^a = \mathcal{V}_u^{a'}$.

(ii) From (i) (when $a = 0$, $a' = \frac{1}{2}$), with A in $\mathcal{R}^+$,

$$\Delta^{1/2} A u = J A u\,.$$

Suppose $x \in \mathcal{V}_u^0$. Then x is the limit of a sequence $\{A_n u\}$ for some sequence $\{A_n\}$ of operators in $\mathcal{R}^+$. Since J is an isometry

$$\Delta^{1/2} A_n u = J A_n u \to J x\,.$$

As $\Delta^{1/2}$ is closed, $x \in \mathcal{D}(\Delta^{1/2})$ and $\Delta^{1/2} x = Jx$. Thus $\mathcal{W}_u^0 \subseteq \mathcal{D}(\Delta^{1/2})$, and $\Delta^{1/2} y = Jy$ for each y in $\mathcal{W}_u^0$. It follows that

$$\|y\| = \|Jy\| = \|\Delta^{1/2} y\|$$

for each y in $\mathcal{W}_u^0$.

(iii) From Theorem 5.6.15(i), there is a common bounding sequence $\{E_n\}$ for H, K, H^2, K^2. By assumption,

$$\langle H^2 E_n x, E_n x\rangle \le \langle K^2 E_n x, E_n x\rangle,$$

so $\|H E_n x\| \le \|K E_n x\|$ for each n. Since $x \in \mathcal{D}(H) \cap \mathcal{D}(K)$,

$$H E_n x = E_n H x \to H x$$

and

$$KE_nx = E_nKx \to Kx\,.$$

Hence $\|Hx\| \le \|Kx\|$.

(iv) Express Δ^a as $\Delta^a(I - E)\hat{+}\Delta^a E$, where E is the spectral projection for Δ corresponding to $[0, 1]$. Then $\|\Delta^a E\| \le 1$ and $\Delta^a(I - E) \le \Delta^{1/2}(I - E)$. We have that ΔE is everywhere defined, bounded, and $\|\Delta E\| \le 1$. Also, $(I-E)\Delta \subseteq \Delta(I-E)$ (and $\Delta(I-E)$ is closed and self-adjoint). By passing to the function representation, we see that $\Delta^a = \Delta^a(I-E)\hat{+}\Delta^a E$, $\|\Delta^a E\| \le 1$, and $[\Delta^a(I-E)]^2 \le [\Delta^{1/2}(I-E)]^2$. If $y \in \mathcal{D}(\Delta^{1/2})$ $(\subseteq \mathcal{D}(\Delta^a))$, then

$$y \in \mathcal{D}(\Delta^{1/2}(I - E)) \subseteq \mathcal{D}(\Delta^a(I - E)),$$

and

$$\Delta^{1/2}(I - E)y = (I - E)\Delta^{1/2}y, \quad \Delta^a(I - E)y = (I - E)\Delta^a y\,.$$

From (iii), we have

$$\|(I - E)\Delta^a y\| = \|\Delta^a(I - E)y\| \le \|\Delta^{1/2}(I - E)y\| = \|(I - E)\Delta^{1/2}y\|\,.$$

Thus, from (ii), if $y \in \mathcal{W}_u^0$,

$$\begin{aligned}\|\Delta^a y\|^2 &= \|(I - E)\Delta^a y\|^2 + \|E\Delta^a y\|^2 \\ &\le \|(I - E)\Delta^{1/2}y\|^2 + \|\Delta^a Ey\|^2 \\ &\le \|\Delta^{1/2}y\|^2 + \|y\|^2 \\ &= 2\|y\|^2\,.\end{aligned}$$

(v) Suppose $x \in \mathcal{V}_u^0$. Then x is the limit in $\mathcal{H}$ of $\{A_n u\}$ for some sequence $\{A_n\}$ of operators in $\mathcal{R}^+$. From (ii), $A_n u - x \in \mathcal{W}_u^0 \subseteq \mathcal{D}(\Delta^{1/2})$; from (iv)

$$\|\Delta^a(A_n u - x)\| \le 2^{\frac{1}{2}}\|A_n u - x\| \to 0,$$

so that $\Delta^a x \in \mathcal{V}_u^a$. Hence $\Delta^a\mathcal{V}_u^0 \subseteq \mathcal{V}_u^a$. Since $\{\Delta^a Au : A \in \mathcal{R}^+\}$ is dense in $\mathcal{V}_u^a$, $\Delta^a\mathcal{V}_u^0$ is dense in $\mathcal{V}_u^a$.

(vi) From the proof of (i), $\langle x, y\rangle \ge 0$ when $x \in \mathcal{V}_u^a$ and $y \in \mathcal{V}_u^{a'}$. Thus $\mathcal{V}_u^a$ and $\mathcal{V}_u^{a'}$ are contained in the dual cones of one another.

Suppose, now, that y is in the dual cone to $\mathcal{V}_u^a$ (that is, $\langle y, x\rangle \ge 0$ for each x in $\mathcal{V}_u^a$). Let $h_n(p)$ be $(2\pi)^{-\frac{1}{2}}(1 - \frac{1}{n}|p|)$ when $|p| \le n$, and let $h_n(p)$ be 0 when $n < |p|$, where n is a positive integer. (See the beginning of the proof of Theorem 3.2.30.) Then, by Theorem 3.2.30 calculations, $\hat{h}_n(t) = (1 - \cos nt)/\pi n t^2$ when $t \ne 0$ and $\hat{h}_n(0) = n/(2\pi)$. Since both h_n and $\hat{h}_n$ are continuous and in $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$, and $h_n(p) = h_n(-p)$, h_n is the Fourier transform of $\hat{h}_n$ (either from Theorem 3.2.30 or direct calculation). From the equation noted in the statement of Theorem 5.6.36,

$$(*) \qquad \langle h_n(\ln\Delta)y, x\rangle = \int_{\mathbb{R}} \hat{h}_n(t)\langle\Delta^{it}y, x\rangle\, dt\,.$$

Since $\Delta^{it}u = u$, from Remark 5.6.32, and $\Delta^{it}A\Delta^{-it} \in \mathcal{R}^+$ when $A \in \mathcal{R}^+$, $\Delta^{it}Au = \Delta^{it}A\Delta^{-it}u$ and $\Delta^{it}\Delta^a Au = \Delta^a\Delta^{it}Au \in \mathcal{V}_u^a$. Hence $\Delta^{it}\mathcal{V}_u^a = \mathcal{V}_u^a$ for each real t, and the unitary operator Δ^{it} maps the dual cone of $\mathcal{V}_u^a$ onto itself. Thus $0 \le \langle\Delta^{it}y, x\rangle$, and since $\hat{h}_n(t) \ge 0$ for each real t, $h_n(\ln\Delta)y$ is in the dual cone of $\mathcal{V}_u^a$ from $(*)$. Since $\{\sqrt{2\pi}\, h_n\}$ is monotone increasing with pointwise limit the constant function

1 on $\mathbb{R}$ and $h \to h(\ln \Delta)$ is a σ-normal homomorphism of $\mathcal{B}$ into the abelian von Neumann algebra generated by Δ (see Theorems 5.2.8 and 5.6.26), $\sqrt{2\pi}\, h_n(\ln \Delta)$ is strong-operator convergent to I. Thus

$$(y_n =)\ \sqrt{2\pi}\, h_n(\ln \Delta) y \to y.$$

We show, now, that $y_n \in \mathcal{V}_u^{a'}$. Since h_n vanishes outside a finite interval, $y_n \in \mathcal{D}(\Delta^t)$ for each real t. To see this, pass to the abelian von Neumann algebra generated by Δ and I and to the representing function algebra $C(X)$ for this von Neumann algebra. Then $h_n(\ln \Delta)$ is represented by $h_n \circ f$ in $C(X)$, where f represents $\ln \Delta$ in $\mathcal{N}(X)$. If q is a point in X at which the function representing Δ takes a value outside the interval $[\exp -n, \exp n]$, then $(h_n \circ f)(q) = 0$. It follows that $\Delta^t \hat{\cdot}\, h_n(\ln \Delta)$ is bounded for each real t. Since $h_n(\ln \Delta)$ is a bounded, everywhere-defined operator, $\Delta^t h_n(\ln \Delta)$ is closed, and densely defined. Thus $\Delta^t h_n(\ln \Delta) = \Delta^t \hat{\cdot}\, h_n(\ln \Delta)$. In particular, $y_n \in \mathcal{D}(\Delta^t)$ for each real t. Thus, with A' in $\mathcal{R}'^{+}$ and A equal to $JA'J$,

$$\begin{aligned}
0 \le \langle \Delta^a A u, y_n \rangle &= \langle \Delta^{-a'} \Delta^{1/2} A u, y_n \rangle \\
&= \langle JAu, \Delta^{-a'} y_n \rangle = \langle JAJu, \Delta^{-a'} y_n \rangle \\
&= \langle A' u, \Delta^{-a'} y_n \rangle,
\end{aligned}$$

from (ii). Hence $\Delta^{-a'} y_n \in \mathcal{V}_u^0$. From (v), $y_n \in \Delta^{a'} \mathcal{V}_u^0 \subseteq \mathcal{V}_u^{a'}$. It follows that $y \in \mathcal{V}_u^{a'}$ and that $\mathcal{V}_u^{a'}$ is the dual cone to $\mathcal{V}_u^a$. In particular, $\mathcal{V}_u^{1/4}$ is its own dual (we say that $\mathcal{V}_u^{1/4}$ is *self-dual*).

Theorem 7. *We adopt the notation of Theorem 6, but write $\mathcal{V}_u$ in place of $\mathcal{V}_u^{1/4}$. Let $\mathfrak{A}_0$ and $\mathcal{B}_0$ be the (strong-operator-dense) * subalgebras of $\mathcal{R}$ and $\mathcal{R}'$, respectively, consisting of elements in reflection sequences. (See Subsection 9.2, Tomita's theorem—a second approach.)*

(i) $A_0 J A_0 J u \in \mathcal{V}_u\ (A_0 \in \mathfrak{A}_0)$.

(ii) $AJAJu \in \mathcal{V}_u\ (A \in \mathcal{R})$.

(iii) $\{\Delta^{1/4} A_0^2 u : A_0 \in (\mathfrak{A}_0)_{\mathrm{h}}\}$ is dense in $\mathcal{V}_u$.

(iv) $\{AJAJu : A \in \mathcal{R}\} = \{A'JA'Ju : A' \in \mathbb{R}'\}$, and this set is dense in $\mathcal{V}_u$.

(v) $AJAJ\mathcal{V}_u \subseteq \mathcal{V}_u\ (A \in \mathcal{R})$.

Proof. (i) Let B in $\mathcal{R}$ be the extension of $\Delta^{-1/4} A_0 \Delta^{1/4}$. Then

$$\begin{aligned}
A_0 J A_0 J u = A_0 J A_0 u = A_0 J S A_0^* u &= A_0 \Delta^{1/2} A_0^* u \\
&= \Delta^{1/4} (\Delta^{-1/4} A_0 \Delta^{1/4})(\Delta^{-1/4} A_0 \Delta^{1/4})^* u \\
&= \Delta^{1/4} B B^* u \in \mathcal{V}_u .
\end{aligned}$$

(ii) It will suffice to show that $AJAJu \in \mathcal{V}_u$ for each A in $(\mathcal{R})_1$. Since $\mathfrak{A}_0$ is a strong-operator-dense * subalgebra of $\mathcal{R}$, A is in the strong-operator closure of $(\mathfrak{A}_0)_1$. Hence $AJAJ$ is in the strong-operator closure of $\{A_0 J A_0 J : A_0 \in (\mathfrak{A}_0)_1\}$. Thus $AJAJu \in \mathcal{V}_u$ from (i).

(iii) Suppose that x in $\mathcal{V}_u$ and a positive ϵ are given. Choose A in $\mathcal{R}^+$ such that $\|x - \Delta^{1/4} A u\| < \frac{1}{2}\epsilon$. From (the proof of) Corollary 5.3.6, there is a B in $(\mathfrak{A}_0)_{\mathrm{h}}$

such that $\|Au - B^2u\| < 2^{-3/2}\epsilon$. From Theorem 6(iv), since Au and B^2u are in $\mathcal{W}_u^0$,

$$\|\Delta^{1/4}Au - \Delta^{1/4}B^2u\| < 2^{1/2}2^{-3/2}\epsilon\,.$$

Hence $\|x - \Delta^{1/4}B^2u\| < \epsilon$. It follows that

$\{\Delta^{1/4}A_0^2u : A_0 \in (\mathfrak{A}_0)_{\mathrm{h}}\}$ is dense in $\mathcal{V}_u$.

(iv) With A in $\mathcal{R}$, $(A' =) JAJ \in \mathcal{R}'$, $JA'J = A$. Thus $AJAJ = JA'JA' = A'JA'J$, and

$$\{AJAJ : A \in \mathcal{R}\} = \{A'JA'J : A' \in \mathcal{R}'\}\,.$$

With A_0 in $(\mathfrak{A}_0)_{\mathrm{h}}$, $\Delta^{1/4}A_0\Delta^{-1/4}$ has a (unique) extension B in $\mathcal{R}$. We have,

$$\begin{aligned} BJBJu = BJBu &= BJSB^*u \\ &= B\Delta^{1/2}(\Delta^{-1/4}A_0\Delta^{1/4})u \\ &= B\Delta^{1/4}A_0u = \Delta^{1/4}A_0^2u\,. \end{aligned}$$

From (ii) and (iii), it follows now that $\{AJAJu : A \in \mathcal{R}\}$ is a dense subset of $\mathcal{V}_u$.

(v) With A, B in $\mathcal{R}$,

$$AJAJBJBJu = ABJAJJBJu = ABJABJu \in \mathcal{V}_u,$$

from (ii). Since $AJAJ$ is continuous, $\{BJBJu : B \in \mathcal{R}\}$ is dense in $\mathcal{V}_u$, and $\mathcal{V}_u$ is closed, $AJAJ\mathcal{V}_u \subseteq \mathcal{V}_u$.

Theorem 8. *We adopt the notation of Theorem 7. Suppose $x \in \mathcal{V}_u$. Then*

(i) $Jx = x$;

(ii) $JE = E'J$ and $JEE' = EE'J$, where E and E' are the projections with ranges $[\mathcal{R}'x]$ and $[\mathcal{R}x]$, respectively;

(iii) x is separating for $\mathcal{R}$ if and only if x is generating for $\mathcal{R}$;

(iv) if x is separating for $\mathcal{R}$ and J' is the modular conjugation corresponding to x, then $J' = J$.

Proof. (i) Note that $J\Delta^{1/4}Au = \Delta^{1/4}Au$, when $A \in \mathcal{R}^+$, from Theorem 6(i). Since J is continuous and $\mathcal{V}_u$ is the norm closure of $\{\Delta^{1/4}Au : A \in \mathcal{R}^+\}$, $Jx = x$.

(ii) From (i),

$$JTx = JTJx \in \mathcal{R}'x, \qquad JT'x = JT'Jx \in \mathcal{R}x,$$

when $T \in \mathcal{R}$ and $T' \in \mathcal{R}'$. Thus J maps $\mathcal{R}x$ isometrically onto $\mathcal{R}'x$; whence J maps $\{\mathcal{R}x\}^\perp$ isometrically onto $\{\mathcal{R}'x\}^\perp$. Hence

$$E'Jy = E'JEy + E'J(I - E)y = JEy$$

for each y in $\mathcal{H}$, and $E'J = JE$. Thus

$$JEE' = E'JE' = E'EJ = EE'J\,.$$

(iii) Note that x is separating for $\mathcal{R}$ if and only if $[\mathcal{R}'x]$ is $\mathcal{H}$, which occurs if and only if E is I. From (ii), E is I if and only if $I = JEJ = E'$. Thus, x is separating for $\mathcal{R}$ if and only if $[\mathcal{R}x]$ is $\mathcal{H}$, that is, if and only if x is generating for $\mathcal{R}$.

(iv) Since x is separating for $\mathcal{R}$, it is generating for $\mathcal{R}$, from (iii), and there is a modular structure associated with x. From (i), $Jx = x$; and of course $J^2 = I$

and $J\mathcal{R}J = \mathcal{R}'$. Moreover, $AJAJx \in \mathcal{V}_u$ for each A in $\mathcal{R}$, from Theorem 7(v). Thus $\langle AJAJx, x\rangle \geq 0$, since $\mathcal{V}_u$ is self-dual (Theorem 6(vi)). From Theorem 3(vi), $J = J'$.

Theorem 9. *We adopt the notation of Theorem 7. Let x, y, and v, be vectors in $\mathcal{V}_u$, and suppose that $0 = \langle x, y\rangle = \langle u, v\rangle$. Let E and E' be the projections with ranges $[\mathcal{R}'x]$ and $[\mathcal{R}x]$, respectively. Then*

(i) $v = 0$;

(ii) JEE' $(= J')$ is the conjugation for $E\mathcal{R}EE'$ $(= \S)$ acting on $EE'(\mathcal{H})$ with generating and separating vector x;

(iii) $\mathcal{V}_x' \subseteq \mathcal{V}_u$, where $\mathcal{V}_x'$ is the self-dual cone for $\{\S, x\}$ (corresponding to $\mathcal{V}_u$ for $\{\mathcal{R}, u\}$);

(iv) $EE'y = 0$.

Proof. (i) Choose A_n in $\mathcal{R}^+$ such that $\{\Delta^{1/4}A_n^2u\}$ tends to v and B' in $\mathcal{B}_0$. Then

$$\|A_nu\|^2 = \langle A_n^2u, u\rangle = \langle \Delta^{1/4}A_n^2u, u\rangle \to \langle v, u\rangle = 0\,.$$

Thus, with C' the extension of $\Delta^{1/4}B'\Delta^{-1/4}$ in $\mathcal{R}'$,

$$\langle \Delta^{1/4}A_n^2u, B'u\rangle \to \langle v, B'u\rangle$$

and

$$\begin{aligned}\langle \Delta^{1/4}A_n^2u, B'u\rangle &= \langle A_n^2u, \Delta^{1/4}B'\Delta^{-1/4}u\rangle\\ &= \langle A_nu, A_nC'u\rangle\\ &= \langle A_nu, C'A_nu\rangle\\ &\to 0\,.\end{aligned}$$

Hence, $\langle v, B'u\rangle = 0$. Since $\mathcal{B}_0u$ is dense in $\mathcal{H}$, $v = 0$.

(ii) Let $\mathcal{K}$ be $EE'(\mathcal{H})$ and J' be JEE'. Then J' is a conjugate linear isometry of $\mathcal{K}$ onto $\mathcal{K}$, from Theorem 8(ii). Moreover,

$$(J')^2 = JEE'JEE' = J^2(EE')^2 = EE',$$

so that J' is involutory. From Theorem 8(i), $J'x = x$, and

$$J'EE'AEE'J' = EE'JAJEE' \qquad (A \in \mathcal{R}),$$

whence $J'\S J' = \S'$. Finally, with A in $\mathcal{R}$,

$$\begin{aligned}\langle EE'AEE'J'EE'AEE'J'x, x\rangle &= \langle AJ'Ax, x\rangle\\ &= \langle E'EJAx, A^*x\rangle\\ &= \langle JE'Ax, E'A^*x\rangle\\ &= \langle JAx, A^*x\rangle \geq 0,\end{aligned}$$

from Theorem 7(v) and self-duality of $\mathcal{V}_u$ and since Ax and A^*x are in $E'(\mathcal{H})$. It follows from Theorem 3(vi) that J' is the modular conjugation for $\{\S, x\}$.

(iii) From Theorem 7(iv), $\{EE'AEE'J'EE'AEE'J'x \; : \; A \in \mathcal{R}\}$ is dense in $\mathcal{V}'_x$. From Theorem 7(v), since $x \in \mathcal{V}_u$

$$\begin{aligned} EE'AEE'J'EE'AEE'J'x &= J'AJ'EE'AEE'x \\ &= JEE'AJEE'Ax \\ &= JEAE'JEAx \\ &= JEAJEEAx \\ &= EAJEAJx \in \mathcal{V}_u \,. \end{aligned}$$

As $\mathcal{V}_u$ is closed, $\mathcal{V}'_x \subseteq \mathcal{V}_u$.

(iv) If $w \in \mathcal{V}'_x$, then $w \in \mathcal{V}_u$ from (iii), so that

$$\langle EE'y, w\rangle = \langle y, EE'w\rangle = \langle y, w\rangle \geq 0,$$

since y, $w \in \mathcal{V}_u$ and $\mathcal{V}_u$ is self-dual from Theorem 6(vi). As $\mathcal{V}'_x$ is self-dual, $EE'y \in \mathcal{V}'_x$. But

$$\langle EE'y, x\rangle = \langle y, EE'x\rangle = \langle y, x\rangle = 0,$$

so that, choosing $EE'y$ for v and the vector x for u in (i), we conclude that $EE'y = 0$.

Theorem 10. *With the notation of Theorem 9, let F and F' be the projections with ranges $[\mathcal{R}'y]$ and $[\mathcal{R}y]$, respectively. Let z be Ey, let z' be $E'y$, and let M, M', N, and N' be the projections with ranges $[\mathcal{R}'z]$, $[\mathcal{R}z]$, $[\mathcal{R}'z']$, and $[\mathcal{R}z']$, respectively. Then*

(i) $JM = N'J$, $JN = M'J$, $C_M = C_{M'} = C_N = C_{N'}$, $M \leq E$, and $N' \leq E'$;

(ii) if $z \neq 0$, there is a non-zero partial isometry U in $\mathcal{R}$ such that $U^*U \leq M$, $UU^* \leq N$, and $U^*Uz \neq 0$;

(iii) if $z \neq 0$ and G' is the projection with range $[\mathcal{R}U^*Uz]$, in the notation of (ii), there is a non-zero partial isometry V' in $\mathcal{R}'$ such that $V'^*V' \leq G' \leq M'$, $V'V'^* \leq N'$, and $V'^*V'U^*Uz \neq 0$;

(iv) if $z \neq 0$, then $UV'z$ is a non-zero vector in $NN'(\mathcal{H})$, and there is an A in $N\mathcal{R}N$ such that

$$0 < \langle UV'z, Az'\rangle = -\tfrac{1}{2}\langle BJBJy, y\rangle,$$

where $B = A^*U - JV'J \in \mathcal{R}$, in the notation of (iii);

(v) $Ey = E'y = 0$ and $EF = E'F' = 0$.

Proof. (i) From Theorem 8(ii), $JE = E'J$. Thus

$$Jz = JEy = E'Jy = E'y = z'\,.$$

Hence $JA'z = JA'Jz'$, and J maps $[\mathcal{R}'z]$ isometrically onto $[\mathcal{R}z']$. Hence $JM = N'J$. Similarly, $JA'z' = JA'Jz$, and $JN = M'J$. Since $A \to JA^*J$ is a $*$ anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$, this mapping preserves central carriers. Moreover, $JPJ = P$ for each central projection P in $\mathcal{R}$. Hence $C_N = C_{M'}$ and $C_M = C_{N'}$. From Proposition 5.5.13, $C_N = C_{N'}$ and $C_M = C_{M'}$. Finally, $M \leq E$ and $N' \leq E'$ since

$$\mathcal{R}'z = \mathcal{R}'Ey = E\mathcal{R}'y, \qquad \mathcal{R}z' = \mathcal{R}E'y = E'\mathcal{R}y\,.$$

(ii) If $z \neq 0$, then M and N are non-zero projections in $\mathcal{R}$. Since $C_M = C_N$, M and N have equivalent, non-zero subprojections from the comparison theorem (Theorem 6.2.7). Thus, there is a partial isometry U in $\mathcal{R}$ such that $0 < U^*U \leq M$ and $UU^* \leq N$. Moreover, $U^*Uz \neq 0$ since

$$[\mathcal{R}'U^*Uz] = [U^*U\mathcal{R}'z] = U^*UM(\mathcal{H}) \neq (0).$$

(iii) If $z \neq 0$, there is U as in (ii), and $U^*Uz \neq 0$. Since $[\mathcal{R}U^*Uz] \subseteq [\mathcal{R}z]$, $G' \leq M'$. Thus $0 \neq C_{G'} \leq C_{M'} = C_{N'}$, and there is a non-zero partial isometry V' in $\mathcal{R}'$ such that $V'^*V' \leq G'$ and $V'V'^* \leq N'$. Moreover, $V'^*V'U^*Uz \neq 0$ since

$$[\mathcal{R}V'^*V'U^*Uz] = [V'^*V'\mathcal{R}U^*Uz] = V'^*V'G'(\mathcal{H}) \neq (0).$$

(iv) If $z \neq 0$, then $V'^*U^*UV'z \neq 0$ from (iii), so $UV'z \neq 0$. Since U has range in $N(\mathcal{H})$ and V' has range in $N'(\mathcal{H})$,

$$NN'UV'z = NUN'V'z = UV'z.$$

Since N and N' have ranges $[\mathcal{R}'z']$ and $[\mathcal{R}z']$, respectively, z' is generating and separating for $N\mathcal{R}NN'$ acting on $NN'(\mathcal{H})$. Hence there is an A in $N\mathcal{R}N$ such that $AN'z'$ $(= Az')$ is near $UV'z$. Multiplying A by a suitable scalar, we may assume that $\langle UV'z, Az'\rangle > 0$. With B as defined,

$$BJBJ = A^*UJA^*UJ + JV'JV' - A^*UV' - JV'A^*UJ,$$

and

$$-\tfrac{1}{2}\langle BJBJy, y\rangle = -\tfrac{1}{2}\langle UJA^*Uy, Ay\rangle - \tfrac{1}{2}\langle y, V'JV'y\rangle + \mathrm{Re}\,\langle UV'y, Ay\rangle.$$

Now $V' = N'V' = E'N'V' = E'V'$ and $U = UM = UME = UE$ from (i), (ii), and (iii). Thus

$$\langle UV'y, Ay\rangle = \langle UEE'V'y, Ay\rangle = \langle UV'Ey, AE'y\rangle = \langle UV'z, Az'\rangle,$$

$$\begin{aligned}\langle y, V'JV'y\rangle &= \langle y, E'V'JE'V'y\rangle = \langle E'y, V'EJV'y\rangle \\ &= \langle EE'y, V'JV'y\rangle = 0,\end{aligned}$$

and

$$\begin{aligned}\langle UJA^*Uy, Ay\rangle &= \langle UEJA^*UEy, Ay\rangle = \langle UJE'A^*UEy, Ay\rangle \\ &= \langle UJA^*UE'Ey, Ay\rangle = 0,\end{aligned}$$

from Theorem 9(iv). Thus

$$0 < \langle UV'z, Az'\rangle = \langle UV'y, Ay\rangle = \mathrm{Re}\,\langle UV'y, Ay\rangle = -\tfrac{1}{2}\langle BJBJy, y\rangle.$$

(v) Since $y \in \mathcal{V}_u$, $BJBJ\mathcal{V}_u \subseteq \mathcal{V}_u$ from Theorem 7(v), and $\mathcal{V}_u$ is self-dual, we have that $0 \leq \langle BJBJy, y\rangle$. This inequality contradicts the conclusion of (iv), if $z \neq 0$. Thus $Ey = z = 0$ and $E'y = z' = Jz = 0$. It follows that $[\mathcal{R}'y] \subseteq (I-E)(\mathcal{H})$ and $[\mathcal{R}y] \subseteq (I-E')(\mathcal{H})$, whence $EF = E'F' = 0$.

Theorem 11. *With the notation of Theorem 7, let $\mathcal{H}_{\mathrm{r}}$ be $\{x : Jx = x\}$. Then*

(i) $\mathcal{H}_{\mathrm{r}}$ is a real Hilbert space relative to the structure imposed by $\mathcal{H}$;

(ii) each element of $\mathcal{H}$ has a decomposition $x_{\mathrm{r}} + ix_{\mathrm{i}}$ with x_{r} and x_{i} in $\mathcal{H}_{\mathrm{r}}$ and this decomposition is unique;

(iii) each element of $\mathcal{H}_{\mathrm{r}}$ has a unique decomposition $x_+ - x_-$, where x_+ and x_- are orthogonal vectors in $\mathcal{V}_u$

(iv) if $x, y \in \mathcal{V}_u$, then

$$\|x-y\|^2 \leq \|\omega_x|\mathcal{R} - \omega_y|\mathcal{R}\| \leq \|x-y\|\|x+y\|,$$

so that $\omega_x|\mathcal{R} = \omega_y|\mathcal{R}$ if and only if $x = y$;

(v) $\{\omega_x|\mathcal{R} : x \in \mathcal{V}_u\}$ is norm closed in $\mathcal{R}^\sharp$.

Proof. (i) With x and y in $\mathcal{H}_{\mathrm{r}}$ and a real, $J(ax+y) = aJx + Jy = ax + y$; thus $\mathcal{H}_{\mathrm{r}}$ is a linear space over $\mathbb{R}$. Since J is continuous, $\mathcal{H}_{\mathrm{r}}$ is a closed (real-linear) subspace of $\mathcal{H}$; hence $\mathcal{H}_{\mathrm{r}}$ is complete. Finally,

$$\langle x, y\rangle = \langle Jx, Jy\rangle = \langle y, J^*Jx\rangle = \langle y, x\rangle,$$

and $\mathcal{H}_{\mathrm{r}}$ is a real Hilbert space.

(ii) Let x_{r} be $(x+Jx)/2$ and let x_{i} be $(x-Jx)/(2i)$. Then $x = x_{\mathrm{r}} + ix_{\mathrm{i}}$, $Jx_{\mathrm{r}} = x_{\mathrm{r}}$, and $Jx_{\mathrm{i}} = x_{\mathrm{i}}$. Thus x_{r} and x_{i} are in $\mathcal{H}_{\mathrm{r}}$. Suppose $x = x'_{\mathrm{r}} + ix'_{\mathrm{i}}$ with x'_{r} and x'_{i} in $\mathcal{H}_{\mathrm{r}}$. Then

$$x + Jx = x'_{\mathrm{r}} + ix'_{\mathrm{i}} + x'_{\mathrm{r}} - ix'_{\mathrm{i}} = 2x'_{\mathrm{r}}$$

and

$$x - Jx = x'_{\mathrm{r}} + ix'_{\mathrm{i}} - x'_{\mathrm{r}} + ix'_{\mathrm{i}} = 2ix'_{\mathrm{i}}.$$

Thus $x'_{\mathrm{r}} = x_{\mathrm{r}}$ and $x'_{\mathrm{i}} = x_{\mathrm{i}}$.

(iii) Suppose $x \in \mathcal{H}_{\mathrm{r}}$. From Proposition 2.2.1, there is an element x_+ in $\mathcal{V}_u$ such that

$$\begin{aligned}\langle x_+, x - x_+\rangle &= \operatorname{Re}\langle x_+, x - x_+\rangle \\ (*) \qquad &\geq \operatorname{Re}\langle y, x - x_+\rangle \\ &= \langle y, x - x_+\rangle \qquad\qquad (y \in \mathcal{V}_u).\end{aligned}$$

Since $\mathcal{V}_u$ is a cone, $ay \in \mathcal{V}_u$ for each positive a when $y \in \mathcal{V}_u$. Thus $\langle y, x - x_+\rangle \leq 0$ for each y in $\mathcal{V}_u$. Since $\mathcal{V}_u$ is self-dual (Theorem 6(vi)), $(x_- =)x_+ - x \in \mathcal{V}_u$. Hence $x = x_+ - x_-$, and with 0 in place of y in $(*)$,

$$0 \geq -\langle x_+, x_-\rangle = \langle x_+, x - x_+\rangle \geq \langle 0, x - x_+\rangle = 0.$$

Thus x_+ and x_- are orthogonal vectors in $\mathcal{V}_u$.

If $x = x'_+ - x'_-$, where x'_+ and x'_- are orthogonal vectors in $\mathcal{V}_u$, then

$$\langle x_-, x_-\rangle = \langle x'_-, x_-\rangle - \langle x'_+, x_-\rangle \leq \langle x'_-, x_-\rangle \leq \|x'_-\|\|x_-\|.$$

Thus $\|x_-\| \leq \|x'_-\|$ and, by symmetry, $\|x'_-\| \leq \|x_-\|$. Hence

$$\|x - x'_+\| = \|x'_-\| = \|x_-\| = \|x - x_+\|.$$

By uniqueness of x_+ (as the vector in $\mathcal{V}_u$ nearest x), we have that $x_+ = x'_+$. It follows that $x_- = x'_-$.

(iv) From (iii), we can express $x - y$ as $v - w$, where v and w are in $\mathcal{V}_u$ and $\langle v, w\rangle = 0$. Let E and F be the projections with ranges $[\mathcal{R}'v]$ and $[\mathcal{R}'w]$, respectively. Since $EF = 0$, from Theorem 10(v), $\|E - F\| \le 1$. Thus, since $\mathcal{V}_u$ is self-dual,

$$\begin{aligned}
&\|\omega_x \,|\, \mathcal{R} - \omega_y \,|\, \mathcal{R}\| \\
&\ge |(\omega_x - \omega_y)(E - F)| \\
&= |\langle (E - F)x, x\rangle - \langle (E - F)y, y\rangle| \\
&= \tfrac{1}{2}|\langle (E - F)(x - y), x + y\rangle + \langle x + y, (E - F)(x - y)\rangle| \\
&= \tfrac{1}{2}|\langle (E - F)(v - w), x + y\rangle + \langle x + y, (E - F)(v - w)\rangle| \\
&= \tfrac{1}{2}|\langle v + w, x + y\rangle + \langle x + y, v + w\rangle| \\
&= \langle v + w, x + y\rangle \\
&= \langle v, x\rangle + \langle w, x\rangle + \langle v, y\rangle + \langle w, y\rangle \\
&\ge \langle v, x\rangle - \langle w, x\rangle - \langle v, y\rangle + \langle w, y\rangle \\
&= \langle v - w, x - y\rangle \\
&= \|x - y\|^2 .
\end{aligned}$$

With A in $\mathcal{R}$,

$$\begin{aligned}
|(\omega_x - \omega_y)(A)| &= |\langle Ax, x\rangle - \langle Ay, y\rangle| \\
&= \tfrac{1}{2}|\langle A(x + y), x - y\rangle + \langle A(x - y), x + y\rangle| \\
&\le \|A\|\|x + y\|\|x - y\|,
\end{aligned}$$

so that $\|\omega_x \,|\, \mathcal{R} - \omega_y \,|\, \mathcal{R}\| \le \|x + y\|\|x - y\|$.

(v) Suppose $x(n) \in \mathcal{V}_u$ and $\{\omega_{x(n)} \,|\, \mathcal{R}\}$ is Cauchy convergent. By virtue of the first inequality of (iv), $\{x(n)\}$ is now Cauchy convergent. Hence $\{x(n)\}$ tends to some y in $\mathcal{V}_u$. The second inequality of (iv) yields that $\{\omega_{x(n)} \,|\, \mathcal{R}\}$ converges to $\omega_y \,|\, \mathcal{R}$. Thus $\{\omega_x \,|\, \mathcal{R} : x \in \mathcal{V}_u\}$ is norm closed in $\mathcal{R}^\sharp$.

Theorem 12. *Let H be a positive invertible (possibly unbounded) operator on a Hilbert space $\mathcal{H}$.*

(i) $H^{1/4}(I + H^{1/2})^{-1}$ is a bounded, everywhere-defined operator on $\mathcal{H}$ and is equal to $(H^{1/4} \hat{+} H^{-1/4})^{-1}$.

(ii) With x and y in $\mathcal{H}$,

$$\langle (H^{1/4} \hat{+} H^{-1/4})^{-1}x, y\rangle = \int_{\mathbb{R}} (e^{\pi t} + e^{-\pi t})^{-1} \langle H^{it/2}x, y\rangle \, dt .$$

(iii) $\Delta^{1/4}(I + \Delta^{1/2})^{-1}\mathcal{V}_u^a \subseteq \mathcal{V}_u^a$ for each a in $[0, \frac{1}{2}]$, with the notation of Theorem 6.

Proof. (i) Passing to the function representation of the abelian von Neumann algebra generated by H, we have that the operators $(I + H^{1/2})^{-1}$ and $H^{1/4} \hat{\cdot} (I + H^{1/2})^{-1}$ are bounded, everywhere-defined operators. But $H^{1/4}(I + H^{1/2})^{-1}$ is closed since $H^{1/4}$ is closed and $(I + H^{1/2})^{-1}$ is bounded. Thus $H^{1/4}(I + H^{1/2})^{-1}$

is bounded and everywhere defined. Again, from the function representation,

$$(H^{1/4}\hat{+}H^{-1/4})^{-1} = H^{1/4}(I + H^{1/2})^{-1}.$$

(ii) The argument is divided into three stages. Consider, first, the case in which H has the form $\sum_{j=1}^{m} a_j F_j$, where $a_1, \dots, a_m$ are positive real numbers and $\{F_1, \dots, F_m\}$ is an orthogonal family of projections with sum I. In this case, from Lemma 9.2.7,

$$\begin{aligned}(H^{1/4}\hat{+}H^{-1/4})^{-1} &= \sum_{j=1}^{m}(a_j^{1/4} + a_j^{-1/4})^{-1}F_j\\ &= \sum_{j=1}^{m}\left(\int_{\mathbb{R}} \frac{a_j^{it/2}\,dt}{e^{\pi t} + e^{-\pi t}}\right)F_j\end{aligned}$$

so that

$$\begin{aligned}\langle (H^{1/4}\hat{+}H^{-1/4})^{-1}x, y\rangle &= \int_{\mathbb{R}} \frac{\langle (\sum_{j=1}^{m} a_j^{it/2}F_j)x, y\rangle\,dt}{e^{\pi t} + e^{-\pi t}}\\ &= \int_{\mathbb{R}} \frac{\langle H^{it/2}x, y\rangle\,dt}{e^{\pi t} + e^{-\pi t}}.\end{aligned}$$

We next consider the case in which H is bounded and has a bounded inverse, and choose positive real numbers a, b such that $aI \le H \le bI$. As in the proof of Lemma 9.2.8, H is the limit in norm of a sequence $\{H_n\}$ of operators, each of the type considered in the preceding paragraph and satisfying $aI \le H_n \le bI$; moreover, H^z and $(H^{1/4} + H^{-1/4})^{-1}$ are the norm limits of the sequences $\{H_n^z\}$, for each complex z, and $\{(H_n^{1/4} + H_n^{-1/4})^{-1}\}$, respectively. From the preceding paragraph,

$$\langle (H_n^{1/4} + H_n^{-1/4})^{-1}x, y\rangle = \int_{\mathbb{R}} \frac{\langle H_n^{it/2}x, y\rangle\,dt}{e^{\pi t} + e^{-\pi t}}.$$

Since $|\langle H_n^{\frac{1}{2}it}x, y\rangle| \le \|x\|\|y\|$ and $\int_{\mathbb{R}}(e^{\pi t} + e^{-\pi t})^{-1}\,dt$ is absolutely integrable, it follows from the dominated convergence theorem that

$$\langle (H^{1/4}\hat{+}H^{-1/4})^{-1}x, y\rangle = \int_{\mathbb{R}} \frac{\langle H^{it/2}x, y\rangle\,dt}{e^{\pi t} + e^{-\pi t}}.$$

Finally, we consider the general case, in which H is unbounded. For each positive integer n, let E_n be the spectral projection for H corresponding to the interval $[n^{-1}, n]$. Since H is a positive invertible operator, the increasing sequence $\{E_n\}$ is strong-operator convergent to I. For a given choice of n, let H_0 be the restriction to $E_n(\mathcal{H})$ of H. Then H_0 is in $\mathcal{B}(E_n(\mathcal{H}))^+$ and has a bounded inverse. When $x, y \in E_n(\mathcal{H})$, from Corollary 5.6.31 and the preceding paragraph,

$$\begin{aligned}\langle (H^{1/4}\hat{+}H^{-1/4})^{-1}x, y\rangle &= \langle (H_0^{1/4} + H_0^{-1/4})^{-1}x, y\rangle\\ &= \int_{\mathbb{R}} \frac{\langle H_0^{it/2}x, y\rangle\,dt}{e^{\pi t} + e^{-\pi t}}\\ &= \int_{\mathbb{R}} \frac{\langle H^{it/2}x, y\rangle\,dt}{e^{\pi t} + e^{-\pi t}}.\end{aligned}$$

For general x and y in $\mathcal{H}$, the preceding equality applies with $E_n x$ and $E_n y$ in place of x and y. Now

$$\begin{aligned}\langle (H^{1/4}\hat{+}H^{-1/4})^{-1}E_n x, E_n y\rangle &= \langle (H^{1/4}\hat{+}H^{-1/4})^{-1}x, E_n y\rangle \\ &\to \langle (H^{1/4}\hat{+}H^{-1/4})^{-1}x, y\rangle\end{aligned}$$

and

$$\langle H^{it/2}E_n x, E_n y\rangle = \langle H^{it/2}x, E_n y\rangle \to \langle H^{it/2}x, y\rangle .$$

From the dominated convergence theorem,

$$\langle (H^{1/4}\hat{+}H^{-1/4})^{-1}x, y\rangle = \int_{\mathbb{R}} \frac{\langle H^{it/2}x, y\rangle\, dt}{e^{\pi t} + e^{-\pi t}} .$$

(iii) Suppose $x \in \mathcal{V}_u^a$ and $y \in \mathcal{V}_u^{a'}$. Then, as in the proof of Theorem 6(vi), $\Delta^{it}x \in \mathcal{V}_u^a$ for each real t and $\langle \Delta^{it}x, y\rangle \geq 0$. From (i) and (ii),

$$\begin{aligned}\langle \Delta^{1/4}(I+\Delta^{1/2})^{-1}x, y\rangle &= \langle (\Delta^{1/4}\hat{+}\Delta^{-1/4})^{-1}x, y\rangle \\ &= \int_{\mathbb{R}} (e^{\pi t}+e^{-\pi t})^{-1}\langle \Delta^{it/2}x, y\rangle\, dt \\ &\geq 0 .\end{aligned}$$

Since $\mathcal{V}_u^a$ is the dual cone to $\mathcal{V}_u^{a'}$, $\Delta^{1/4}(I+\Delta^{1/2})^{-1}x \in \mathcal{V}_u^a$. Hence

$$\Delta^{1/4}(I+\Delta^{1/2})^{-1}\mathcal{V}_u^a \subseteq \mathcal{V}_u^a .$$

Theorem 13. *With the notation of Theorem 7, let ω be a normal linear functional on $\mathcal{R}$ such that $0 \leq \omega \leq \omega_u|\mathcal{R}$. From Proposition 7.3.5, there is an operator H' in $(\mathcal{R}'^{+})_1$ such that $\omega = \omega_{u,H'u}|\mathcal{R}$.*

(i) Suppose $x \in \mathcal{D}(\Delta^{-1/2}) \cap \mathcal{V}_u$ and

$$(*) \qquad \omega = \tfrac{1}{2}(\omega_{u,x} + \omega_{x,u})|\mathcal{R}.$$

Then $x = 2(I+\Delta^{1/2})^{-1}H'u$.

(ii) With x as in (i),

$$(**) \qquad x = 2\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4}H'u.$$

(iii) With x defined by (**), $\Delta^{-1/4}H'u \in \mathcal{V}_u$ and $x \in \mathcal{V}_u$.

(iv) Define x by (**). Then $x \in \mathcal{D}(\Delta^{-1/2}) \cap \mathcal{V}_u$ and (*) holds.

(v) With x as in (iv),

$$u - x = 2\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4}(I-H')u \in \mathcal{V}_u.$$

Proof. (i) By assumption and since $x \in \mathcal{D}(\Delta^{-1/2}) = \mathcal{D}(F)$, for each A in $\mathcal{R}$,

$$\begin{aligned}2\omega(A) &= \langle Au, x\rangle + \langle Ax, u\rangle = \langle Au, x\rangle + \langle x, SAu\rangle \\ &= \langle Au, x\rangle + \langle Au, Fx\rangle = \langle Au, x\rangle + \langle Au, \Delta^{1/2}Jx\rangle .\end{aligned}$$

But $x \in \mathcal{V}_u$, so that $Jx = x$ from Theorem 8(i). Thus

$$2\omega(A) = \langle Au, (I+\Delta^{1/2})x\rangle .$$

By choice of H', $\omega(A) = \langle Au, H'u\rangle$. Since u is generating for $\mathcal{R}$, $2H'u = (I+\Delta^{1/2})x$ and

$$x = 2(I+\Delta^{1/2})^{-1}H'u\,.$$

(ii) Note that $\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4} \subseteq (I+\Delta^{1/2})^{-1}$ and

$$\mathcal{R}'u \subseteq \mathcal{D}(F) = \mathcal{D}(\Delta^{-1/2}) = \mathcal{D}(\Delta^{-1/4}\Delta^{-1/4}) \subseteq \mathcal{D}(\Delta^{-1/4})\,.$$

From Theorem 12(i), $\Delta^{1/4}(I+\Delta^{1/2})^{-1}$ is bounded and everywhere defined, so that

$$\mathcal{R}'u \subseteq \mathcal{D}(\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4})\,.$$

Thus

$$x = 2(I+\Delta^{1/2})^{-1}H'u = 2\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4}H'u\,.$$

(iii) From (ii), $H'u \in \mathcal{D}(\Delta^{-1/4})$. Now

$$\Delta^{-1/4}H'u = JJ\Delta^{-1/4}JJH'Ju = J\Delta^{1/4}Hu,$$

where $H = JH'J \in \mathcal{R}^+$. By definition, $\Delta^{1/4}Hu \in \mathcal{V}_u$. From Theorem 8(i),we have that $J\Delta^{1/4}Hu = \Delta^{1/4}Hu$. Thus

$$\Delta^{-1/4}H'u = \Delta^{1/4}Hu \in \mathcal{V}_u\,.$$

By definition of x and Theorem 12(iii),

$$x = 2\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4}H'u \in \mathcal{V}_u\,.$$

(iv) From (iii), $x \in \mathcal{V}_u$. Now $x = 2(I+\Delta^{1/2})^{-1}H'u$, and

$$(I+\Delta^{1/2})^{-1}\Delta^{-1/2} \subseteq \Delta^{-1/2}(I+\Delta^{1/2})^{-1}\,.$$

Since $H'u \in \mathcal{D}(\Delta^{-1/2})$, $H'u \in \mathcal{D}(\Delta^{-1/2}(I+\Delta^{1/2})^{-1})$. It follows that $x \in \mathcal{D}(\Delta^{-1/2})$. At the same time, $(I+\Delta^{1/2})x = 2H'u$. Thus

$$\begin{aligned} 2\omega(A) &= \langle Au, 2H'u\rangle = \langle Au, (I+\Delta^{1/2})x\rangle \\ &= \langle Au, x\rangle + \langle Au, \Delta^{1/2}Jx\rangle = \langle Au, x\rangle + \langle Au, Fx\rangle \\ &= \langle Au, x\rangle + \langle x, A^*u\rangle = (\omega_{u,x} + \omega_{x,u})(A), \end{aligned}$$

for each A in $\mathcal{R}$.

(v) Since $0 \le H' \le I$, $I - H' \in \mathcal{R}'^+$, and the argument of (iii) applies with $I - H'$ in place of H' to show that

$$2(I+\Delta^{1/2})^{-1}(I-H')u = 2\Delta^{1/4}(I+\Delta^{1/2})^{-1}\Delta^{-1/4}(I-H')u \in \mathcal{V}_u\,.$$

As $(I+\Delta^{1/2})u = 2u$, $u = 2(I+\Delta^{1/2})^{-1}u$ and

$$u - x = 2(I+\Delta^{1/2})^{-1}(I-H')u \in \mathcal{V}_u\,.$$

Theorem 14. *With the notation of Theorem 13,*

(i) there is a y in $\mathcal{V}_u$ such that $u - y \in \mathcal{V}_u$ and

$$\omega_u|\mathcal{R} - \omega = \tfrac{1}{2}(\omega_{u,y} + \omega_{y,u})|\mathcal{R}\,;$$

(ii)

$$u - \tfrac{1}{2}y\,(=z) \in \mathcal{V}_u, \qquad \omega_z|\mathcal{R} - \omega = \omega_{(1/2)y}|\mathcal{R}$$

and

$$\|\omega_u|\mathcal{R} - \omega\| = \langle u, y\rangle$$

where y is as in (i);

(iii) with y and z as in (ii)

$$\|\omega_z|\mathcal{R} - \omega\| \le \tfrac{1}{4}\|\omega_u|\mathcal{R} - \omega\|\,;$$

(iv) with x' in $\mathcal{V}_u$ such that $\omega \le \omega_{x'}|\mathcal{R}$, E and E' the projections whose ranges are $[\mathcal{R}'x']$ and $[\mathcal{R}x']$, respectively, and $\mathcal{R}_0$ the von Neumann algebra $E\mathcal{R}EE'$ acting on $EE'(\mathcal{H})$ $(= \mathcal{H}_0)$, we have that x' is generating and separating for $\mathcal{R}_0$, $\mathcal{V}_{x'} \subseteq \mathcal{V}_u$, the equation

$$\omega_0(EAEE') = \omega(A) \qquad (A \in \mathcal{R})$$

defines a positive normal linear functional ω_0 on $\mathcal{R}_0$, and there is a vector z' in $\mathcal{V}_{x'}$, such that $\omega \le \omega_{z'}|\mathcal{R}$ and

$$\|\omega_{z'}|\mathcal{R} - \omega\| \le \tfrac{1}{4}\|\omega_{x'}|\mathcal{R} - \omega\|,$$

(v) there is a sequence $\{u(n)\}$ in $\mathcal{V}_u$ with u as $u(0)$ such that, $\omega \le \omega_{u(n)}|\mathcal{R}$,

$$\|\omega_{u(n)}|\mathcal{R} - \omega\| \le \tfrac{1}{4}\|\omega_{u(n-1)}|\mathcal{R} - \omega\|,$$

and $\{u(n)\}$ converges to some v in $\mathcal{V}_u$ such that $\omega_v|\mathcal{R} = \omega$;

(vi) the set of (normal) linear functionals ω' on $\mathcal{R}$ such that $0 \le a\omega' \le \omega_u|\mathcal{R}$ for some positive a is a norm-dense subset of the set of all vector functionals on $\mathcal{R}$, and each positive normal functional on $\mathcal{R}$ has a representation as $\omega_{v'}|\mathcal{R}$ for a unique v' in $\mathcal{V}_u$.

Proof. (i) Since $0 \le \omega \le \omega_u \,|\, \mathcal{R}$, we have

$$0 \le \omega_u \,|\, \mathcal{R} - \omega \le \omega_u \,|\, \mathcal{R},$$

and we may apply Theorem 13(iv) and (v) to $\omega_u \,|\, \mathcal{R} - \omega$ (in place of ω). Hence there is a y as described.

(ii) Since $\frac{1}{2}y$ and $u - y$ are in $\mathcal{V}_u$ and $\mathcal{V}_u$ is a cone,

$$(z =)\, u - \tfrac{1}{2}y = u - y + \tfrac{1}{2}y \in \mathcal{V}_u\,.$$

Note that

$$\begin{aligned}\omega_z \,|\, \mathcal{R} - \omega &= \omega_u \,|\, \mathcal{R} + \omega_{\frac{1}{2}y} \,|\, \mathcal{R} - \tfrac{1}{2}(\omega_{u,y} + \omega_{y,u}) \,|\, \mathcal{R} - \omega \\ &= \omega_u \,|\, \mathcal{R} + \omega_{\frac{1}{2}y} \,|\, \mathcal{R} - \omega_u \,|\, \mathcal{R} + \omega - \omega = \omega_{\frac{1}{2}y} \,|\, \mathcal{R}\end{aligned}$$

by choice of y, and that

$$\|\omega_u \,|\, \mathcal{R} - \omega\| = (\omega_u \,|\, \mathcal{R} - \omega)(I) = \tfrac{1}{2}(\langle u, y\rangle + \langle y, u\rangle) = \langle u, y\rangle$$

since $u, y \in \mathcal{V}_u$ and $\langle u, y\rangle = \langle y, u\rangle \ge 0$.

(iii) Since y and $u - y$ are in $\mathcal{V}_u$ and $\mathcal{V}_u$ is self-dual,

$$0 \le \langle y, y\rangle \le \langle u, y\rangle\,.$$

But from (ii),

$$\|\omega_z \,|\, \mathcal{R} - \omega\| = \omega_{\frac{1}{2}y}(I) = \tfrac{1}{4}\langle y, y\rangle \le \tfrac{1}{4}\langle u, y\rangle = \tfrac{1}{4}\|\omega_u \,|\, \mathcal{R} - \omega\|\,.$$

(iv) In the present case, x' takes the place of x in Theorem 9; we conclude from (ii) and (iii) of that theorem that x' is generating and separating for $\mathcal{R}_0$ acting on $\mathcal{H}_0$ and $\mathcal{V}_{x'} \subseteq \mathcal{V}_u$. (We use $\mathcal{V}_{x'}$ in place of $\mathcal{V}'_x$ of Theorem 9(iii).) Since $0 \le \omega \le \omega_{x'} \,|\, \mathcal{R}$, the support of ω is contained in E, from Remark 7.2.6, so that $\omega(A) = \omega(EAE)$ for each A in $\mathcal{R}$. Now $E'E$ has central carrier E (relative to $\mathcal{R}'E$), from Proposition 5.5.13, so that the mapping

$$EAE \to EAEE' \qquad (A \in \mathcal{R})$$

is a * isomorphism of $E\mathcal{R}E$ onto $E\mathcal{R}EE'$ from Proposition 5.5.5. Thus the equation $\omega_0(EAEE') = \omega(A)$ $(A \in \mathcal{R})$ defines a positive normal linear functional on $E\mathcal{R}EE'$. In addition, $\omega_0 \le \omega_{x'} \,|\, \mathcal{R}_0$. By applying the conclusion of (iii) to ω_0, $\mathcal{R}_0$, x', and $\mathcal{V}_{x'}$, we see that there is a vector z' in $\mathcal{V}_{x'}$ such that

$$\omega_0 \le \omega_{z'} \,|\, \mathcal{R}_0, \quad \|\omega_{z'} \,|\, \mathcal{R}_0 - \omega_0\| \le \tfrac{1}{4}\|\omega_{x'} \,|\, \mathcal{R}_0 - \omega_0\| .$$

Thus, with H in $\mathcal{R}^+$, since $EE'z' = z'$,

$$\omega(H) = \omega_0(EHEE') \le \omega_{z'}(EHEE') = \omega_{z'}(H),$$

and $\omega \le \omega_{z'} \,|\, \mathcal{R}$. In addition, since $\omega \le \omega_{x'} \,|\, \mathcal{R}$,

$$\begin{aligned} \|\omega_{z'} \,|\, \mathcal{R} - \omega\| &= (\omega_{z'} - \omega)(I) = \omega_{z'}(EE') - \omega(E) \\ &= (\omega_{z'} - \omega_0)(EE') = \|\omega_{z'} \,|\, \mathcal{R}_0 - \omega_0\| \\ &\le \tfrac{1}{4}\|\omega_{x'} \,|\, \mathcal{R}_0 - \omega_0\| = \tfrac{1}{4}\|\omega_{x'} \,|\, \mathcal{R} - \omega\| . \end{aligned}$$

(v) Let $u(0)$ be u and $u(1)$ be z (of (iii)). Suppose we have found $u(0), \ldots, u(n)$ with the properties described in the statement of this theorem. Then $u(n) \in \mathcal{V}_u$ and $\omega \le \omega_{u(n)} \,|\, \mathcal{R}$. From (iv), with $u(n)$ in place of x', there is a $u(n+1)$ (replacing z') in $\mathcal{V}_u$ such that $\omega \le \omega_{u(n+1)} \,|\, \mathcal{R}$ and

$$\|\omega_{u(n+1)} \,|\, \mathcal{R} - \omega\| \le \tfrac{1}{4}\|\omega_{u(n)} \,|\, \mathcal{R} - \omega\| .$$

The sequence $\{u(n)\}$ is constructed by this inductive process. It follows that $\{\omega_{u(n)} \,|\, \mathcal{R}\}$ converges to ω. From Theorem 11(iv), $\{u(n)\}$ is Cauchy convergent and therefore tends to a vector v in $\mathcal{V}_u$. Again, from Theorem 11(iv), $\{\omega_{u(n)} \,|\, \mathcal{R}\}$ tends to $\omega_v \,|\, \mathcal{R}$. Thus $\omega = \omega_v \,|\, \mathcal{R}$, and from Theorem 11(iv), v is the only such vector in $\mathcal{V}_u$.

(vi) If $A' \in \mathcal{R}'$ and $H \in \mathcal{R}^+$, then $0 \le A'^*A'H \le \|A'\|^2 H$ and

$$\begin{aligned} \omega_{A'u}(H) &= \langle HA'u, A'u\rangle = \langle A'^*A'Hu, u\rangle \\ &\le \|A'\|^2\langle Hu, u\rangle = \|A'\|^2\omega_u(H) . \end{aligned}$$

Since $\{A'u : A' \in \mathcal{R}'\}$ is dense in $\mathcal{H}$, the set $\S$ of positive (normal) linear functionals ω' such that $a\omega' \le \omega_u \,|\, \mathcal{R}$ for some positive scalar a is norm dense in the set of all vector functionals on $\mathcal{R}$. Since u is separating for $\mathcal{R}$, all positive normal linear functionals on $\mathcal{R}$ are vector functionals, from Theorem 7.2.3. Now each element of $\S$ has the form $\omega_{v'} \,|\, \mathcal{R}$ for some v' in $\mathcal{V}_u$, from (v), and the set of positive (normal) linear functionals on $\mathcal{R}$ that are representable in this form is a norm-closed subset of $\mathcal{R}^\sharp$, from Theorem 11(v). Thus each positive normal linear functional on $\mathcal{R}$ has the form $\omega_{v'} \,|\, \mathcal{R}$ for some v' in $\mathcal{V}_u$, and v' is unique from Theorem 11(iv).

Theorem 15. *Let $\mathcal{R}$ and $\S$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with separating and generating unit vectors u and v, respectively, and let ϕ be a * isomorphism of $\mathcal{R}$ onto $\S$. Let $\mathcal{V}_u$ and $\mathcal{V}_v$ be the respective self-dual cones for $\mathcal{R}$ and $\S$ corresponding to u and v.*

(i) With x in $\mathcal{V}_u$ a separating or generating vector for $\mathcal{R}$, we have that $\mathcal{V}_x = \mathcal{V}_u$.

(ii) There is a unique unitary transformation U of $\mathcal{H}$ onto $\mathcal{K}$ such that $UAU^{-1} = \phi(A)$ and $Uu' = v$ for some u' in $\mathcal{V}_u$.

(iii) With U as in (ii), $U\mathcal{V}_u = \mathcal{V}_v$.

(iv) With ω and ω' normal states of $\mathcal{R}$ and $\S$, respectively, denote by u_ω and $v_{\omega'}$ the (unique) vectors in $\mathcal{V}_u$ and $\mathcal{V}_v$ whose corresponding vector states are ω and ω', respectively. Then $Uu_{\omega'\circ\phi} = v_{\omega'}$ for each normal state ω' of $\S$, with U as in (ii).

(v) Suppose $\mathcal{R} = \S$ and $\mathcal{H} = \mathcal{K}$. There is a unique unitary operator U' in $\mathcal{R}'$ such that $U'u_\omega = v_\omega$ for each normal state ω of $\mathcal{R}$.

(vi) With the assumption of (v), J the modular conjugation operator for $(\mathcal{R}, u)$ and J' the modular conjugation operator for $(\mathcal{R}, v)$, there is a unitary operator V in $\mathcal{R}$ such that $VAV^* = JJ'AJ'J$ for all A in $\mathcal{R}$.

Proof. (i) From Theorem 8(iii), x is both generating *and* separating for $\mathcal{R}$ when $x \in \mathcal{V}_u$ and it is *either* generating *or* separating for $\mathcal{R}$. Thus, from Theorem 9(iii), $\mathcal{V}_x \subseteq \mathcal{V}_u$, where $\mathcal{V}_x$ is the self-dual cone corresponding to $(\mathcal{R}, x)$. Theorem 8(iv) assures us that the modular conjugations corresponding to x and u are the same. Thus u is a "real" element relative to $\mathcal{V}_x$ (that is, is in $\mathcal{H}_{\mathrm{r}}$ relative to the modular conjugation for $(\mathcal{R}, x)$) in the sense of Theorem 11. From (iii) of that theorem, $u = u_+ - u_-$, where u_+ and u_- are orthogonal elements of $\mathcal{V}_x$ (and, hence, of $\mathcal{V}_u$). But $u - 0$ is another decomposition of u as a difference of orthogonal elements of $\mathcal{V}_u$. The uniqueness clause of Theorem 11(iii) allows us to conclude, now, that $u_- = 0$ and $u = u_+ \in \mathcal{V}_x$. From Theorem 9(iii), again, $\mathcal{V}_u \subseteq \mathcal{V}_x$. Hence $\mathcal{V}_x = \mathcal{V}_u$.

(ii) Since v is separating for $\S$, $\omega_v \,|\, \S$ is faithful (on $\S$)and $(\omega_v \,|\, \S) \circ \phi$ is faithful on $\mathcal{R}$. From Theorem 14(vi), there is a (unique) vector u' in $\mathcal{V}_u$ such that $\omega_{u'} \,|\, \mathcal{R} = (\omega_v \,|\, \S) \circ \phi$. As $\omega_{u'} \,|\, \mathcal{R}$ is faithful, u' is a separating vector for $\mathcal{R}$. From Theorem 8(iii), u' is generating for $\mathcal{R}$. From Exercise 7.6.23, the mapping $Au' \to \phi(A)v$ $(A \in \mathcal{R})$ extends to a unitary transformation U of $\mathcal{H}$ onto $\mathcal{K}$ such that, for each A in $\mathcal{R}$, $UAU^{-1} = \phi(A)$. Choosing I for A, we have that $Uu' = v$.

Suppose, now, that U' is a unitary transformation of $\mathcal{H}$ onto $\mathcal{K}$ such that $U'AU'^{-1} = \phi(A)$ $(A \in \mathcal{R})$ and $U'x = v$ for some vector x in $\mathcal{V}_u$. Then, with A in $\mathcal{R}$,

$$U'Ax = U'AU'^{-1}U'x = \phi(A)v, \quad Ax = U'^{-1}\phi(A)v, \quad x = U'^{-1}v\,.$$

Now

$$\begin{aligned}[(\omega_v \,|\, \S) \circ \phi](A) &= \langle \phi(A)v, v\rangle = \langle AU'^{-1}v, U'^{-1}v\rangle \\ &= \langle Ax, x\rangle = \omega_x(A)\,.\end{aligned}$$

But u' is the only vector in $\mathcal{V}_u$ whose corresponding functional is $(\omega_v \,|\, \S) \circ \phi$. Thus $u' = x$ and $U'Au' = \phi(A)v = UAu'$ $(A \in \mathcal{R})$. Hence $U = U'$.

(iii) From (i), with u' as in (ii), $\mathcal{V}_{u'} = \mathcal{V}_u$. Since U is a unitary transformation of $\mathcal{H}$ onto $\mathcal{K}$ taking u' onto v such that $U\mathcal{R}U^{-1} = \S$, we have that

$$U\mathcal{V}_u = U\mathcal{V}_{u'} = \mathcal{V}_v .$$

(iv) Let U be as in (ii) and note that $U^{-1}v_{\omega'} \in \mathcal{V}_u$ from (iii). Now, with A in $\mathcal{R}$,

$$\begin{aligned}\langle AU^{-1}v_{\omega'}, U^{-1}v_{\omega'}\rangle &= \langle UAU^{-1}v_{\omega'}, v_{\omega'}\rangle \\ &= \langle \phi(A)v_{\omega'}, v_{\omega'}\rangle \\ &= (\omega' \circ \phi)(A) .\end{aligned}$$

But $u_{\omega'\circ\phi}$ is the only vector in $\mathcal{V}_u$ whose corresponding functional is $\omega' \circ \phi$. Thus $u_{\omega'\circ\phi} = U^{-1}v_{\omega'}$ and $Uu_{\omega'\circ\phi} = v_{\omega'}$.

(v) Under the present assumption, let U' be the (unique) unitary operator of (ii) corresponding to the identity automorphism of $\mathcal{R}$. Then $U'AU'^{-1} = A$ $(A \in \mathcal{R})$; whence $U' \in \mathcal{R}'$. Moreover, from (iv), $U'u_\omega = v_\omega$ for each normal state ω of $\mathcal{R}$.

(vi) Since $A \to J'A^*J'$ and $A' \to JA'^*J$ are * anti-isomorphisms of $\mathcal{R}$ onto $\mathcal{R}'$ and $\mathcal{R}'$ onto $\mathcal{R}$, respectively, and $(J'A^*J')^* = J'AJ'$, the mapping

$$A \to JJ'AJ'J \qquad (A \in \mathcal{R})$$

is a * automorphism of $\mathcal{R}$. Let U' be the unitary operator in $\mathcal{R}'$ described in (v) and (from Theorem 14(vi)) let u' be the (unique) vector in $\mathcal{V}_u$ whose corresponding vector state on $\mathcal{R}$ is $\omega_v \,|\, \mathcal{R}$. From (v), $U'u' = v$. From Theorem 8(iv), the modular conjugation for $(\mathcal{R}, u')$ is J. (Of course u' is separating and generating for $\mathcal{R}$ since v is.) As $U'\mathcal{R}U'^* = \mathcal{R}$ and $U'u' = v$, we have that $U'JU'^* = J'$. Let V be $JU'J$. Then V is a unitary operator in $\mathcal{R}$, and for each A in $\mathcal{R}$,

$$\begin{aligned}VAV^* &= JU'JAJU'^*J = J(U'JU'^*)U'AU'^*(U'JU'^*)J \\ &= JJ'U'AU'^*J'J = JJ'AU'U'^*J'J \\ &= JJ'AJ'J .\end{aligned}$$

Hence the mapping $A \to JJ'AJ'J$ $(A \in \mathcal{R})$ is an *inner* * automorphism of $\mathcal{R}$ (implemented by V).

Theorem 16. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a generating and separating vector u. Let ψ be a * automorphism of $\mathcal{R}$ and U_ψ be the (unique) unitary operator on $\mathcal{H}$ (described in Theorem 15(ii) and (iii)) such that $\mathcal{U}_\psi\mathcal{V}_u = \mathcal{V}_u$ and*

$$\psi(A) = U_\psi AU_\psi^* \qquad (A \in \mathcal{R}).$$

*With ψ' another * automorphism of $\mathcal{R}$, we have that $U_{\psi\psi'} = U_\psi U_{\psi'}$ and the mapping $\psi \to U_\psi$ is a unitary representation of the group of * automorphisms of $\mathcal{R}$.*

Proof. Note that

$$U_{\psi\psi'}AU_{\psi\psi'}^* = (\psi\psi')(A) = U_\psi U_{\psi'}AU_{\psi'}^*U_\psi^* .$$

Now $U_\psi U_{\psi'}\mathcal{V}_u = U_\psi\mathcal{V}_u = \mathcal{V}_u$, from Theorem 15(iii). Thus, from the uniqueness assertion of Theorem 15(ii), $U_{\psi\psi'} = U_\psi U_{\psi'}$, and the mapping $\psi \to U_\psi$ is a unitary representation of the automorphism group of $\mathcal{R}$ on $\mathcal{H}$.

Appendix. The Friedrichs Extension

Proposition 1′. *Let A_0 be a closed, densely defined operator acting on a Hilbert space $\mathcal{H}$, and suppose $\langle A_0x, x\rangle \geq 0$ for each x in the domain $\mathcal{D}(A_0)$ of A_0.*

(i) Then A_0 and $A_0 + I$ are closed symmetric operators on $\mathcal{H}$.

(ii) Suppose A is a positive self-adjoint extension of A_0. Then $A+I$ is a positive self-adjoint extension of $A_0 + I$, $A + I$ is a one-to-one linear transformation with range $\mathcal{H}$, and the inverse B of $A + I$ is in $(\mathcal{B}(\mathcal{H}))^+$.

(iii) With B as in (ii), y in $\mathcal{H}$, and x in $\mathcal{D}(A_0)$ $(= \mathcal{D}(A_0 + I))$,

$$(*) \qquad \langle x, y\rangle = \langle (A_0 + I)x, By\rangle$$

and $By \in \mathcal{D}(A_0^*)$.

Proof. (i) From the second relation in Proposition 2.1.7, with x and y in $\mathcal{D}(A_0)$, A_0x in place of u and A_0y in place of v, we deduce 2.4(3) (p.102) with A_0 in place of T. Since each of the inner products on the right-hand side of 2.4(3) is real (when T is replaced by A_0), the vector entries can be interchanged in each of these inner products, yielding the right-hand side of the second relation of Proposition 2.1.7 with x, y, A_0x, A_0y, replacing u, v, x, y, respectively. Thus $\langle A_0x, y\rangle = \langle x, A_0y\rangle$, when $x, y \in \mathcal{D}(A_0)$. Hence $A_0 \subseteq A_0^*$.

Since $\langle (A_0 + I)x, x\rangle \geq 0$ for each x in $\mathcal{D}(A_0)$ $(= \mathcal{D}(A_0 + I))$, $A_0 + I$ is also symmetric. If $x_n \in \mathcal{D}(A_0)$, $x_n \to x$, and $(A_0+I)x_n \to y$, then $A_0x_n \to y-x$. Since A_0 is closed, $x \in \mathcal{D}(A_0)$ and $A_0x = y - x$. Hence $x \in \mathcal{D}(A_0+I)$ and $(A_0+I)x = y$. It follows that $A_0 + I$ is closed.

(ii) Since $A + I$ is closed (as just argued for $A_0 + I$), $A + I = A\hat{+}I$, and $A + I$ is self-adjoint. Moreover,

$$\langle (A + I)x, x\rangle = \langle Ax, x\rangle + \|x\|^2 \geq \|x\|^2 \geq 0,$$

for each x in $\mathcal{D}(A)(= \mathcal{D}(A + I))$; $A + I$ is a positive self-adjoint operator with null space (0). Hence $A + I$ is a one-to-one linear transformation with range dense in $\mathcal{H}$ (from Exercise 2.8.45, the closure of the range of $A + I$ $(= (A + I)^*)$ is the orthogonal complement of the null space of $A + I$). From Lemma 2.7.9, $A + I$ has closed range. Hence $\mathcal{H}$ is the range of $A + I$.

If $y = (A + I)x$ and B is the inverse mapping to $A + I$, then

$$0 \leq \|By\|^2 = \|x\|^2 \leq \langle x, (A + I)x\rangle = \langle By, y\rangle \leq \|By\|\|y\|$$

so that $B \in (\mathcal{B}(\mathcal{H}))_1^+$.

(iii) Since $B \in (\mathcal{B}(\mathcal{H}))_1^+$, B is self-adjoint. By definition of B, $B(A_0 + I)x = x$ (for each x in $\mathcal{D}(A_0 + I)$). Thus

$$\langle x, y\rangle = \langle B(A_0 + I)x, y\rangle = \langle (A_0 + I)x, By\rangle,$$

with x and y as described in the statement of this exercise. Thus $By \in \mathcal{D}((A_0+I)^*)$ and $(A_0 + I)^*By = y$. We show that $(A_0 + I)^* = A_0^* + I$ — more generally, that $(T + S)^* = T^* + S^*$ when S is bounded. Suppose $v \in \mathcal{D}((T + S)^*)$ and $u \in \mathcal{D}(T)$ $(= \mathcal{D}(T + S))$. Then

$$\langle Tu, v\rangle = \langle (T + S)u, v\rangle - \langle Su, v\rangle = \langle u, (T + S)^*v - S^*v\rangle,$$

so that $v \in \mathcal{D}(T^*)$ and $T^*v = (T+S)^*v - S^*v$. It follows that $v \in \mathcal{D}(T^* + S^*)$ and $(T^* + S^*)v = (T+S)^*v$. Hence

$$(T+S)^* \subseteq T^* + S^*.$$

Since the reverse inclusion

$$T^* + S^* \subseteq (T+S)^*$$

is valid, in general,

$$(T+S)^* = T^* + S^*$$

when S is bounded. It follows that

$$By \in \mathcal{D}(A_0^*) \ (= \mathcal{D}(A_0^* + I) = \mathcal{D}((A_0 + I)^*)).$$

Theorem 2′. *With A_0 as in Proposition 1′, define $\langle u, v\rangle'$, for each pair of vectors u, v in $\mathcal{D}(A_0)$, to be $\langle (A_0 + I)u, v\rangle$ and let $\mathcal{D}'$ be the completion of $\mathcal{D}(A_0)$ relative to the definite inner product $(u, v) \to \langle u, v\rangle'$ on $\mathcal{D}(A_0)$.*

(i) The "identity" mapping on $\mathcal{D}(A_0)$ has a (unique) bounded extension ι mapping $\mathcal{D}'$ into $\mathcal{H}$, ι is one-to-one, and $\|\iota\| \le 1$.

(ii) With y in $\mathcal{H}$, $x \to \langle x, y\rangle$ $(x \in \mathcal{D}(A_0))$ extends to a bounded linear functional on $\mathcal{D}'$ of norm not exceeding $\|y\|$.

(iii) There is a vector By in $\mathcal{D}(A_0^*)$ satisfying $(*)$ of Proposition 1′(iii).

(iv) $B \in (\mathcal{B}(\mathcal{H}))_1^+$.

(v) B is a one-to-one mapping and its inverse A_1 is a self-adjoint extension of $A_0 + I$.

(vi) $A_1 - I$ (which we denote by 'A') is a positive self-adjoint extension of A_0, and $\mathcal{D}(A) \subseteq \mathcal{D}(A_0^*) \subseteq \iota(\mathcal{D}')$.

Proof. (i) Note that with x in $\mathcal{D}(A_0)$,

$$\|x\|^2 = \langle x, x\rangle \le \langle x, x\rangle + \langle A_0x, x\rangle = \|x\|'^2$$

so that the identity mapping of $\mathcal{D}(A_0)$ onto itself has a (unique) bounded extension ι mapping $\mathcal{D}'$ into $\mathcal{H}$, and $\|\iota\| \le 1$. Choose x_n in $\mathcal{D}(A_0)$ tending to z' in $\mathcal{D}'$. Assume that $\iota(z') = 0$. Since

$$\|x_n - \iota(z')\| = \|\iota(x_n) - \iota(z')\| \le \|x_n - z'\|' \to 0,$$

we have that $\|x_n\| \to 0$. Thus, for each m,

$$\begin{aligned}\langle z', x_m\rangle' &= \lim_n \langle x_n, x_m\rangle' = \lim_n \langle (A_0 + I)x_n, x_m\rangle \\ &= \lim_n \langle x_n, (A_0 + I)^*x_m\rangle = 0,\end{aligned}$$

since $x_m \in \mathcal{D}(A_0 + I) \subseteq \mathcal{D}((A_0 + I)^*)$ from Proposition 1′(i). But

$$\langle z', z'\rangle' = \lim_m \langle z', x_m\rangle' = 0,$$

so that $z' = 0$ and ι is one-to-one.

(ii) Since $|\langle x, y\rangle| \le \|x\|\|y\| \le \|x\|'\|y\|$ from (i), when $x \in \mathcal{D}(A_0)$ and $y \in \mathcal{H}$, we see that the functional $x \to \langle x, y\rangle$ on $\mathcal{D}(A_0)$ has bound not exceeding $\|y\|$ relative to the norm $x \to \|x\|'$. This functional extends (uniquely) to a functional of norm not exceeding $\|y\|$ on $\mathcal{D}'$.

(iii) From (ii) and Riesz's representation theorem (Theorem 2.3.1), there is a (unique) vector z' in $\mathcal{D}'$ such that $\langle x, y\rangle = \langle x, z'\rangle'$ for each x in $\mathcal{D}(A_0)$. Let By be $\iota(z')$. Choose x_n in $\mathcal{D}(A_0)$ so that $\{x_n\}$ tends to z' (in $\mathcal{D}'$). Then

$$\|x_n - \iota(z')\| = \|\iota(x_n) - \iota(z')\| \le \|x_n - z'\|' \to 0.$$

Thus

$$\begin{aligned}\langle x, y\rangle = \langle x, z'\rangle' = \lim_n \langle x, x_n\rangle' &= \lim_n \langle (A_0 + I)x, x_n\rangle \\ &= \langle (A_0 + I)x, \iota(z')\rangle = \langle (A_0 + I)x, By\rangle.\end{aligned}$$

(iv) From (i), (ii), and (iii) (and with the notation of the solution to (iii)), we have that $\|By\| = \|\iota(z')\| \le \|z'\|' \le \|y\|$. Hence $\|B\| \le 1$. Choose x_n in $\mathcal{D}(A_0)$ so that $\|x_n - z'\|'$ tends to 0. Then from (i),

$$\|x_n - By\| = \|\iota(x_n - z')\| \le \|x_n - z'\|' \to 0,$$

and from (iii), by choice of z',

$$\begin{aligned}\langle By, y\rangle = \lim_n \langle x_n, y\rangle = \lim_n \langle x_n, z'\rangle' &= \lim_n \langle x_n, \iota^{-1}(By)\rangle' \\ = \langle z', \iota^{-1}(By)\rangle' &= \|\iota^{-1}(By)\|'^2 \ge 0,\end{aligned}$$

or note that $\lim_n \langle x_n, z'\rangle' = \langle z', z'\rangle' \ge 0$. Thus $B \in (\mathcal{B}(\mathcal{H}))_1^+$.

(v) If $(0 \ne)\ y \in \mathcal{H}$, then $By \in \iota(\mathcal{D}')$ and

$$0 \ne \langle x, y\rangle = \langle x, \iota^{-1}(By)\rangle'$$

for some x in (the dense manifold) $\mathcal{D}(A_0)$. So $\iota^{-1}(By) \ne 0$. From (i), ι is a one-to-one mapping, whence $By \ne 0$. From the discussion following Theorem 7.2.1, with B in place of T (and now, the null space is (0)), the inverse A_1 to B is a self-adjoint operator with domain contained in $\iota(\mathcal{D}')$.

If $x, u \in \mathcal{D}(A_0)$, then from (iii) and Proposition 1′(i),

$$\langle u, x\rangle' = \langle (A_0 + I)u, x\rangle = \langle u, (A_0 + I)x\rangle = \langle u, \iota^{-1}(B(A_0 + I)x)\rangle'.$$

Since $\mathcal{D}(A_0)$ is dense in $\mathcal{D}'$, $x = \iota^{-1}(B(A_0 + I)x)$. From (i), $x = \iota(x) = B(A_0 + I)x$, whence

$$A_1 x = A_1 B(A_0 + I)x = (A_0 + I)x.$$

Thus A_1 is a self-adjoint extension of $A_0 + I$.

(vi) As in the proof of Proposition 1′(ii), $A_1 - I$ is self-adjoint. Since $A_0 + I \subseteq A_1$, $A_0 \subseteq A_1 - I\ (= A)$. Since $0 \le B \le I$, $B(I - B) \ge 0$. Hence

$$\begin{aligned}\langle ABy, By\rangle = \langle A_1 By, By\rangle - \langle By, By\rangle &= \langle y, By\rangle - \langle By, By\rangle \\ = \langle (I - B)y, By\rangle &= \langle B(I - B)y, y\rangle \ge 0.\end{aligned}$$

But $\mathcal{D}(A) = \mathcal{D}(A_1)$, and $\mathcal{D}(A_1)$ is the range of B. Thus $0 \le A$ and $\mathcal{D}(A) \subseteq \iota(\mathcal{D}')$.

Theorem 3′. *With the notation of Theorem 2′, we have that A is the* unique *positive self-adjoint extension (the* Friedrichs extension*) of A_0 whose domain is contained in $\iota(\mathcal{D}')$.*

Proof. From Proposition 1′(ii) and (iii), if A' is a positive self-adjoint extension of A_0, then there is an operator B' with the properties of B in that proposition. Thus $\langle (A_0 + I)x, (B - B')y\rangle = 0$ for each x in $\mathcal{D}(A_0)$. If, in addition, we assume that the domain of A' is contained in $\iota(\mathcal{D}')$, then there is a vector u' in $\mathcal{D}'$ such that $\iota(u') = (B - B')y$. Let $\{x_n\}$ be a sequence of vectors in $\mathcal{D}(A_0)$ tending to u' (in $\mathcal{D}'$). Then

$$\|x_m - (B - B')y\| = \|x_m - \iota(u')\| \leq \|x_m - u'\|' \to 0$$

$$\begin{aligned}\langle x_n, u' \rangle' &= \lim_m \langle x_n, x_m \rangle' = \lim_m \langle (A_0 + I)x_n, x_m \rangle \\ &= \langle (A_0 + I)x_n, (B - B')y \rangle = 0.\end{aligned}$$

Hence $\langle u', u' \rangle' = \lim_n \langle x_n, u' \rangle' = 0$, and $u' = 0$. It follows that $\iota(u') = (B - B')y = 0$ and $B = B'$. Since B and B' are the inverse mappings to A and A', respectively, $A = A'$.

Theorem 4$'$. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and A_0 be an operator affiliated with $\mathcal{R}$. Suppose that $\langle A_0 x, x \rangle \geq 0$ for each x in $\mathcal{D}(A_0)$. Then the Friedrichs extension of A_0 is affiliated with $\mathcal{R}$.*

Proof. From Proposition 1$'$(i), A_0 is symmetric. By definition of "affiliation," A_0 is closed. Theorems 2$'$ and 3$'$ guarantee the existence and uniqueness, respectively, of the Friedrichs extension A of A_0. Let U' be a unitary operator in $\mathcal{R}'$. Then $U'AU'^*$ is a positive self-adjoint extension of $U'A_0U'^*$ and $\mathcal{D}(U'AU'^*) \subseteq U'(\iota(\mathcal{D}'))$. Since $A_0 \ \eta \ \mathcal{R}$, $U'A_0U'^* = A_0$. Since A is unique (Theorem 3$'$), it remains to show that $U'(\iota(\mathcal{D}')) \subseteq \iota(\mathcal{D}')$.

Suppose $z \in \iota(\mathcal{D}')$ and $\iota(z') = z$ (with z' in $\mathcal{D}'$). Then $\{x_n\}$ tends to z' for some sequence $\{x_n\}$ in $\mathcal{D}(A_0)$. Since $A_0 \ \eta \ \mathcal{R}$, $U'(\mathcal{D}(A_0)) = \mathcal{D}(A_0)$ and $U'x_n \in \mathcal{D}(A_0)$. Now

$$\begin{aligned}\|U'x_n - U'x_m\|'^2 &= \langle (A_0 + I)U'(x_n - x_m), U'(x_n - x_m) \rangle \\ &= \langle (A_0 + I)(x_n - x_m), (x_n - x_m) \rangle \\ &= \|x_n - x_m\|'^2 \to 0\end{aligned}$$

as $n, m \to \infty$ since $\{x_n\}$ converges in $\mathcal{D}'$. Thus $\{U'x_n\}$ converges in $\mathcal{D}'$ to some u' and $\{U'x_n\}$ converges in $\mathcal{H}$ to $\iota(u')$. Since $\{x_n\}$ tends to z in $\mathcal{H}$, $\{U'x_n\}$ tends to $U'z$ in $\mathcal{H}$. Thus $U'z = \iota(u') \in \iota(\mathcal{D}')$ and $U'(\iota(\mathcal{D}')) \subseteq \iota(\mathcal{D}')$.

References

[A1] H. Araki, Positive cone, Radon–Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita–Takesaki theory, in "C-algebras and Their Applications to Statistical Mechanics and Quantum Field Theory" Proc. Internat. School of Physics "Enrico Fermi," Course LX, Varenna, D. Kastler, ed., 1973, pp. 64–100, North-Holland Publ., Amsterdam, 1976.*

[A2] H. Araki, Positive cones for von Neumann algebras, in "Operator Algebras and Applications" Proc. of Symposia in Pure Math., Vol. 38, Part 2, R. Kadison, ed., 1980, pp. 5–15 Amer. Math. Soc., Providence, 1982.

[C] A. Connes, Charactérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, Ann. Inst. Fourier, Grenoble 24 4, 1974, pp. 121–155.

[F] K. Friedrichs, Spektraltheorie halbbeschränkten Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren, Math. Ann. 109, 1934, pp. 465–487.

[H1] U. Haagerup, The standard form of von Neumann algebras, Preprint, Mathematical Institute, University of Copenhagen, 1973.

[H2] U. Haagerup, The standard form of von Neumann algebras, Math. Scand. 37, 1975, pp. 271–283.

[KR1–4] R. Kadison and J. Ringrose, "Fundamentals of the Theory of Operator Algebras." Vol. 1, 4th Printing, Vols. 2,3,4, 2nd Printing, Amer. Math. Soc., Providence, RI, 1997.

[T] M. Takesaki, "Tomita's Theory of Modular Hilbert Algebras and Its Applications." Lecture Notes in Mathematics, Vol. 128, Springer-Verlag, Heidelberg, 1970.

MATHEMATICS DEPARTMENT, UNIVERSITY OF PENNSYLVANIA