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On tensor products of von Neumann algebras

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Dedicated to Masamichi Takesaki for his many fundamental contributions to Non-commutative Analysis of the occasion of his sixtieth birthday

1. Introduction

Our goal in this article is the proof of a basic result for tensor products of von Neumann algebras. We study the question of when a von Neumann subalgebra $\mathscr{B}$ of the (von Neumann-algebra) tensor product $\mathscr{R}\bar{\otimes}\mathscr{S}$ of von Neumann algebras $\mathscr{R}$ and $\mathscr{S}$ "splits" as a tensor product $\mathscr{R}_0\bar{\otimes}\mathscr{S}_0$ of von Neumann subalgebras $\mathscr{R}_0$ of $\mathscr{R}$ and $\mathscr{S}_0$ of $\mathscr{S}$. We give a general answer to this question in Theorem B. Our principal result (Theorem A), the culmination of a series of arguments, asserts that this splitting *always* occurs when $\mathscr{R}$ is a factor (center consisting of scalar multiples of the identity I) and $\mathscr{B}$ contains $\mathscr{R}$ (rather, $\mathscr{R}\bar{\otimes}\mathbb{C}I$, where $\mathbb{C}$ is the complex numbers). This was proved in [Ge] for the case where $\mathscr{R}$ is a factor of type II_1 and $\mathscr{S}$ is a finite von Neumann algebra (in the sense of Murray and von Neumann [M-vN]). It is the key, in [Ge], to answering a question raised by Popa [P1] concerning maximal injective subalgebras of a von Neumann algebra. In the last section, we apply our general splitting result to an extension of this answer and then relate that extension to the powerful result of Connes [C] on the equivalence of hyperfiniteness and injectivity.

Our splitting results are not hard to verify for finite dimensional matrix algebras. The problems posed on passage to infinite dimensional, analytic-topological structures are another matter altogether. Forming the tensor product of finite dimensional algebraic structures is a basic and well-understood construction. For infinite dimensional, analytic-topological algebraic structures, this same construction, though still basic, acquires a bewildering complexity. Since Grothendieck's seminal work [Gr], it has been apparent that the infinite dimensional commutative case involves a host of subtleties and analytic difficulties. In the non-commutative, infinite dimensional case, the difficulties and subtleties are magnified manyfold. Assertions that are easily proved in finite dimensions may become false or very difficult to prove when transferred to that case. Some examples of these difficulties are instructive and useful for our later purposes.

If $\mathcal{M}$ is a central, simple algebra over $\mathbb{C}$ acting as linear transformations of a finite dimensional space $\mathcal{H}$ (over $\mathbb{C}$) into itself and $\mathcal{M}'$ is the set of linear transformations of $\mathcal{H}$ into itself each of which commutes with all the transformations in $\mathcal{M}$ (we call $\mathcal{M}'$ the *commutant* of $\mathcal{M}$), then $\mathcal{B}(\mathcal{H})$, the algebra of all linear transformations of $\mathcal{H}$ into itself, is generated, as an algebra, by $\mathcal{M}$ and $\mathcal{M}'$ and is isomorphic to $\mathcal{M} \otimes \mathcal{M}'$. The most direct infinite dimensional analogue of this fact results from allowing $\mathcal{H}$ to be an infinite dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ to be the algebra of all *bounded* linear transformations of $\mathcal{H}$ into itself (*operators*), and $\mathcal{M}$ to be one of the Murray–von Neumann factors [M-vN]. Under these circumstances, $\mathcal{M}$ and $\mathcal{M}'$ generate, algebraically, an algebra isomorphic to the *algebraic* tensor product $\mathcal{M} \otimes \mathcal{M}'$ of $\mathcal{M}$ and $\mathcal{M}'$. (This fact is non-trivial – see [M-vN] or [K1] for an extension of it.) The *ultraweak* closure (which is the "appropriate" closure) of this generated algebra is, indeed, $\mathcal{B}(\mathcal{H})$. But $\mathcal{B}(\mathcal{H})$ is not, in general, the von Neumann-algebra, tensor product of $\mathcal{M}$ and $\mathcal{M}'$. It is this tensor product precisely when $\mathcal{M}$ has a minimal idempotent – that is, when $\mathcal{M}$ is a "factor of type I." A large part of the point to non-commutative analysis over the past sixty years involves the fact that there are factors with no minimal idempotents – those of "types II and III" (loosely, those that do and those that do not admit a trace-like functional).

If we study *norm-closed* algebras of operators stable under the adjoint operation acting on a Hilbert space $\mathcal{H}$ (the so-called C*-algebras), tensor-product subtleties abound. Takesaki [Ta1] was the first to note, by clever example, that two C*-subalgebras of $\mathcal{B}(\mathcal{H})$ may generate, algebraically, their algebraic tensor product whose norm closure is just one of the many C*-algebras that are candidates for the role of C*-tensor product. Each of these has a norm that restricts to the original norm on the C*-algebras and is a "crossnorm" on the (dense) algebraic tensor product (that is, $\|A \otimes B\| = \|A\|\|B\|$ when A and B are in the respective algebras). Takesaki's example makes use of the C*-algebras generated by the left and right regular representations of the free (non-abelian) group $\mathbb{F}_2$ on two generators (acting on $l_2(\mathbb{F}_2)$). Certain C*-algebras (the so-called *nuclear* algebras), when tensored with any other C*-algebra allow just one cross-norm and a single C*-tensor product. (Among these are the commutative C*-algebras and the Glimm algebras [Gl]. See [K-R II; Sect. 11.3].) Very little of this is easy and all of it is crucial to the structural analysis of C*-algebras. (An excellent account of these matters appears in [E-L].)

Our arguments will make use of a range of operator (especially, von Neumann)-algebra techniques. (We refer to [K-R I–IV] as our primary reference for terminology and results.) Chief among these are the meaning process introduced by Dixmier (the "Dixmier Process" – see [K-R II; Sect. 8.3]) and its refinements (thoroughly developed in [H], [H-Z], and [R]), and the conditional-expectation, slice-mapping results of Tomiyama [To1, To2]. (See also [K-R IV; Exercises 8.7.23–24, 12.4.36].)

The main slice-mapping results make critical use of another basic fact about tensor products of von Neumann algebras. If $\mathscr{R}$ acts on $\mathscr{H}$ and $\mathscr{S}$ acts on $\mathscr{K}$, then $\mathscr{R}\bar{\otimes}\mathscr{S}$ acts on $\mathscr{H}\otimes\mathscr{K}$. For finite dimensional algebras $\mathscr{R}$ and $\mathscr{S}$, it is easily verified that $(\mathscr{R}\otimes\mathscr{S})' = \mathscr{R}'\otimes\mathscr{S}'$ – "the commutant of the tensor product is the tensor product of the communtants." The corresponding formula holds for von Neumann algebras (and the von Neumann-algebra tensor product), but remained an open problem for nearly twenty years – finally yielding to the powerful Tomita–Takesaki theory [T, Ta2]. Our Theorem A is a stronger result than the commutant formula (as demonstrated in [Ge]), but it relies on the slice-mapping results, which are tantamount to the commutant formula.

For the reader's convenience, we review the slice-mapping and conditional-expectation techniques in the next section. Sect. 3 contains the statements and proofs of our main results.

The first-named author wishes to express his gratitude to Uffe Haagerup for his hospitality during a part of the research on this article. In particular, discussions with him about his earlier work [H-Z] on the substance of Lemmas D and E (in the C*-algebra case) were very valuable – although the arguments for these lemmas proceed by quite different methods.

2. Preliminaries

Our account of the (non-commutative) conditional-expectation and slice-mapping results is drawn directly from [K-R II, IV]. Tomiyama [To1] shows that an idempotent Φ of norm 1 from a C*-algebra $\mathfrak{A}$ onto a C*-subalgebra $\mathfrak{B}$ is a conditional expectation:

(i) Φ is linear and positive, $\Phi(A) \geqq 0$ when $A \geqq 0$;

(ii) Φ is a $\mathfrak{B}$-bimodule mapping on $\mathfrak{A}$, $\Phi(B_1AB_2) = B_1\Phi(A)B_2$ for all B_1, B_2 in $\mathfrak{B}$ and all A in $\mathfrak{A}$.

We say that a conditional expectation Ψ of a von Neumann algebra $\mathscr{T}$ onto a von Neumann subalgebra $\mathscr{S}$ is *proper* when $\Psi(T)$ is in the ultraweak closure of the convex hull of $\{UTU^* : U \text{ unitary in } \mathscr{T}\}$ for each T in $\mathscr{T}$. In the next section, we shall show (Theorem C) that each von Neumann algebra has a proper conditional expectation onto its center – and this result will be one of the keys to our proof of Theorem A.

The tensor product $\mathscr{R}\bar{\otimes}\mathscr{S}$ of two von Neumann algebras $\mathscr{R}$ and $\mathscr{S}$ admits certain mappings, *slice mappings* [To2], that play a central role in our arguments. Suppose that ρ and σ are normal (that is, ultraweakly continuous) linear functionals on $\mathscr{R}$ and $\mathscr{S}$, respectively. There is a unique normal functional $\rho\otimes\sigma$ (their *product*) on $\mathscr{R}\bar{\otimes}\mathscr{S}$ satisfying the condition that $(\rho\otimes\sigma)(A\otimes B) = \rho(A)\sigma(B)$ for each A in $\mathscr{R}$ and B in $\mathscr{S}$. For this, there are representations of $\mathscr{R}$ and $\mathscr{S}$ on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, such that for each such ρ, there are vectors x and y in $\mathscr{H}$ for which $\rho(A) = \langle Ax, y\rangle$ for all A in $\mathscr{R}$. Similarly, for each such σ, there are vectors u and v in $\mathscr{K}$ such that $\sigma(B) = \langle Bu, v\rangle$ for all B in $\mathscr{S}$. Then $\rho\otimes\sigma$ corresponds to the two vectors

$x \otimes u$ and $y \otimes v$ in $\mathscr{H} \otimes \mathscr{K}$, the Hilbert space tensor product of $\mathscr{H}$ and $\mathscr{K}$, and $(\rho \otimes \sigma)(T) = \langle T(x \otimes u), y \otimes v \rangle$ for each T in $\mathscr{R} \bar{\otimes} \mathscr{S}$.

We denote by $\mathscr{R}_\#$ the linear space of all normal linear functionals on $\mathscr{R}$. With ρ in $\mathscr{R}_\#$ and $\|\rho\|$ the bound of ρ, the function $\rho \to \|\rho\|$ is a norm on $\mathscr{R}_\#$ with respect to which it is a Banach space. Of course, each A in $\mathscr{R}$ gives rise to a linear functional $\hat{A}$ on $\mathscr{R}_\#$ ($\hat{A}(\rho) = \rho(A)$). Each linear functional Γ on $\mathscr{R}_\#$ corresponds to an operator on $\mathscr{H}$, for with x and y in $\mathscr{H}$ corresponding to ρ (as before), the mapping $(x, y) \to \Gamma(\rho)$ is a bounded (conjugate-)bilinear functional on $\mathscr{H}$. The Riesz representation of such bilinear functionals gives us an operator T in $\mathscr{B}(\mathscr{H})$ corresponding to Γ. With the aid of von Neumann Double Commutant Theorem [vN] (see, also, [K-R; Theorem 5.3.1]), we see that $T \in \mathscr{R}$. Thus $\mathscr{R}$ is the norm dual of $\mathscr{R}_\#$. (A basic result of Sakai [S] characterizes the von Neumann algebras as those C*-algebras that are norm duals, and shows that the *predual* must be $\mathscr{R}_\#$.) The w*-topology on $\mathscr{R}$ (as the dual of $\mathscr{R}_\#$) coincides with the ultraweak topology on $\mathscr{R}$.

With ρ in $\mathscr{R}_\#$ and T in $\mathscr{R} \bar{\otimes} \mathscr{S}$, the mapping $\sigma \to (\rho \otimes \sigma)(T)$ is a bounded linear functional on $\mathscr{S}_\#$, hence, an element $\Psi_\rho(T)$ in $\mathscr{S}$. Symmetrically, we construct an operator $\Phi_\sigma(T)$ in $\mathscr{R}$. By definition

$$(*) \qquad \sigma(\Psi_\rho(T)) = (\rho \otimes \sigma)(T) = \rho(\Phi_\sigma(T)) \quad (T \in \mathscr{R} \bar{\otimes} \mathscr{S}).$$

From $(\rho \otimes \sigma)(A \otimes B) = \rho(A)\sigma(B)$, we conclude that $\Psi_\rho(A \otimes B) = \rho(A)B$ and $\Phi_\sigma(A \otimes B) = \sigma(B)A$ when $A \in \mathscr{R}$ and $B \in \mathscr{S}$. From its definition, and since we are dealing with ultraweakly continuous functionals ρ and σ, the mappings Ψ_ρ and Φ_σ are ultraweakly continuous. Again, the linearity of ρ and σ (combined with $(*)$) result in the linearity of Ψ_ρ and Φ_σ.

With A and B in $\mathscr{R}$, define $\rho'(C)$ to be $\rho(ACB)$. Studying ρ' in conjunction with $(*)$, we have that

$$\Phi_\sigma((A \otimes I)(R \otimes S)(B \otimes I)) = A\Phi_\sigma(R \otimes S)B \quad (R \in \mathscr{R}, S \in \mathscr{S})$$

and, thence, that

$$\Phi_\sigma((A \otimes I)T(B \otimes I)) = A\Phi_\sigma(T)B \quad (T \in \mathscr{R} \bar{\otimes} \mathscr{S}).$$

Symmetrically, with C and D in $\mathscr{S}$,

$$\Psi_\rho((I \otimes C)T(I \otimes D)) = C\Psi_\rho(T)D \quad (T \in \mathscr{R} \bar{\otimes} \mathscr{S}).$$

The mappings Ψ_ρ and Φ_σ, just described, are referred to as *slice mappings* (*of* $\mathscr{R} \bar{\otimes} \mathscr{S}$ *onto* $\mathscr{S}$ *and* $\mathscr{R}$ *corresponding to* ρ *and* σ, *respectively*). It will be convenient to have notation for the mappings $\tilde{\Psi}_\rho$ and $\tilde{\Phi}_\sigma$ defined by

$$\tilde{\Psi}_\rho(T) = I \otimes \Psi_\rho(T), \qquad \tilde{\Phi}_\sigma(T) = \Phi_\sigma(T) \otimes I \quad (T \in \mathscr{R} \bar{\otimes} \mathscr{S}).$$

We refer to $\tilde{\Psi}_\rho$ and $\tilde{\Phi}_\sigma$ as *tensor-slice mappings*. When ρ and σ are states (of $\mathscr{R}$ and $\mathscr{S}$, respectively – positive and 1 at I) in $\mathscr{R}_\#$ and $\mathscr{S}_\#$, then $\tilde{\Psi}_\rho$ and $\tilde{\Phi}_\sigma$ fulfill conditions (i) and (ii) (of this section) and are conditional expectations.

With $\mathscr{R}_0$ and $\mathscr{S}_0$ von Neumann subalgebras of $\mathscr{R}$ and $\mathscr{S}$ and R_0 and S_0 elements of $\mathscr{R}_0$ and $\mathscr{S}_0$, respectively, we have that

$$\Psi_\rho(R_0 \otimes S_0) = \rho(R_0)S_0 \in \mathscr{S}_0, \qquad \Phi_\sigma(R_0 \otimes \mathscr{S}_0) = \sigma(S_0)R_0 \in \mathscr{R}_0 .$$

By linearity and ultraweak continuity, Ψ_ρ and Φ_σ map $\mathscr{R}_0 \bar{\otimes} \mathscr{S}_0$ into $\mathscr{S}_0$ and $\mathscr{R}_0$, respectively.

From the Slice Mapping Theorem [To2], if $\Phi_\sigma(T) \in \mathscr{R}_0$ and $\Psi_\rho(T) \in \mathscr{S}_0$ for each σ in $\mathscr{S}_\#$ and ρ in $\mathscr{R}_\#$, with T in $\mathscr{R} \bar{\otimes} \mathscr{S}$, then $T \in \mathscr{R}_0 \bar{\otimes} \mathscr{S}_0$. This follows from a careful choice of vector functionals on $\mathscr{R}$ and $\mathscr{S}$ involving A' in $\mathscr{R}_0'$ and the properties of the slice mappings corresponding to these functionals. The crucial step puts T in $(\mathscr{R}_0' \bar{\otimes} \mathscr{S}_0')'$, which is $\mathscr{R}_0 \bar{\otimes} \mathscr{S}_0$ from the commutant formula for tensor products ([Ta2], [Ta3], [R-vD], [K-R II; Theorem 11.2.16]) and the Double Commutant Theorem [vN].

The splitting problem for tensor products of C*-algebras, in general, remains open. It would seem to require a development of slice-mapping theory for such tensor products. An interesting and clear account of such a development and the problems in it that remain open are to be found in [W]. An important extension of slice-mapping techniques to ultraweakly closed subspaces is studied in [Kr].

3. Main results

In this section, we prove the following theorem.

Theorem A. *If $\mathscr{M}$ is a factor, $\mathscr{S}$ is a von Neumann algebra, and $\mathscr{B}$ is a von Neumann subalgebra of $\mathscr{M} \bar{\otimes} \mathscr{S}$ that contains $\mathscr{M} \otimes \mathbb{C}I$, then $\mathscr{B} = \mathscr{M} \otimes \mathscr{T}$, where $\mathscr{T}$ is a von Neumann subalgebra of $\mathscr{S}$.*

Toward this end, we gather the following results.

Theorem B. *The von Neumann subalgebras of a tensor product of von Neumann algebras that split are precisely those that are stable under all tensor-slice mappings.*

Proof. Let $\mathscr{B}$ be a von Neumann subalgebra of the tensor product $\mathscr{R} \bar{\otimes} \mathscr{S}$ of the von Neumann algebras $\mathscr{R}$ and $\mathscr{S}$. Let ρ and σ be elements of $\mathscr{R}_\#$ and $\mathscr{S}_\#$, respectively, and $\tilde{\Psi}_\rho$ and $\tilde{\Phi}_\sigma$ their associated tensor-slice mappings.

Suppose, first, that $\mathscr{B} = \mathscr{R}_0 \bar{\otimes} \mathscr{S}_0$, where $\mathscr{R}_0$ and $\mathscr{S}_0$ are von Neumann subalgebras of $\mathscr{R}$ and $\mathscr{S}$, respectively. With T in $\mathscr{B}$, we have that $\tilde{\Psi}_\rho(T) \in \mathbb{C}I \bar{\otimes} \mathscr{S}_0$ and $\tilde{\Phi}_\sigma(T) \in \mathscr{R}_0 \bar{\otimes} \mathbb{C}I$ from the Slice Mapping Theorem (see [K-R IV; Exercise 12.4.36(iv)]). Thus $\mathscr{B}$ is stable under $\tilde{\Psi}_\rho$ and $\tilde{\Phi}_\sigma$.

Suppose, now, that $\mathscr{B}$ is stable under $\tilde{\Psi}_\rho$ and $\tilde{\Phi}_\sigma$, for all ρ in $\mathscr{R}_\#$ and σ in $\mathscr{S}_\#$. Let $\mathscr{R}_0$ be $\{R_0 \in \mathscr{R} : R_0 \otimes I \in \mathscr{B}\}$ and $\mathscr{S}_0$ be $\{S_0 \in \mathscr{S} : I \otimes S_0 \in \mathscr{B}\}$. Then $\mathscr{R}_0 \bar{\otimes} \mathscr{S}_0 \subseteq \mathscr{B}$. If $T \in \mathscr{B}$, then $\tilde{\Phi}_\sigma(T) \in \mathscr{B} \cap (\mathscr{R} \bar{\otimes} \mathbb{C}I)$, whence $\Phi_\sigma(T) \in \mathscr{R}_0$. Similarly, $\Psi_\rho(T) \in \mathscr{S}_0$. From the Slice Mapping Theorem (see [K-R IV; Exercise 12.4.36(v)]), $T \in \mathscr{R}_0 \bar{\otimes} \mathscr{S}_0$. Thus $\mathscr{B} = \mathscr{R}_0 \bar{\otimes} \mathscr{S}_0$, and $\mathscr{B}$ splits. □

From Theorem B, it will suffice to show that the tensor-slice mappings associated with elements of the predual of $\mathscr{M}$ map $\mathscr{B}$ into $\mathscr{B}$ in order to prove

Theorem A. Of course, the tensor-slice mappings associated with elements of the predual of $\mathscr{S}$ map $\mathscr{B}$ into $\mathscr{B}$ since they map $\mathscr{M} \bar{\otimes} \mathscr{S}$ into $\mathscr{M} \bar{\otimes} \mathbb{C}I$ which is contained in $\mathscr{B}$ by hypothesis. Our strategy is to show that, with ρ in $\mathscr{M}_{\#}$, and T in $\mathscr{B}$, we can approximate $\tilde{\Psi}_\rho(T)$ ultraweakly by convex combinations of operators of the form $(U \otimes I)T(U \otimes I)^*$, with U a unitary operator in $\mathscr{M}$. For notational convenience, we denote by $\tilde{\alpha}$ the mapping that carries T in $\mathscr{M} \bar{\otimes} \mathscr{S}$ to this combination and by $\tilde{\mathscr{D}}$ the family of all such mappings. If we achieve this approximation, the $\tilde{\Psi}_\rho(T)$ lies in $\mathscr{B}$ when $T \in \mathscr{B}$, for each $\tilde{\alpha}(T)$ is in $\mathscr{B}$ (recall that $\mathscr{M} \bar{\otimes} \mathbb{C}I \subseteq \mathscr{B}$), whence their ultraweak closure point $\tilde{\Psi}_\rho(T)$ lies in $\mathscr{B}$.

Our plan for achieving such an approximation relies on the fact that $\tilde{\Psi}_\rho$ maps, in effect, by "applying ρ to the $\mathscr{M}$ component of the 'tensor representation' of an operator" and this same effect can be obtained by the Dixmier Process applied to this same component. (This is somewhat oversimplified.) The meaning of this is easily understood in the case of operators in $\mathscr{M} \bar{\otimes} \mathscr{S}$ that are finite sums of simple tensors. It is less clear when applied to operators that are ultraweak limits of such sums but not themselves such sums; a "forbidden" interchange of limits is involved. It is somehow magically banished by Lemma F. The implementation of this line of argument motivates the results that follow.

With $\mathscr{R}$ a von Neumann algebra, we denote by $\mathscr{B}(\mathscr{R})$ the family of all bounded linear transformations of $\mathscr{R}$ into itself and by $\mathscr{B}_1(\mathscr{R})$ the closed unit ball of this Banach space (we write '$\mathscr{B}_1$' when there is no danger of mistaking which space is involved). We shall consider this Banach space topologized with the "point-ultraweak topology" and refer to this topology as "the v-topology" (so we shall speak of "v-limits" and "v-convergence"). A subbase for the open sets of this topology consists of those subsets of $\mathscr{B}(\mathscr{R})$ that map a given element of $\mathscr{R}$ into a given ultraweakly open subset of $\mathscr{R}$. A net $\{\beta_a\}_{a \in \mathbb{A}}$ in $\mathscr{B}(\mathscr{R})$ v-converges to β when $\{\beta_a(A)\}$ is ultraweakly convergent to $\beta(A)$ for each A in $\mathscr{R}$. From [K2; Sect. 2, Theorem] (with $\mathscr{R}$ for $\mathscr{S}$ and $\mathscr{S}'$ there), we have that $\mathscr{B}_1$ is v-compact. For this, we need the fact that the closed unit ball of $\mathscr{R}$ is ultraweakly compact – which also follows from [K2; Sect. 2, Theorem].

Theorem C. *Each von Neumann algebra $\mathscr{R}$ admits a proper conditional expectation onto its center. If $\mathscr{R}$ is finite, this proper conditional expectation is unique and coincides with the normalized center-valued trace on $\mathscr{R}$.*

Proof. Let $\mathscr{D}$ be the subset of $\mathscr{B}_1(\mathscr{R})$ consisting of those mappings α of the form

$$\alpha(A) = \sum_{j=1}^{n} a_j U_j A U_j^* \quad (A \in \mathscr{R})$$

where $a_j > 0, \sum_{j=1}^{n} a_j = 1$, and U_j in $\mathscr{R}$ is unitary. From [K-R II; Lemma 8.3.3], if $\{A_1, \ldots, A_n\}$, a finite subset of $\mathscr{R}$, and a positive ε are given, there is an α in $\mathscr{D}$ and elements $C_1, \ldots, C_n$ in the center $\mathscr{C}$ of $\mathscr{R}$ such that $\|\alpha(A_j) - C_j\| < \varepsilon$ $(j = 1, \ldots, n)$. Let $\|\alpha(\mathbb{F})\|_{\mathscr{C}}$ be $\max\{\|\alpha(A) + \mathscr{C}\| : A \in \mathbb{F}\}$ for each finite subset

$\mathbb{F}$ of $\mathscr{R}$ and each α in $\mathscr{D}$, where $\|\alpha(A)+\mathscr{C}\|$ is the norm of $\alpha(A)+\mathscr{C}$ in the quotient Banach space $\mathscr{R}/\mathscr{C}$. To each (non-empty) $\mathbb{F}$ in the family $\mathscr{F}$ of finite subsets of $\mathscr{R}$, we associate an $\alpha_{\mathbb{F}}$ in $\mathscr{D}$ such that $\|\alpha_{\mathbb{F}}(\mathbb{F})\|_{\mathscr{C}} < |\mathbb{F}|^{-1}$, where $|\mathbb{F}|$ is the cardinality of $\mathbb{F}$. With $\mathscr{F}$ partially ordered by inclusion, the net $\{\alpha_{\mathbb{F}}\}_{\mathbb{F}\in\mathscr{F}}$ in $\mathscr{B}_1$ has a ν-convergent, cofinal subnet $\{\alpha_{\mathbb{F}}\}_{\mathbb{F}\in\mathscr{F}_0}$ since $\mathscr{B}_1$ is ν-compact.

Let β be the ν-limit of $\{\alpha_{\mathbb{F}}\}_{\mathbb{F}\in\mathscr{F}_0}$. We show, first, that $\beta(A)\in\mathscr{C}$ for each A in $\mathscr{R}$. Given a unit vector x in $\mathscr{H}$, the Hilbert space on which $\mathscr{R}$ acts, an element B of norm 1 in $\mathscr{R}$, and a positive integer n, there is a subset $\mathbb{F}$ in $\mathscr{F}_0$ containing A with more than n elements such that

$$|\langle(\alpha_{\mathbb{F}}(A)-\beta(A))Bx,x\rangle| < \frac{1}{n}, \quad |\langle(\alpha_{\mathbb{F}}(A)-\beta(A))x,B^*x\rangle| < \frac{1}{n}$$

since $\{\alpha_{\mathbb{F}}\}_{\mathbb{F}\in\mathscr{F}_0}$ is cofinal in $\{\alpha_{\mathbb{F}}\}_{\mathbb{F}\in\mathscr{F}}$ and $\{\alpha_{\mathbb{F}}(A)\}_{\mathbb{F}\in\mathscr{F}_0}$ is ultraweakly convergent to $\beta(A)$. By choice of the mapping $\alpha_{\mathbb{F}}$, there is a C in $\mathscr{C}$ such that $\|\alpha_{\mathbb{F}}(A)-C\| < \frac{1}{n}$ since $A\in\mathbb{F}$ and $\|\alpha_{\mathbb{F}}(\mathbb{F})\|_{\mathscr{C}} < |\mathbb{F}|^{-1} < \frac{1}{n}$. Thus

$$\begin{aligned}|\langle(\beta(A)B-B\beta(A))x,x\rangle| &\leqq |\langle(\beta(A)-\alpha_{\mathbb{F}}(A))Bx,x\rangle| + |\langle(\alpha_{\mathbb{F}}(A)-C)Bx,x\rangle| \\ &\quad + |\langle(C-\alpha_{\mathbb{F}}(A))x,B^*x\rangle| + |\langle(\alpha_{\mathbb{F}}(A)-\beta(A))x,B^*x\rangle| \\ &< \frac{4}{n}.\end{aligned}$$

It follows that $\beta(A)B = B\beta(A)$ for all A and B in $\mathscr{R}$. Hence $\beta(A)\in\mathscr{C}$ for each A in $\mathscr{R}$.

With C in $\mathscr{C}$, $\alpha_{\mathbb{F}}(C) = C$ for each $\mathbb{F}$ in $\mathscr{F}$. Since $\{\alpha_{\mathbb{F}}(C)\}_{\mathbb{F}\in\mathscr{F}_0}$ tends ultraweakly to $\beta(C)$, we have that $\beta(C) = C$ for each C in $\mathscr{C}$. In particular, $\beta(I) = I$. If ρ is a state of $\mathscr{C}$, then $(\rho\circ\beta)(I) = 1$ and $\|\rho\circ\beta\| = 1$ since $\beta\in\mathscr{B}_1$. Thus $\rho\circ\beta$ is a state of $\mathscr{R}$, and β is a positive linear mapping of $\mathscr{R}$ into $\mathscr{C}$. (The positivity of β follows, also, from the fact that each $\alpha_{\mathbb{F}}$ is a positive linear mapping of $\mathscr{R}$ into $\mathscr{R}$.) Hence β is a conditional expectation of $\mathscr{R}$ onto $\mathscr{C}$ [To1]. (See also [K-R IV; Exercises 10.5.85–86].) Since $\beta(A)$ is the ultraweak limit of $\{\alpha_{\mathbb{F}}(A)\}_{\mathbb{F}\in\mathscr{F}_0}$ for each A in $\mathscr{R}$, β is a proper conditional expectation of $\mathscr{R}$ onto $\mathscr{C}$.

Suppose, now, that $\mathscr{R}$ is a finite von Neumann algebra and τ is its (unique) normalized, center-valued trace. Since $\tau(\alpha(A)) = \tau(A)$ for each α in $\mathscr{D}$ and each A in $\mathscr{R}$, τ is an ultraweakly continuous mapping of $\mathscr{R}$ into $\mathscr{R}$, and $\beta(A)$ is the ultraweak limit of $\{\alpha_{\mathbb{F}}(A)\}_{\mathbb{F}\in\mathscr{F}_0}$, we have that $\beta(A) = \tau(\beta(A)) = \tau(A)$ for each A in $\mathscr{R}$, and $\beta = \tau$. □

Lemma D. *If $\mathscr{M}$ is a countably decomposable factor of type* III *and β is a state of $\mathscr{M}$, then $\{\beta\circ\alpha : \alpha\in\mathscr{D}\}$ is a convex, w^*-dense subset of the state space of $\mathscr{M}$.*

Proof. Since the family $\mathscr{D}$ is a convex subset of positive linear mappings in $\mathscr{B}_1(\mathscr{M})$ that map I to I, the set $\{\beta\circ\alpha : \alpha\in\mathscr{D}\}$ is a convex set of states of $\mathscr{M}$. If T in $\mathscr{M}$ is not self-adjoint, then $T = A+iB$ with A and B self-adjoint and B non-zero. From [K-R; Lemma 1], there is a sequence $\{\alpha_n\}$ of mappings in $\mathscr{D}$ such that $\{\alpha_n(A)\}$ converges in norm to aI for some real a, and $\alpha_n(B)$ converges to bI for some real, non-zero b. Thus $\{(\beta \circ \alpha_n)(T)\}$ converges to $a + ib$. If $(\beta \circ \alpha)(T)$ is real for all α in $\mathscr{D}$, then T is self-adjoint.

Suppose, now, that A is self-adjoint and A is not positive. Then there is some a in $\mathrm{sp}(A)$ such that $a < 0$. From [K-R IV; Exercise 8.7.11], there is a sequence $\{\alpha_n'\}$ of mappings α_n' in $\mathscr{D}$ such that $\{\alpha_n'(A)\}$ converges in norm to aI. It follows that $\{(\beta \circ \alpha_n')(A)\}$ converges to a. Thus if $(\beta \circ \alpha)(A) \geqq 0$ for each α in $\mathscr{D}$, then $A \geqq 0$. In sum, then, if $(\beta \circ \alpha)(T) \geqq 0$ for each α in $\mathscr{D}$, then T is self-adjoint and positive. From [K3; Theorem (2.2)], the convex family of states $\{\beta \circ \alpha : \alpha \in \mathscr{D}\}$ is a "full family" of states and is w*-dense in the state space of $\mathscr{M}$. □

Lemma E. *If $\mathscr{M}$ is a countably decomposable factor of type* III *and ρ is a state of $\mathscr{M}$, the mapping $\tilde{\rho}$ of $\mathscr{M}$ into its center, defined at A as $\rho(A)I$, is the v-limit of a net $\{\alpha_a\}_{a \in \mathbb{A}}$ of elements α_a of $\mathscr{D}$.*

Proof. From Theorem C (and its proof), there is a state β of $\mathscr{M}$ such that $\tilde{\beta}$ is the v-limit of a net $\{\alpha_{\mathbb{F}}\}_{\mathbb{F} \in \mathscr{F}_0}$ of mappings $\alpha_{\mathbb{F}}$ in $\mathscr{D}$ such that $\|\alpha_{\mathbb{F}}(\mathbb{F})\|_{\mathscr{C}} < |\mathbb{F}|^{-1}$, where $\mathscr{C}$ is the center $\{zI : z \in \mathbb{C}\}$ of $\mathscr{M}$ and $\tilde{\beta}(A) = \beta(A)I$ for each A in $\mathscr{M}$.

Suppose that a finite subset $\{A_1, \dots, A_n\}$ $(= \mathbb{F}')$ of $\mathscr{M}$, vectors $x_1, \dots, x_m$, $y_1, \dots, y_m$ in $\mathscr{H}$, the Hilbert space on which $\mathscr{M}$ acts, and a positive ε are given. We shall find α in $\mathscr{D}$ such that

$$|\langle(\alpha(A_j) - \tilde{\rho}(A_j))x_k, y_k\rangle| < \varepsilon \quad (j = 1, \dots, n;\ k = 1, \dots, m)\,. \tag{1}$$

From Lemma D, $\{\beta \circ \alpha : \alpha \in \mathscr{D}\}$ is w*-dense in the state space of $\mathscr{M}$ since $\mathscr{M}$ is a countably decomposable factor of type III. Thus we may choose α_1 in $\mathscr{D}$ such that

$$|\rho(A_j) - \beta(\alpha_1(A_j))| < \frac{\varepsilon}{2c} \quad (j = 1, \dots, n)$$

where $c = 1 + \max\{\|x_k\|\|y_k\| : k = 1, \dots, m\}$. It follows that

$$|\langle(\tilde{\rho}(A_j) - \tilde{\beta}(\alpha_1(A_j)))x_k, y_k\rangle| < \frac{\varepsilon}{2} \quad (j = 1, \dots, n;\ k = 1, \dots, m)\,. \tag{2}$$

Using the net $\{\alpha_{\mathbb{F}}\}_{\mathbb{F} \in \mathscr{F}_0}$, we choose α_2 in $\mathscr{D}$ such that

$$|\langle(\alpha_2(\alpha_1(A_j)) - \tilde{\beta}(\alpha_1(A_j)))x_k, y_k\rangle| < \frac{\varepsilon}{2} \quad (j = 1, \dots, n;\ k = 1, \dots, m)\,. \tag{3}$$

Letting α (in $\mathscr{D}$) be $\alpha_2 \circ \alpha_1$ and combining (2) and (3), we have (1).

Let $\mathbb{A}$ be the set of ordered triples $\langle \mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3 \rangle$ where $\mathbb{F}_1$ is a finite subset of $\mathscr{M}$, $\mathbb{F}_2$ and $\mathbb{F}_3$ are finite subsets of $\mathscr{H}$ with the same cardinality. Partially order $\mathbb{A}$ by the relation

$$\langle \mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3 \rangle \leqq \langle \mathbb{F}_1', \mathbb{F}_2', \mathbb{F}_3' \rangle$$

when $\mathbb{F}_j \subseteq \mathbb{F}_j'$ $(j = 1, 2, 3)$. Then $\langle \mathbb{A}, \leqq \rangle$ is a directed set. For each a in $\mathbb{A}$, choose α_a in $\mathscr{D}$ such that (1) holds with $|\mathbb{F}_1|^{-1}$ in place of ε, where $\mathbb{F}_1$ is

$\{A_1,\ldots,A_n\}$, $\mathbb{F}_2$ is $\{x_1,\ldots,x_m\}$, and $\mathbb{F}_3$ is $\{y_1,\ldots,y_m\}$. Then $\tilde{\rho}$ is the ν-limit of $\{\alpha_a\}_{a\in\mathbb{A}}$. □

Lemma F. *Let $\mathscr{R}$ and $\mathscr{S}$ be von Neumann algebras, ρ in $\mathscr{R}_\#$, $\mathscr{D}$ the family described in the proof of Theorem C, $\tilde{D}$ the set of mappings $\alpha\otimes\iota$ $(=\tilde{\alpha})$ with α in $\mathscr{D}$ and ι the identity transform of $\mathscr{S}$ onto itself, and $\{\tilde{\alpha}_a\}_{a\in\mathbb{A}}$ a net in $\tilde{\mathscr{D}}$ ν-convergent to Ψ in $\mathscr{B}_1(\mathscr{R}\bar{\otimes}\mathscr{S})$ such that $\Psi(A\otimes I)=\rho(A)I\otimes I$ for each A in $\mathscr{R}$. Then $\Psi=\tilde{\Psi}_\rho$.*

Proof. With T in $\mathscr{R}\bar{\otimes}\mathscr{S}$ and σ in $\mathscr{S}_\#$, note that

$$\begin{aligned}\tilde{\Phi}_\sigma(\Psi(T)) &= \lim_a \tilde{\Phi}_\sigma(\tilde{\alpha}_a(T)) = \lim_a \tilde{\alpha}_a(\tilde{\Phi}_\sigma(T))\\ &= \Psi(\tilde{\Phi}_\sigma(T)) = \rho(\Phi_\sigma(T))I\otimes I\end{aligned}$$

since $\{\tilde{\alpha}_a(T)\}$ and $\{\tilde{\alpha}_a(\tilde{\Phi}_\sigma(T))\}$ are ultraweakly convergent to $\Psi(T)$ and $\Psi(\tilde{\Phi}_\sigma(T))$, respectively, and $\tilde{\Phi}_\sigma$ is an ultraweakly continuous $\mathscr{R}\bar{\otimes}\mathbb{C}I$-bimodule mapping of $\mathscr{R}\bar{\otimes}\mathscr{S}$ onto $\mathscr{R}\bar{\otimes}\mathbb{C}I$. By definition of Φ_σ,

$$\rho(\Phi_\sigma(T)) = (\rho\otimes\sigma)(T) = \sigma(\Psi_\rho(T))\,.$$

Thus

$$\begin{aligned}\tilde{\Phi}_\sigma(\Psi(T)) &= \rho(\Phi_\sigma(T))I\otimes I = I\otimes\sigma(\Psi_\rho(T))I\\ &= \tilde{\Phi}_\sigma(I\otimes\Psi_\rho(T)) = \tilde{\Phi}_\sigma(\tilde{\Psi}_\rho(T))\,.\end{aligned}$$

It follows that $\tilde{\Phi}_\sigma(\Psi(T)-\tilde{\Psi}_\rho(T))=0$ for each σ in $\mathscr{S}_\#$. But then, for each η in $\mathscr{R}_\#$,

$$(\eta\otimes\sigma)(\Psi(T)-\tilde{\Psi}_\rho(T)) = \eta(\Phi_\sigma(\Psi(T)-\tilde{\Psi}_\rho(T))) = \eta(0) = 0\,.$$

Since $\{\eta\otimes\sigma : \eta\in\mathscr{R}_\#,\ \sigma\in\mathscr{S}_\#\}$ generates a norm-dense subspace of $(\mathscr{R}\bar{\otimes}\mathscr{S})_\#$, we conclude that $\Psi(T)=\tilde{\Psi}_\rho(T)$ for all T in $\mathscr{R}\bar{\otimes}\mathscr{S}$. Thus $\Psi=\tilde{\Psi}_\rho$. □

The lemma that follows is subsumed in Theorem A, but is useful in dealing with the case where $\mathscr{M}$ is a factor of type I. We recall that a subalgebra $\mathscr{B}$ of a von Neumann algebra $\mathscr{R}$ is said to be *normal* in a von Neumann algebra $\mathscr{R}$ when $(\mathscr{B}'\cap\mathscr{R})'\cap\mathscr{R}=\mathscr{B}$ (that is, $\mathscr{B}$ is equal to its second relative commutant in $\mathscr{R}$).

Lemma G. *If $\mathscr{M}$ is a factor, $\mathscr{S}$ a von Neumann algebra, and $\mathscr{B}$ a normal von Neumann subalgebra of $\mathscr{M}\bar{\otimes}\mathscr{S}$ that contains $\mathscr{M}\bar{\otimes}\mathbb{C}I$, then $\mathscr{B}=\mathscr{M}\bar{\otimes}\mathscr{T}$ for some von Neumann subalgebra $\mathscr{T}$ of $\mathscr{S}$.*

Proof. Since $\mathscr{M}\bar{\otimes}\mathbb{C}I\subseteq\mathscr{B}$, we have that

$$\mathscr{B}'\cap(\mathscr{M}\bar{\otimes}\mathscr{S})\subseteq(\mathscr{M}\bar{\otimes}\mathbb{C}I)'\cap(\mathscr{M}\bar{\otimes}\mathscr{S})=\mathbb{C}I\bar{\otimes}\mathscr{S}\,.$$

Thus $\mathscr{B}'\cap(\mathscr{M}\bar{\otimes}\mathscr{S})=\mathbb{C}I\bar{\otimes}\mathscr{S}_0$ for some von Neumann subalgebra $\mathscr{S}_0$ of $\mathscr{S}$. Let $\mathscr{T}$ be $\mathscr{S}_0'\cap\mathscr{S}$. By assumption,

$$\begin{aligned}\mathscr{B} &= (\mathscr{B}'\cap(\mathscr{M}\bar{\otimes}\mathscr{S}))'\cap(\mathscr{M}\bar{\otimes}\mathscr{S})\\ &= (\mathbb{C}I\bar{\otimes}\mathscr{S}_0)'\cap(\mathscr{M}\bar{\otimes}\mathscr{S}) = \mathscr{M}\bar{\otimes}\mathscr{T}\,.\end{aligned}$$

□

Proof of Theorem A. We show, first, that it suffices to consider the case where $\mathcal{M}$ is countably decomposable. Given ρ in $\mathcal{M}_\#$, we must show that $\mathcal{B}$ is stable under $\tilde{\Psi}_\rho$. Let E be the support projection for ρ in $\mathcal{M}$, $\mathcal{M}_0$ be the von Neumann algebra $E\mathcal{M}E$ (acting on the range of E), $\tilde{E}$ be $E \otimes I$, and $\mathcal{B}_0$ be $\mathcal{B} \cap (\mathcal{M}_0 \bar{\otimes} \mathcal{S})$. We note that $\tilde{E}(\mathcal{M} \bar{\otimes} \mathcal{S})\tilde{E}$ is $\mathcal{M}_0 \bar{\otimes} \mathcal{S}$ since $A \to \tilde{E}A\tilde{E}$ is a strong-operator continuous linear mapping of $\mathcal{M} \bar{\otimes} \mathcal{S}$ into itself that maps each $T \otimes S$ into $\mathcal{M}_0 \bar{\otimes} \mathcal{S}$ and leaves $T \otimes S$ fixed when $T \in \mathcal{M}_0$. As $\tilde{\Psi}_\rho$ and $A \to \tilde{\Psi}_\rho(\tilde{E}A\tilde{E})$ are linear and ultraweakly continuous on $\mathcal{M} \bar{\otimes} \mathcal{S}$ and

$$\begin{aligned}\tilde{\Psi}_\rho(\tilde{E}(T \otimes S)\tilde{E}) &= \tilde{\Psi}_\rho((ETE) \otimes S) \\ &= \rho(ETE)I \otimes S = \rho(T)I \otimes S = \tilde{\Psi}_\rho(T \otimes S)\,,\end{aligned}$$

we have that $\tilde{\Psi}_\rho(A) = \tilde{\Psi}_\rho(\tilde{E}A\tilde{E})$ for each A in $\mathcal{M} \bar{\otimes} \mathcal{S}$.

Let ρ_0 be the restriction of ρ to $\mathcal{M}_0$. Suppose $\tilde{\Psi}_{\rho_0}(T_0) \in \mathcal{B}_0$ for each T_0 in $\mathcal{B}_0$. With T in $\mathcal{B}$, $\tilde{E}T\tilde{E} \in \mathcal{B} \cap (\mathcal{M}_0 \bar{\otimes} \mathcal{S}) = \mathcal{B}_0$. Thus

$$\tilde{\Psi}_\rho(T) = \tilde{\Psi}_\rho(\tilde{E}T\tilde{E}) = \tilde{\Psi}_{\rho_0}(\tilde{E}T\tilde{E}) \in \mathcal{B}_0 \subseteq \mathcal{B}\,.$$

Hence it suffices to consider $\tilde{\Psi}_{\rho_0}$ in place of Ψ_ρ and $\mathcal{M}_0$ and $\mathcal{B}_0$ in place of $\mathcal{M}$ and $\mathcal{B}$, respectively. As ρ_0 is faithful, we have that $\mathcal{M}_0$ is countably decomposable. We may restrict our attention to the case where $\mathcal{M}$ is countably decomposable.

(i) If $\mathcal{M}$ is a factor of type I and $\mathcal{H}$ is the Hilbert space on which $\mathcal{S}$ acts, then $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{H})$ is a factor of type I, and

$$\mathcal{M} \bar{\otimes} \mathbb{C}I \subseteq \mathcal{B} \subseteq \mathcal{M} \bar{\otimes} \mathcal{S} \subseteq \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{H})\,.$$

The von Neumann Double Commutant Theorem assures us, therefore, that all von Neumann subalgebras are normal in $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{H})$. From Lemma G, $\mathcal{B} = \mathcal{M} \bar{\otimes} \mathcal{T}$ for some von Neumann subalgebra $\mathcal{T}$ of $\mathcal{B}(\mathcal{H})$. From

$$\mathbb{C}I \bar{\otimes} \mathcal{T} \subseteq \mathcal{B} \subseteq \mathcal{M} \bar{\otimes} \mathcal{S}\,,$$

we have that

$$\mathbb{C}I \bar{\otimes} \mathcal{T} \subseteq (\mathcal{M} \bar{\otimes} \mathbb{C}I)' \cap (\mathcal{M} \bar{\otimes} \mathcal{S}) = \mathbb{C}I \bar{\otimes} \mathcal{S}\,.$$

Thus $\mathcal{T} \subseteq \mathcal{S}$, and $\mathcal{B} = \mathcal{M} \bar{\otimes} \mathcal{T}$ with $\mathcal{T}$ a von Neumann subalgebra of $\mathcal{S}$.

(ii) Suppose $\mathcal{M}$ is a factor of type II_1 and $\mathcal{M}$ acting on the Hilbert space $\mathcal{H}$ is the GNS representation of $\mathcal{M}$ corresponding to its (unique, normalized) trace τ. There is a generating (unit) vector u for $\mathcal{M}$ in $\mathcal{H}$ such that $\tau(A) = \langle Au, u\rangle$ for each A in $\mathcal{M}$. Since u is separating for $\mathcal{M}$ (τ is faithful on $\mathcal{M}$), for each ρ of norm 1 in $\mathcal{M}_\#$, there is a pair of unit vectors (not unique) x, y in $\mathcal{H}$ such that $\rho(A) = \langle Ax, y\rangle$ for each A in $\mathcal{M}$ [K-R II; Corollary 7.3.3]. As u is generating for $\mathcal{M}$, there are sequences $\{T_n\}$ and $\{S_n\}$ of operators in $\mathcal{M}$ such that $\{T_n u\}$ tends to x and $\{S_n u\}$ tends to y. Let $\rho_n(A)$ be $\langle AT_n u, S_n u\rangle$.

Then $\rho_n(A)$ is $\tau(AT_nS_n^*)$ and $\{\rho_n\}$ tends in norm to ρ. Thus elements of the form τ_B, where $\tau_B(A) = \tau(AB)$ for each A in $\mathcal{M}$, form a norm-dense linear subspace of $\mathcal{M}_\#$. From the solution to [K-R IV; Exercise 12.4.36], it follows that $\{\tilde{\Psi}_{\rho_n}(T)\}$ tends in norm to $\tilde{\Psi}_\rho(T)$ for each T in $\mathcal{M} \bar{\otimes} \mathcal{S}$. Hence $\mathcal{B}$ is stable under $\tilde{\Psi}_\rho$ if it is stable under all $\tilde{\Psi}_{\rho_n}$. Thus it suffices to prove that $\mathcal{B}$ is stable under $\tilde{\Psi}_{\tau_B}$ for each B in $\mathcal{M}$.

The proof of Lemma E, with τ in place of β and ρ, shows that there is a net $\{\alpha_a\}_{a\in A}$ of mappings α_a in $\mathcal{D}$ that ν-converges to $\tilde{\tau}$, where $\tilde{\tau}(A) = \tau(A)I$ for each A in $\mathcal{M}$. If $\alpha_a(A)$ is $\sum_{j=1}^n a_jU_jAU_j^*$ for each A in $\mathcal{M}$, let $\tilde{\alpha}_a(T)$ be $\sum_{j=1}^n a_j(U_j \otimes I)T(U_j \otimes I)^*$ for each T in $\mathcal{M} \bar{\otimes} \mathcal{S}$. Then $\tilde{\alpha}_n$ is a positive linear mapping that leaves $I \otimes I$ fixed. Thus $\|\tilde{\alpha}_a\| = 1$. By ν-compactness of $\mathcal{B}_1(\mathcal{M} \bar{\otimes} \mathcal{S})$, $\{\tilde{\alpha}_a\}_{a\in A}$ has a cofinal subnet $\{\tilde{\alpha}_a\}_{a\in A_0}$ that is ν-convergent to some Ψ in $\mathcal{B}_1(\mathcal{M} \bar{\otimes} \mathcal{S})$.

Since $\{\alpha_a(A)\}_{a\in A}$ is ultraweakly convergent to $\tilde{\tau}(A)$ for each A in $\mathcal{M}$, the same is true for the cofinal subnet $\{\alpha_a(A)\}_{a\in A_0}$. Thus $\{\tilde{\alpha}_a(A \otimes I)\}_{a\in A_0}$ is ultraweakly convergent to $\tilde{\tau}(A) \otimes I$ and to $\Psi(A \otimes I)$ for each A in $\mathcal{M}$. From Lemma F, $\Psi = \tilde{\Psi}_\tau$. With T in $\mathcal{B}$, $\tilde{\alpha}_a(T) \in \mathcal{B}$ since $\mathcal{M} \bar{\otimes} \mathbb{C}I \subseteq \mathcal{B}$. As $\mathcal{B}$ is ultraweakly closed, $\tilde{\Psi}_\tau(T)$ $(= \Psi(T)) \in \mathcal{B}$. At the same time, with $\mathcal{B}$ in $\mathcal{M}$, $(B \otimes I)T \in \mathcal{B}$, whence $\tilde{\Psi}_\tau((B \otimes I)T) \in \mathcal{B}$. We conclude the proof for the case where $\mathcal{M}$ is a factor of type II_1 by noting that $\tilde{\Psi}_\tau((B \otimes I)T) = \tilde{\Psi}_{\tau_B}(T)$. This follows from the linearity and ultraweak continuity of the mappings $\tilde{\Psi}_{\tau_B}$ and $T \to \tilde{\Psi}_\tau((B \otimes I)T)$ and the equality

$$\tilde{\Psi}_\tau((B \otimes I)(A \otimes I)) = \tau(BA)I \otimes I = \tau_B(A)I \otimes I = \tilde{\Psi}_{\tau_B}(A \otimes I) \quad (A \in \mathcal{M}).$$

(iii) Suppose that $\mathcal{M}$ is a factor of type II_∞. From [K-R II; Theorem 6.7.10], $\mathcal{M}$ is (isomorphic to) the tensor product of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and a factor $\mathcal{N}$ of type II_1. From (i) of this proof, since $\mathcal{B}$ contains $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathbb{C}I$ (where $\mathbb{C}I$ is the tensor product of the scalars in $\mathcal{N}$ and in $\mathcal{S}$), $\mathcal{B}$ splits as $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{T}_0$, where $\mathcal{T}_0$ is a von Neumann subalgebra of $\mathcal{N} \bar{\otimes} \mathcal{S}$. We show that $\mathcal{N} \bar{\otimes} \mathbb{C}I \subseteq \mathcal{T}_0$. Since

$$\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{N} \bar{\otimes} \mathbb{C}I = \mathcal{M} \bar{\otimes} \mathbb{C}I \subseteq \mathcal{B} = \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{T}_0,$$

it suffices to note that if $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{R}_1 \subseteq \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{R}_2$ for von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\mathcal{B}(\mathcal{K})$, then $\mathcal{R}_1 \subseteq \mathcal{R}_2$. (With R_1 in $\mathcal{R}_1$, $I \otimes R_1 \in \mathcal{B}(H) \bar{\otimes} \mathcal{R}_1$; hence $I \otimes R_1 \in \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{R}_2$ and $I \otimes \mathcal{R}_1$ commutes with $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathbb{C}I$. Thus $I \otimes R_1 \in \mathbb{C}I \bar{\otimes} \mathcal{R}_2$.)

From (ii), it follows, now, that $\mathcal{T}_0$ splits as $\mathcal{N} \bar{\otimes} \mathcal{T}$ for some von Neumann subalgebra $\mathcal{T}$ of $\mathcal{S}$. Thus

$$\mathcal{B} = \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{T}_0 = \mathcal{B} \bar{\otimes} (\mathcal{N} \bar{\otimes} \mathcal{T}) = (\mathcal{B} \bar{\otimes} \mathcal{N}) \bar{\otimes} \mathcal{T} = \mathcal{M} \bar{\otimes} \mathcal{T}.$$

(iv) We conclude the proof by establishing our result when $\mathcal{M}$ is a factor of type III. As noted at the beginning of this proof, we may assume that $\mathcal{M}$ is countably decomposable. From Lemma E, with ρ a normal state of $\mathcal{M}$, $\tilde{\rho}$ is the ν-limit of a net $\{\alpha_a\}_{a\in A}$ of elements α_a of $\mathcal{D}$. As in the proof of (ii) (the penultimate paragraph), there is a cofinal subnet $\{\tilde{\alpha}_a\}_{a\in A_0}$ of $\{\tilde{\alpha}_a\}_{a\in A}$ that is ν-convergent to some Ψ in $\mathcal{B}_1(\mathcal{M} \bar{\otimes} \mathcal{S})$. Since $\{\alpha_a(A)\}_{a\in A}$ is ultraweakly convergent to $\rho(A)I$ for each A in $\mathscr{M}$, the same is true for the cofinal subnet $\{\alpha_a(A)\}_{a\in\Lambda_0}$. Thus $\{\tilde{\alpha}_a(A\otimes I)\}_{a\in\Lambda_0}$ is ultraweakly convergent to $\tilde{\rho}(A)\otimes I$ and to $\Psi(A\otimes I)$, for each A in $\mathscr{M}$. From Lemma F, $\Psi = \tilde{\Psi}_\rho$.

With T in $\mathscr{B}$, $\tilde{\Psi}_\rho(T) = \Psi(T)$ is the ultraweak limit of $\{\tilde{\alpha}_a(T)\}_{a\in\Lambda_0}$. Since $\mathscr{M}\bar{\otimes}\mathbb{C}I \subseteq \mathscr{B}$ and $T\in\mathscr{B}$, $\tilde{\alpha}(T)\in\mathscr{B}$ for each α in $\mathscr{D}$. Thus $\tilde{\Psi}_\rho(T)$ $(=\Psi(T))$ $\in\mathscr{B}$, as $\mathscr{B}$ is ultraweakly closed. □

4. An application

In [P1], Popa asks: If $\mathscr{M}_1$ and $\mathscr{M}_2$ are factors of type II_1 and $\mathscr{B}_1$ and $\mathscr{B}_2$ are maximal injective subalgebras of $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively, is $B_1\bar{\otimes}\mathscr{B}_2$ maximal injective in $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$? He also asks if this is true when we assume that $\mathscr{M}_1 = \mathscr{B}_1$ (that is, that $\mathscr{M}_1$ is the hyperfinite II_1 factor). In [Ge], this latter question is answered affirmatively using the splitting theorem for the case where $\mathscr{M}$ and $\mathscr{S}$ are finite.

Of course, one can ask Popa's questions when the types of the factors $\mathscr{M}_1$ and $\mathscr{M}_2$ are not constrained to be finite. Following the pattern of the argument in [Ge], we show:

Theorem H. *If $\mathscr{M}_1$ and $\mathscr{M}_2$ are factors, $\mathscr{M}_1$ is injective, and $\mathscr{B}_2$ is a maximal injective von Neumann subalgebra of $\mathscr{M}_2$, then $\mathscr{M}_1\bar{\otimes}\mathscr{B}_2$ is maximal injective in $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$.*

Proof. Suppose $\mathscr{B}$ is a maximal injective von Neumann subalgebra of $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ containing $\mathscr{M}_1\bar{\otimes}\mathscr{B}_2$. Then $\mathscr{B}$ contains $\mathscr{M}_1\bar{\otimes}\mathbb{C}I$. The hypotheses of Theorem A are satisfied, and $\mathscr{B}$ splits as $\mathscr{M}_1\bar{\otimes}\mathscr{T}$, where $\mathscr{T}$ is a von Neumann subalgebra of $\mathscr{M}_2$. With ρ in the predual of $\mathscr{M}_1$, σ in the predual of $\mathscr{M}_2$, and T in $(\mathscr{M}_1\bar{\otimes}\mathscr{T})\cap(\mathbb{C}I\bar{\otimes}\mathscr{M}_2)$, we have that $\Psi_\rho(T)\in\mathscr{T}$ since $T\in\mathscr{M}_1\bar{\otimes}\mathscr{T}$ and $\phi_\sigma(T)$ is a scalar since $T\in\mathbb{C}I\bar{\otimes}\mathscr{M}_2$. From the Slice Mapping Theorem, $T\in\mathbb{C}I\bar{\otimes}\mathscr{T}$. Thus

$$\mathbb{C}I\bar{\otimes}\mathscr{B}_2\subseteq(\mathscr{M}_1\bar{\otimes}\mathscr{B}_2)\cap(\mathbb{C}I\bar{\otimes}\mathscr{M}_2)\subseteq(\mathscr{M}_1\bar{\otimes}\mathscr{T})\cap(\mathbb{C}I\bar{\otimes}\mathscr{M}_2)\subseteq\mathbb{C}I\bar{\otimes}\mathscr{T},$$

and $\mathscr{B}_2\subseteq\mathscr{T}$.

The tensor-slice mapping corresponding to a normal state of $\mathscr{M}_1$ is a conditional expectation of $\mathscr{M}_1\bar{\otimes}\mathscr{T}$ $(=\mathscr{B})$ onto $\mathbb{C}I\bar{\otimes}\mathscr{T}$. Since $\mathscr{B}$ is a assumed to be injective, there is a conditional expectation of $\mathscr{B}(\mathscr{H}_1\otimes\mathscr{H}_2)$ onto $\mathscr{M}_1\bar{\otimes}\mathscr{T}$ (where $\mathscr{M}_1$ acts on $\mathscr{H}_1$ and $\mathscr{M}_2$ acts on $\mathscr{H}_2$). Composing these conditional expectations, we have a conditional expectation of $\mathscr{B}(\mathscr{H}_1\otimes\mathscr{H}_2)$ onto $\mathbb{C}I\bar{\otimes}\mathscr{T}$. Thus $\mathbb{C}I\bar{\otimes}\mathscr{T}$, and hence $\mathscr{T}$, are injective. By assumption, $\mathscr{B}_2$ is maximal injective in $\mathscr{M}_2$. Hence $\mathscr{B}_2=\mathscr{T}$, and $\mathscr{B}=\mathscr{M}_1\bar{\otimes}\mathscr{B}_2$. It follows that $\mathscr{M}_1\bar{\otimes}\mathscr{B}_2$ is a maximal injective von Neumann subalgebra of $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$. □

The key question underlying the celebrated result of Connes [C] (see also [Haa] and [P2]) was that of uniqueness of the hyperfinite factor $\mathscr{M}$ of type II_∞ on a separable Hilbert space. Is $\mathscr{M}$ the tensor product of the hyperfinite II_1

factor and a factor of type I_∞? If we express $\mathcal{M}$ as $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, where $\mathcal{M}_1$ is a factor of type II_1 and $\mathcal{M}_2$ is a factor of type I_∞, this question becomes that of whether or not $\mathcal{M}_1$ is hyperfinite. In any event, there is a maximal hyperfinite factor $\mathcal{M}_0$ in $\mathcal{M}_1$ (from [F-K]). Since $\mathcal{M}_2$ is of type I_∞, it is hyperfinite. Once we know that $\mathcal{M}_0 \bar{\otimes} \mathcal{M}_2$ is maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, we have that $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 = \mathcal{M}_0 \bar{\otimes} \mathcal{M}_2$, since $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ is injective by assumption. Again, from the Slice Mapping Theorem, $\mathcal{M}_1 = \mathcal{M}_0$, whence $\mathcal{M}_1$ is injective. Thus $\mathcal{M}_1$ is hyperfinite. From Theorem H, $\mathcal{M}_0 \bar{\otimes} \mathcal{M}_2$ *is* maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. Of course, we are using [C], and of course, there is a brief proof of the uniqueness using [C]. Our point is that Theorem H, in very restricted circumstances (where $\mathcal{M}$, itself, is hyperfinite), appears as a prominent component in a natural proof of uniqueness. In this sense, the full result (Theorem H), may be viewed as an "extension" of that aspect of the Connes uniqueness result.

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