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Notes on the Gelfand–Neumark Theorem

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Dedicated to Irving Kaplansky and Irving Segal with gratitude and respect.

ABSTRACT. The Gelfand–Neumark Theorem, the GNS construction and some of their consequences over the past fifty years are studied.

1. Introduction

In 1943, a paper **[G-N]**, written by I. M. Gelfand and M. Neumark, "On the imbedding of normed rings into the ring of operators in Hilbert space," appeared (in English) in Mat. Sbornik (see previous paper). From the vantage point of a fifty year history, it is safe to say that that paper changed the face of modern analysis. Together with the monumental "Rings of operators" series **[M-vN I, II, III, IV]**, authored by F. J. Murray and J. von Neumann, it introduced "non-commutative analysis," the vast area of mathematics that provides the mathematical model for quantum physics.

The founders of the theory underlying quantum mechanics (Schrödinger and Heisenberg, primarily) were groping their way toward this mathematics ("wave" and "matrix" mechanics). With his magnificent volume **[D]**, P. A. M. Dirac all but invents the operator algebra and uses Hilbert space techniques to produce powerful conclusions in physics. Of course, simultaneously with his introduction of "rings of operators," von Neumann's book **[vN2]** appeared, providing a model for "quantum measurement" and some of the fundamentals of quantum statistical mechanics.

Extremely knowledgeable and vitally interested in quantum physics, I. E. Segal, who had been developing commutative and non-commutative harmonic analysis in the Hilbert space context, recognized the construction buried in the Gelfand–Neumark paper — a construction that is basic and crucial for the subject of operator algebras. Just after publication of his "Postulates for quantum

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mechanics" [**S1**], Segal published his groundbreaking "Irreducible operator algebras" [**S2**] in which that construction is sharpened and made explicit and then used in one of the earliest general studies of (infinite-dimensional) unitary representations of (non-commutative) locally compact groups.

A statement of the Gelfand–Neumark theorem follows.

THEOREM (GELFAND– NEUMARK 1943). *If $\mathfrak{A}$ is an algebra over the complex numbers $\mathbb{C}$ with unit I, with a norm $A \to \| A \|$ relative to which it is a Banach space for which $\| AB \| \leq \| A \| \; \| B \|$, and $\| I \| = 1$ ($\mathfrak{A}$ is a Banach algebra), and with a mapping (involution) $A \to A^*$ such that*

i) $(aA + B^*) = \bar{a}A + B^*$,
ii) $(AB)^* = B^*A^*$,
iii) $(A^*)^* = A$,
iv) $\| A^*A \| = \| A^* \| \; \| A \|$,
v) *$A^*A + I$ has an inverse (in $\mathfrak{A}$) for each A in $\mathfrak{A}$,*
vi) $\| A^* \| = \| A \|$,

then there is an isomorphism φ of $\mathfrak{A}$ with a norm-closed subalgebra $\mathfrak{B}$ of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ such that $\varphi(A^*) = \varphi(A)^*$, where $\varphi(A)^*$ is the adjoint (in $\mathcal{B}(\mathcal{H})$) of $\varphi(A)$. Moreover, $\| \varphi(A) \| = \| A \|$ for all A in $\mathfrak{A}$.

Gelfand and Neumark conjecture, in their paper, that conditions (v) and (vi) are superfluous, that is, derivable from the others. They were proved right ten years later on (v) and seventeen years later on (vi). (Compare [**F,K-V,Sc**] and [**G-K**].) In the following, we present a complete proof of the Gelfand-Neumark theorem without assuming conditions (v) and (vi). Before beginning, let us define an element A of our algebra $\mathfrak{A}$ to be *self-adjoint, normal, unitary, positive*, or *regular*, when $A = A^*$, $AA^* = A^*A$, $AA^* = A^*A = I$, $A = A^*$ and the spectrum $\mathrm{sp}(A)$ of A consists of non-negative real numbers, or A has a two- sided inverse in $\mathfrak{A}$, respectively. We say that a linear functional on $\mathfrak{A}$ that takes the value 1 at I and is real and non-negative on positive elements is a *state* of $\mathfrak{A}$.

In our argument, we shall make use of a standard form of the Hahn–Banach theorem (see, for example, [**K-R I; Theorem 1.6.1**]). In addition, we use some basic facts from the theory of (complex) Banach algebras, notably those concerning $\mathrm{r}(A)$, the *spectral radius* of A. We recall that

$$\mathrm{r}(A) = \sup\{|a| : a \in \mathrm{sp}(A)\}$$

and that $\mathrm{r}(A) = \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}} \leq \| A \|$. Moreover, $I + A$ is regular if $\| A \| < 1$, $\mathrm{sp}(p(A)) = p(\mathrm{sp}(A))$ ($= \{p(a) : a \in \mathrm{sp}(A)\}$) for each polynomial p, and $\mathrm{sp}(B)$ is a closed, non-null subset of the disk of radius $\| B \|$ with center 0 in the complex numbers $\mathbb{C}$. (See [**K-R I; Sections 3.1 and 3.2**] for accounts of these results.)

PROOF. (Assuming only conditions (i) – (iv).) If $A = A^*$, then $\| A^2 \| = \| A \|^2$. Thus $\| A^{2^n} \| = \| A \|^{2^n}$, and $\| A \| = \mathrm{r}(A)$. Since $p(A)$ is self-adjoint for each real polynomial p,

$$\| p(A) \| = \mathrm{r}(p(A)) = \sup\{|p(a)| : a \in \mathrm{sp}(A)\}. \tag{1}$$

If p is complex, then $p = p_1 + ip_2$, with p_1 and p_2 real, and

$$\begin{aligned} \mathrm{r}((p_1^2 + p_2^2)(A)) &\le \mathrm{r}((p_1 - ip_2)(A)) \cdot \mathrm{r}((p_1 + ip_2)(A)) \\ &\le \| [(p_1 + ip_2)(A)]^* \| \, \| (p_1 + ip_2)(A) \| \\ &= \| (p_1^2 + p_2^2)(A) \| \\ &= \mathrm{r}((p_1^2 + p_2^2)(A)). \end{aligned}$$

It follows that we must have equality at each stage of the preceding inequality. Since

$$\begin{aligned} \mathrm{r}((p_1 - ip_2)(A)) &\le \| [(p_1 + ip_2)(A)]^* \| , \\ \mathrm{r}((p_1 + ip_2)(A)) &\le \| (p_1 + ip_2)(A) \| , \end{aligned}$$

we must have equality in each of these last inequalities. Hence (1) holds for all complex polynomials p as well. It follows that, with A a self-adjoint element in $\mathfrak{A}$, the mapping carrying $p(A)$ onto the polynomial p on $\mathrm{sp}(A)$ is an isometric isomorphism of the subalgebra of complex polynomials of A into $C(\mathrm{sp}(A))$, the algebra of continuous, complex-valued functions on $\mathrm{sp}(A)$, and has a (unique) isometric extension to the closure $\mathfrak{A}(A)$ mapping it onto the closure $\mathcal{P}$ of the polynomials in $C(\mathrm{sp}(A))$. (Note that we have not established, as yet, that $\mathrm{sp}(A)$ is a subset of the reals $\mathbb{R}$, so we cannot apply the Stone–Weierstrass theorem to conclude, at this point, that $\mathcal{P} = C(\mathrm{sp}(A))$.)

With a in $\mathrm{sp}(A)$, the mapping that assigns $g(a)$ to g in $\mathcal{P}$ is a linear functional of norm 1 on $\mathcal{P}$ that assigns 1 to the image of I in $\mathcal{P}$ and a to the image of A. Composed with the isomorphism, this linear functional gives rise to a linear functional ρ_0 of norm 1 on $\mathfrak{A}(A)$ such that $\rho_0(I) = 1$ and $\rho_0(A) = a$. Using the Hahn–Banach theorem, we extend ρ_0 to a linear functional ρ of norm 1 on $\mathfrak{A}$. Suppose B is a self-adjoint element in $\mathfrak{A}$. We show that $\rho(B)$ is real. Assume the contrary. By adding a suitable real multiple of I to B, we produce a self-adjoint element of $\mathfrak{A}$ to which ρ assigns a non-zero, purely imaginary value. We may assume that $\rho(B) = ib$ with b a non-zero real number. We may assume, further, that $b > 0$, otherwise, we replace B by $-B$. Note that $c \in \mathrm{sp}(C)$ if and only if $\bar{c} \in \mathrm{sp}(C^*)$, whence $\mathrm{r}(C) = \mathrm{r}(C^*)$ for each element C in $\mathfrak{A}$. Thus, for each real t

$$\begin{aligned} b^2 + 2bt + t^2 = |\rho(B + itI)|^2 &\le \| B + itI \|^2 \\ &= [\mathrm{r}(B + itI)]^2 = (\mathrm{r}(B + itI))(\mathrm{r}([B + itI]^*)) \\ &= \| B + itI \| \, \| B - itI \| = \| B^2 + t^2 I \| \\ &\le \| B^2 \| + t^2. \end{aligned}$$

From this inequality, we draw the contradiction

$$t \le \frac{\| B^2 \|}{2b} - \frac{b}{2},$$

for each real t. It follows that ρ assumes real values on each self-adjoint element in $\mathfrak{A}$. In particular, $\rho(A) = a$ is real, and the spectrum of each self-adjoint element consists of real numbers. Thus $\mathrm{sp}(A) \subseteq \mathbb{R}$, and $\mathcal{P} = C(\mathrm{sp}(A))$. With B self-adjoint in $\mathfrak{A}(A)$, this argument also shows that B maps to a real-valued function in $C(\mathrm{sp}(A))$. Representing an element T of $\mathfrak{A}(A)$ as $T_1 + iT_2$, where $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$, we see that if the function representing T in $C(\mathrm{sp}(A))$ is real, then T is self-adjoint. Since the (real) algebra of real-valued functions in $C(\mathrm{sp}(A))$ is the norm closure of the algebra of polynomials with real coefficients on $\mathrm{sp}(A)$, the family of self-adjoint elements in $\mathfrak{A}(A)$ is the norm closure of the polynomials in A with real coefficients; in particular, this family is a norm-closed, real algebra. We have, too, that A^2 is positive when A is self-adjoint, for if a is real and positive, then $A^2 + a^2 I = (A - iaI)(A + iaI)$, a product of regular elements.

We mention, as a brief historical note, that the argument showing that $\rho(B)$ is real is an adaptation, to the present circumstances, of the celebrated "Arens Trick." Published during the Second World War, the Gelfand–Neumark article **[G-N]** did not reach the USA until 1946. In proving that the spectrum of a self-adjoint element consists of real numbers, it cited a lemma that was not known to the young mathematicians in the USA reading the article. This lemma turned out to be the (even more celebrated) lemma establishing the existence of the Silov boundary. Richard Arens managed to circumvent that lemma by finding the clever argument whose essence is presented above.

We show, next, that the functional ρ is a state of $\mathfrak{A}$. Suppose, to the contrary, that B is a positive element of $\mathfrak{A}$ and $\rho(B) < 0$. As $\mathrm{sp}(B - \|B\| I) = \mathrm{sp}(B) - \|B\| I$ and $\mathrm{sp}(B)$ consists of non-negative, real numbers, we have that

$$\| B - \|B\| I \| = \mathrm{r}(B - \|B\| I) \le \|B\| .$$

But

$$|\rho(B - \|B\| I)| = |\rho(B) - \|B\| \,| > \|B\| \ge \| B - \|B\| I \| ,$$

contradicting the fact that $\|\rho\| = 1$. Thus ρ is a state of $\mathfrak{A}$, $\rho(A) = a$, and ρ has norm 1. It follows, now, that each positive B in $\mathfrak{A}(A)$ maps to a positive function in $C(\mathrm{sp}(A))$. As noted before, an element of $\mathfrak{A}(A)$ that maps to a real-valued function in $C(\mathrm{sp}(A))$ is self-adjoint. If the image takes on only non-negative values, then the element must be positive since the spectrum of the element is a subset of the range of the representing function. (If an element of $\mathfrak{A}(A)$ has no inverse in $\mathfrak{A}$, then its representing function has no inverse in $C(\mathrm{sp}(A))$.) Thus the image of an element of $\mathfrak{A}(A)$ is a non-negative, real-valued function in $C(\mathrm{sp}(A))$ if and only if the element is positive.

If $A_1, \dots, A_n$ are positive elements in $\mathfrak{A}$ and $a \in \mathrm{sp}(A_1 + \cdots + A_n)$, then (as just established) there is a state ρ of $\mathfrak{A}$ such that

$$a = \rho(A_1 + \cdots + A_n) = \rho(A_1) + \cdots + \rho(A_n) \ge 0.$$

It follows that $A_1 + \cdots + A_n$ is positive. Since

$$\mathrm{r}(A_1 + \cdots + A_k) = \| (A_1 + \cdots + A_k) \|$$

and $\mathrm{sp}(A_1 + \cdots + A_k)$ is closed ($k \neq n$) and consists of real, non-negative numbers, $\| (A_1 + \cdots + A_k) \| \in \mathrm{sp}(A_1 + \cdots + A_k)$. From what we have just proved, there is a norm 1 state ρ of $\mathfrak{A}$ such that

$$\| (A_1 + \cdots + A_k) \| = \rho(A_1 + \cdots + A_k) \leq \rho(A_1 + \cdots + A_n) \leq \| (A_1 + \cdots + A_n) \| .$$

Continuing our historical notes, we have just concluded the proof, in a reasonably straightforward manner, of the fact that was missing for so long. It replaces the argument of Kelley–Vaught **[K-V]** or of Fukamiya **[F]** of this same fact that Kaplansky lacked in his derivation of condition (v) ("symmetry") from (i) – (iv). The remaining argument, that A^*A is positive, is Kaplansky's (published in Mathematical Reviews **[Sc]**) and will appear in the next paragraph. It is one of the (many) small historical ironies that Kaplansky's argument is (at least to this author) the far cleverer part of the proof.

Using the isomorphism of $\mathfrak{A}(A^*A)$ with $C(\mathrm{sp}(A^*A))$, we pass to the function f representing A^*A (which is, as a matter of fact, the polynomial x restricted to $\mathrm{sp}(A^*A)$). Decompose f as $f^+ - f^-$, where f^+ and f^- are functions in $C(\mathrm{sp}(A^*A))$ taking only non-negative, real values such that $f^+f^- = 0$. From the argument of the preceding paragraphs, there are positive elements B and C in $\mathfrak{A}(A^*A)$ such that $A^*A = B - C$ and $BC = 0$. It follows that $(AC)^*(AC) = -C^3$, which is negative. Making use of the fact that, in a Banach algebra with unit, the spectra of TS and ST, each with 0 adjoined, coincide (see, for example, **[K-R I; Proposition 3.2.8]**) we have that $(AC)(AC)^*$ is negative. As before, we decompose AC as $A_1 + iA_2$, with A_1 and A_2 self-adjoint elements in $\mathfrak{A}(A^*A)$. Since

$$0 \geq (AC)(AC)^* + (AC)^*(AC) = 2(A_1^2 + A_2^2) \geq 0,$$

we have that

$$0 = \| A_1^2 + A_2^2 \| \geq \| A_j \|^2 \quad (j = 1, 2).$$

Thus AC is 0 as are C^3 $(= -(AC)^*(AC))$ and C. It follows that A^*A is a positive element for each A in $\mathfrak{A}$ (in particular, $A^*A + I$ is regular).

Again, with f representing A^*A in $C(\mathrm{sp}(A^*A))$, f has a (unique) positive square root in $C(\mathrm{sp}(A^*A))$, which represents a positive element $(A^*A)^{\frac{1}{2}}$ in $\mathfrak{A}(A^*A)$ whose square is A^*A. If A is regular, so are A^* and $(A^*A)^{\frac{1}{2}}$. The element $A(A^*A)^{-\frac{1}{2}}$ $(= U)$ is unitary since

$$UU^* = A(A^*A)^{-1}A^* = AA^{-1}(A^*)^{-1}A^* = I$$

and

$$U^*U = (A^*A)^{-\frac{1}{2}}A^*A(A^*A)^{-\frac{1}{2}} = I.$$

Thus each regular element A in $\mathfrak{A}$ has the "polar decomposition" UH where H is the positive element $(A^*A)^{\frac{1}{2}}$ and U is the unitary element $A(A^*A)^{-\frac{1}{2}}$.

We prove, next, that each unitary element U in $\mathfrak{A}$ has norm 1. Decompose U as $A + iB$ with A and B self-adjoint elements of $\mathfrak{A}$. Since U and U^* commute, $U - U^*$ $(= 2iB)$ and $U + U^*$ $(= 2A)$ commute; hence A and B commute. Thus $I = UU^* = A^2 + B^2$. Hence $\| A^2 \| = \| A \|^2 \leq 1$ and $\| B \| \leq 1$. It follows that $\| U \|$ and $\| U^* \|$ do not exceed 2. Now

$$1 = \| I \| = \| U^*U \| = \| U^* \| \, \| U \| ,$$

whence $\| U^* \| = \| U \|^{-1}$. Suppose $\| U \| < 1$. Then $\| U^n \| \leq \| U \|^n < \frac{1}{2}$ when n is suitably large. But, then, $\| (U^n)^* \| = \| U^n \|^{-1} > 2$, contradicting the just-noted fact that each unitary element in $\mathfrak{A}$ has norm not exceeding 2. Thus $\| U \| \geq 1$ for *each* unitary element U in $\mathfrak{A}$. Since $\| U^* \| = \| U \|^{-1}$, we conclude that $\| U \| = 1$.

With A a regular element in $\mathfrak{A}$ and UH its polar decomposition,

$$\| A \| = \| UH \| \leq \| U \| \, \| (A^*A)^{\frac{1}{2}} \| = (\| A^* \| \, \| A \|)^{\frac{1}{2}}.$$

Thus $\| A \| \leq \| A^* \|$, for each regular element A in $\mathfrak{A}$. Since A^* is regular, $\| A^* \| \leq \| A \|$, and $\| A \| = \| A^* \|$ for each regular element A in $\mathfrak{A}$. It follows that $\| H \| = (\| A^* \| \, \| A \|)^{\frac{1}{2}} = \| A \|$ for each regular A in $\mathfrak{A}$. If $\| A \| \leq 1$, then $\| H \| \leq 1$. Passing to the representation of $\mathfrak{A}(H)$ as $C(\mathrm{sp}(H))$, we see that $I - H^2$ is positive and has a positive square root $(I - H^2)^{\frac{1}{2}}$ in $\mathfrak{A}(H)$. Again from this representation (or by direct computation), $H + i(I - H^2)^{\frac{1}{2}}$ is a unitary element V in $\mathfrak{A}$ and $H - i(I - H^2)^{\frac{1}{2}}$ is V^*. Thus $H = \frac{1}{2}(V + V^*)$, and $A = UH = \frac{1}{2}(UV + UV^*)$. It follows that each regular element in the unit ball of $\mathfrak{A}$ is the mean of two unitary elements of $\mathfrak{A}$.

At this point, we have developed enough information about the structure of $\mathfrak{A}$ to recapture Theorem 1 of [**K-P**]. We state and prove this result in our context, before continuing the proof of the Gelfand–Neumark theorem.

THEOREM. *If* $\| S \| < 1 - 2n^{-1}$ *for some* S *in* $\mathfrak{A}$, *with* n *an integer greater than 2, then there are unitary elements* $U_1, \ldots, U_n$ *in* $\mathfrak{A}$ *such that* $S = \frac{1}{n}(U_1 + \ldots + U_n)$.

PROOF. Let T be an element of $\mathfrak{A}$ of norm less than 1 and V be a unitary element of $\mathfrak{A}$. Then $\| V^*T \| \leq \| T \| < 1$, whence $I + V^*T$ and $V(I + V^*T)$ $(= V + T)$ are regular elements of $\mathfrak{A}$. Since $\frac{1}{2}(V + T)$ is a regular element in the unit ball of $\mathfrak{A}$, we have proved that $\frac{1}{2}(V + T)$ is the mean of two unitary elements in $\mathfrak{A}$. Thus there are unitary elements W_1 and W_2 in $\mathfrak{A}$ such that $V + T = W_1 + W_2$. It follows that, for each positive integer n, there are unitary elements $U_1, \ldots, U_n$ and $V_1, \ldots, V_{n-1}$ $(= U_n)$ in $\mathfrak{A}$ such that

$$\begin{aligned} V + (n-1)T &= U_1 + V_1 + (n-2)T \\ &= U_1 + U_2 + V_2 + (n-3)T \\ &= \cdots \\ &= U_1 + U_2 + \cdots + U_n. \end{aligned} \tag{2}$$

Under the assumption that $\|S\| < 1 - 2n^{-1}$ (and $n \geq 3$), we have that

$$\|(n-1)^{-1}(nS - I)\| \leq (n-1)^{-1}(n\|S\| + 1) < 1.$$

Thus we may use $(n-1)^{-1}(nS - I)$ in place of T and I in place of V in (2). With these choices, we have that $S = \frac{1}{n}(U_1 + \cdots + U_n)$. □

We note that the Russo–Dye theorem **[R-D]** is an immediate corollary of the preceding result, for each element A in the unit ball of $\mathfrak{A}$ is the norm limit of $(1 - 3n^{-1})A$, and $(1 - 3n^{-1})A$ is a mean of n unitary elements of $\mathfrak{A}$. The preceding result grew out of a short proof of the Russo–Dye theorem shown to this author by L. T. Gardner **[G]** following a lecture in Toronto. Shortly after that, while visiting the University of Copenhagen, the author lectured on Gardner's proof to a seminar. A day after the lecture, the author and G. K. Pedersen recognized that a key element in Gardner's argument ($V + T$ is a sum of two unitary elements) could be applied, as indicated, to give the strong form of the Russo-Dye result just proved.

We resume the proof of the Gelfand–Neumark theorem.

Proof (continued). If A is an element of norm less than 1 in $\mathfrak{A}$ such that $\|A\| < \|A^*\|$, then there is a positive a such that $\|aA\| < 1$ and $\|aA^*\| > 1$. At the same time, there are unitary elements $U_1, \ldots, U_n$ in $\mathfrak{A}$, for some positive integer n, such that $aA = \frac{1}{n}(U_1 + \cdots + U_n)$. Thus $aA^* = \frac{1}{n}(U_1^* + \cdots + U_n^*)$, and $\|aA^*\| \leq 1$. From this contradiction, it follows that $\|A^*\| \leq \|A\| < 1$. But then $\|A\| = \|(A^*)^*\| \leq \|A^*\| \leq \|A\|$. Thus $\|A\| = \|A^*\|$ when A is an element of $\mathfrak{A}$ of norm less than 1. Multiplying by a suitable positive scalar, we have the same result for an arbitrary element in $\mathfrak{A}$. This completes the derivation of condition (vi) from (i) – (iv).

The remainder of the proof of the Gelfand–Neumark theorem proceeds along relatively standard lines with the aid of the apparatus of states that has been developed. We use the GNS (Gelfand–Neumark–Segal) construction, as formulated by Segal in **[S2]**. With ρ a state of $\mathfrak{A}$, let $\langle A, B\rangle_\rho'$ be $\rho(B^*A)$ for each A and B in $\mathfrak{A}$. Note that $\langle\, ,\rangle_\rho'$ is a positive, semi-definite inner product on $\mathfrak{A}$. Suppose $\langle A, A\rangle_\rho' = 0$ for some A in $\mathfrak{A}$ (A is a "null vector"). Then $\langle BA, BA\rangle_\rho' = \langle A, B^*BA\rangle_\rho' = 0$ for each B in $\mathfrak{A}$ (by applying the Schwarz inequality to $\langle\, ,\, \rangle_\rho'$). Thus BA is a null vector. With A and B null vectors,

$$\langle A+B, A+B\rangle_\rho' = \langle A, A\rangle_\rho' + \langle A, B\rangle_\rho' + \langle B, A\rangle_\rho' + \langle B, B\rangle_\rho' = 0,$$

whence $A + B$ is a null vector. Thus the set $\mathcal{K}_\rho$ of null vectors is a left ideal in $\mathfrak{A}$ (the "left kernel" of ρ).

Let $\langle A + \mathcal{K}_\rho, B + \mathcal{K}_\rho\rangle_\rho$ be $\rho(B^*A)$ for each A and B in $\mathfrak{A}$. Then $\langle\, ,\, \rangle_\rho$ is a positive, definite inner product on the quotient vector space $\mathfrak{A}/\mathcal{K}_\rho$. Let $\mathcal{H}_\rho$ be the completion of $\mathfrak{A}/\mathcal{K}_\rho$ relative to the norm induced on it by the inner product $\langle\, ,\, \rangle_\rho$. Define the operator $\pi_\rho'(A)$ on $\mathfrak{A}/\mathcal{K}_\rho$ by $\pi_\rho'(A)(B + \mathcal{K}_\rho) = AB + \mathcal{K}_\rho$. Then $\pi_\rho'(A)$ is well defined (since $\mathcal{K}_\rho$ is a left ideal) and is a linear transformation of

$\mathfrak{A}/\mathcal{K}_\rho$ into itself. Note that

$$\|\pi'_\rho(A)(B+\mathcal{K}_\rho)\|_\rho^2 = \langle \pi'_\rho(A)(B+\mathcal{K}_\rho), \pi'_\rho(A)(B+\mathcal{K}_\rho)\rangle = \rho(B^*A^*AB).$$

As $\|A^*A\|I - A^*A + aI$ is regular for each positive a, we see that $\|A^*A\|I - A^*A$ is positive and, hence, the square of some (positive) self-adjoint element C in $\mathfrak{A}$. Thus

$$B^*(\|A^*A\|I - A^*A)B = B^*C^2B = (CB)^*(CB) \geq 0,$$

and

$$\|\pi'_\rho(A)(B+\mathcal{K}_\rho)\|_\rho^2 = \rho(B^*A^*AB) \leq \|A^*A\|\rho(B^*B) = \|A\|^2\|B+\mathcal{K}_\rho\|_\rho^2.$$

It follows that $\pi'_\rho(A)$ is bounded and $\|\pi'_\rho(A)\| \leq \|A\|$. Thus $\pi'_\rho(A)$ extends (uniquely) to a bounded linear transformation $\pi_\rho(A)$ of $\mathcal{H}_\rho$ into itself with the same bound as
$\pi'_\rho(A)$. Since $\pi'_\rho(A+B) = \pi'_\rho(A) + \pi'_\rho(B)$, we have that

$$\pi_\rho(A+B) = \pi_\rho(A) + \pi_\rho(B)$$

for each A and B in $\mathfrak{A}$. Similarly,

$$\pi_\rho(AB) = \pi_\rho(A)\pi_\rho(B)$$

for each A and B in $\mathfrak{A}$. With B and C in $\mathfrak{A}$, we see that

$$\langle \pi'_\rho(A^*)(B+\mathcal{K}_\rho), C+\mathcal{K}_\rho\rangle_\rho = \rho(C^*A^*B) = \langle B+\mathcal{K}_\rho, \pi'_\rho(A)(C+\mathcal{K}_\rho)\rangle_\rho.$$

Thus

$$\langle \pi_\rho(A^*)(x), y\rangle_\rho = \langle x, \pi_\rho(A)(y)\rangle_\rho$$

for each x and y in $\mathcal{H}_\rho$. It follows that $\pi_\rho(A^*) = \pi_\rho(A)^*$ for each A in $\mathfrak{A}$. We say that π_ρ is a * *representation* of $\mathfrak{A}$ on $\mathcal{H}_\rho$, the GNS representation constructed from $\mathfrak{A}$ and the state ρ on $\mathfrak{A}$.

To complete the proof, we construct the direct sum representation π of the GNS representations π_ρ. For each A in $\mathfrak{A}$, $\pi(A)$ acts on $\mathcal{H}$, the direct sum $\sum \oplus \mathcal{H}_\rho$ of the representation Hilbert spaces $(\pi(A)(\{x_\rho\}) = \{\pi_\rho(A)x_\rho\})$. With A in $\mathfrak{A}$, $\|A^*A\| = \mathrm{r}(A^*A) \in \mathrm{sp}(A^*A)$, since A^*A has spectrum that is closed and consists of non-negative real numbers. It follows that there is a state ρ of $\mathfrak{A}$ such that $\rho(A^*A) = \|A^*A\|$. If we denote by 'x_ρ' the unit vector $I + \mathcal{K}_\rho$ in $\mathcal{H}_\rho$, then

$$\langle \pi_\rho(T)x_\rho, x_\rho\rangle = \rho(I^*TI) = \rho(T) \quad (T \in \mathfrak{A}).$$

Thus

$$\|\pi_\rho(A)x_\rho\|^2 = \langle \pi_\rho(A^*A)x_\rho, x_\rho\rangle = \rho(A^*A) = \|A^*A\| = \|A\|^2.$$

Hence $\|A\| \leq \|\pi_\rho(A)\|$. But we have already established the reverse inequality (for all states of $\mathfrak{A}$). For the particular ρ we have constructed, $\|\pi_\rho(A)\| = \|A\|$. It follows that

$$\|\pi(A)\| = \sup\{\|\pi_{\rho'}(A)\| : \rho' \text{ a state of } \mathfrak{A}\} = \|A\|.$$

Hence π is an isometric * isomorphism of $\mathfrak{A}$. $\square$

2. Completely positive mappings and Stinespring's theorem

In the summer of 1951, while I was visiting the University of Chicago for a few months, Paul Halmos reported the following result to me, just proved by his student Errett Bishop.

THEOREM. *If $\{A_n\}$ is a sequence of positive operators on a Hilbert space $\mathcal{H}$ such that $\sum_{n=1}^{\infty}\langle A_n x, y\rangle = \langle x, y\rangle$ for each x and y in $\mathcal{H}$ (that is, $\sum A_n$ is weak-operator convergent to I), then there is a Hilbert space $\mathcal{K}$, containing $\mathcal{H}$, and a sequence $\{E_n\}$ of mutually orthogonal projections E_n on $\mathcal{K}$, with sum I, such that EE_nE is A_nE for each n, where E is the projection of $\mathcal{K}$ onto $\mathcal{H}$.*

We say that the *compression* of each E_n to $\mathcal{H}$ is A_n. As Halmos reported it to me, Bishop had done this "by hand." Thinking about what Bishop might have put into the argument helped me to finish something I needed and was struggling with at that time.

THEOREM. *If η is a linear mapping from one C^*-algebra $\mathfrak{A}$ to another that transforms each positive operator to a positive operator (we say that η is a* positive linear mapping) *and $\eta(I) \le I$, then $\eta(A)^2 \le \eta(A^2)$, for each self-adjoint A in $\mathfrak{A}$.*

A few days after hearing it from Halmos and completing the above, I mentioned Bishop's result to Segal. He recognized, almost instantly, that it was a consequence of a result of Neumark [**N**]. In effect, the Neumark result states that a positive-operator-valued measure on a Hilbert space is the compression of a standard, projection-valued measure (a *spectral* measure) on a larger space. In Bishop's case, the space is the set of natural numbers and A_n is the measure of $\{n\}$. Apparently, Segal had suggested to his student, Stinespring, that Stinespring generalize the Neumark result to the non-commutative case.

To see what this means, we note that, using Riesz–Markov, a regular Borel measure on a compact space and the corresponding positive linear functional on $C(X)$, the continuous, complex-valued functions on X, which represents integration relative to that measure, are equivalent — one can pass from one to the other. So Neumark's positive-operator-valued measure on, say, a compact Hausdorff space X amounts to a linear mapping η from $C(X)$ to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space on which the operator values of the measure act. The assumption that the operator values are positive translates to the assumption that η is a positive linear mapping of $C(X)$ into $\mathcal{B}(\mathcal{H})$. What happens to the conclusion of Neumark's theorem in this setting? A measurable set in X corresponds to its characteristic function which is idempotent, and the improved (spectral) projection-valued measure of the theorem assigns to this function (or set) a projection — so, a self-adjoint idempotent. Moreover, orthogonal projections are assigned to disjoint sets (that is, to characteristic functions with product 0). It is just a short jump from this analysis to realizing that the output of the Neumark theorem in the continuous function framework should be a homomorphism φ of $C(X)$ into $\mathcal{B}(\mathcal{K})$ (for which $\varphi(\bar{f}) = \varphi(f)^*$), where $\mathcal{H} \subseteq \mathcal{K}$, such that $\varphi(f)$ compressed to $\mathcal{H}$ is $\eta(f)$ for each f in $C(X)$. In other words, the positive linear mapping η is the composition of a representation (of $C(X)$ on $\mathcal{K}$) and a compression (to $\mathcal{H}$).

Now, it was nice enough to "modernize" the Neumark theorem by taking it from the measure theory framework to the $C(X)$ framework, but Stinespring's goal was to make it non-commutative — to replace $C(X)$ by a C*-algebra $\mathfrak{A}$. If we look at the Neumark proof and understand its essential elements, we can apply it to the $C(X)$ case and, then, to the case where an arbitrary C*-algebra replaces $C(X)$. Doing that, we see that the argument amounts to applying a GNS construction to the appropriate structure. For the remainder of this section, we detail that construction and the basics of the Stinespring concept of *completely positive mappings.* We begin with an example that underscores the need for such a concept.

EXAMPLE 2.1. Let η be the mapping of $M_n(\mathbb{C})$ into itself that assigns to each matrix $[a_{jk}]$ its transpose matrix (whose (j,k) entry is a_{kj}). Then η is a $*$ anti-automorphism of $M_n(\mathbb{C})$; η is a positive linear mapping of $M_n(\mathbb{C})$ into itself. When $n \geq 2$, the (unique) linear mapping $\eta \otimes \iota$ of $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$ into itself that assigns $\eta(A) \otimes B$ to $A \otimes B$ is *not* a positive linear mapping.

To see this, note that, since matrices are multiplied by a scalar and added on an entry-by-entry basis, η is a linear mapping of $M_n(\mathbb{C})$ onto itself. The (j,k) entry of $\eta([a_{jk}][b_{jk}])$ is the (k,j) entry of $[a_{jk}][b_{jk}]$, namely $\sum_{r=1}^{n} a_{kr}b_{rj}$, which is the (j,k) entry of the matrix $\eta([b_{jk}])\eta([a_{jk}])$. Hence

$$\eta([a_{jk}][b_{jk}]) = \eta([b_{jk}])\eta([a_{jk}]),$$

and η is an anti-automorphism of $M_n(\mathbb{C})$.

The (j,k) entry of $\eta([a_{jk}]^*)$ is the (k,j) entry of $[a_{jk}]^*$, which is $\bar{a}_{jk}$; while the (j,k) entry of $\eta([a_{jk}])^*$ is the complex conjugate of the (k,j) entry of $\eta([a_{jk}])$, namely $\bar{a}_{jk}$. Thus η is a $*$ anti-automorphism of $M_n(\mathbb{C})$.

Each $*$ anti-homomorphism φ of a C*-algebra $\mathfrak{A}$ is a positive linear mapping; for with A a positive element of $\mathfrak{A}$,

$$\varphi(A) = \varphi((A^{\frac{1}{2}})^2) = \varphi(A^{\frac{1}{2}})^2 \geq 0.$$

In particular, η is a positive linear mapping.

Each element of the algebra $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$ has a representation as a 2×2 matrix with entries from $M_n(\mathbb{C})$ — the matrix representing $A \otimes B$ is

$$\begin{bmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{bmatrix},$$

where $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and that of $(\eta \otimes \iota)(A \otimes B)$ $(= \eta(A) \otimes B)$ is

$$\begin{bmatrix} b_{11}\eta(A) & b_{12}\eta(A) \\ b_{21}\eta(A) & b_{22}\eta(A) \end{bmatrix}.$$

That is, the effect of $\eta \otimes \iota$ on the 2×2 matrix over $M_n(\mathbb{C})$ representing $A \otimes B$ is to transpose each $n \times n$ entry. Since this same process applied to all elements of $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$ is a linear mapping, it is the linear mapping $\eta \otimes \iota$.

Let T be the matrix with 0 at each entry except for the entries with the positions labeled as $(1,1), \dots, (n+1, n+1), (n, n+1)$, and $(n+1, n)$ at which entries the value 1 appears (T is a $2n \times 2n$ matrix). Then $T \geq 0$, but $(\eta \otimes \iota)(T)$ has 1 at the $(1, 2n)$ entry and 0 at the $(2n, 2n)$ entry; so that $(\eta \otimes \iota)(T)$ is not positive, and $\eta \otimes \iota$ is not a positive linear mapping. □

DEFINITION 2.2. A positive linear mapping η of a C*-algebra $\mathfrak{A}$ is said to be *completely positive* when, for each positive integer $n, \eta \otimes \iota_n$, the (unique) linear mapping whose value at $A \otimes B$ is $\eta(A) \otimes B$ for each A in $\mathfrak{A}$ and each B in $M_n(\mathbb{C})$, is positive.

We prove the following elementary facts about completely positive mappings.

PROPOSITION 2.3. *(i) η is completely positive when η is a * homomorphism.*

(ii) η is completely positive when $\eta(A) = TAT^*$ for each A in $\mathfrak{A}$, where $\mathfrak{A}$ acts on the Hilbert space $\mathcal{H}$ and T is a given bounded linear transformation of $\mathcal{H}$ into another Hilbert space $\mathcal{K}$.

(iii) η is completely positive when η is a composition of completely positive mappings.

*(iv)η is completely positive when $\eta(A) = T\varphi(A)T^*$, where φ is a * homomorphism of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ and T is a bounded linear transformation of the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{K}$.*

(v) Not each positive linear mapping of a C-algebra is completely positive.*

PROOF. (i) We show that $\eta \otimes \iota_n$ is a * homomorphism of $\mathfrak{A} \otimes M_n(\mathbb{C})$ for each positive integer n when η is a * homomorphism. To see this, it suffices to show that

$$(\eta \otimes \iota_n)(RS) = (\eta \otimes \iota_n)(R)(\eta \otimes \iota_n)(S)$$

for all R and S in some set of linear generators for $\mathfrak{A} \otimes M_n(\mathbb{C})$. Now, for all A_1, A_2 in $\mathfrak{A}$ and B_1, B_2 in $M_n(\mathbb{C})$,

$$\begin{aligned}(\eta \otimes \iota_n)((A_1 \otimes B_1)(A_2 \otimes B_2)) &= (\eta \otimes \iota_n)(A_1A_2 \otimes B_1B_2) \\ &= \eta(A_1A_2) \otimes B_1B_2 \\ &= \eta(A_1)\eta(A_2) \otimes B_1B_2 \\ &= (\eta(A_1) \otimes B_1)(\eta(A_2) \otimes B_2) \\ &= (\eta \otimes \iota_n)(A_1 \otimes B_1)(\eta \otimes \iota_n)(A_2 \otimes B_2).\end{aligned}$$

Hence $\eta \otimes \iota_n$ is a * homomorphism and is, therefore, a positive linear mapping. Thus η is completely positive when η is a * homomorphism.

(ii) Each element A of $\mathfrak{A} \otimes M_n(\mathbb{C})$ has a representation as an $n \times n$ matrix $[A_{jk}]$ with entries A_{jk} from $\mathfrak{A}$, and $(\eta \otimes \iota_n)(A)$ has $[\eta(A_{jk}]$ as its representing matrix. Thus, if $[A_{jk}]$ is positive and η arises from the linear transformation T as described in the statement of this proposition, $(\eta \otimes \iota_n)(A)$ has $[TA_{jk}T^*]$ as its representing matrix. But

$$[TA_{jk}T^*] = [T_{jk}][A_{jk}][T_{jk}]^*,$$

where $T_{jj} = T$ for j in $\{1, \dots, n\}$ and $T_{jk} = 0$ when $j \neq k$. Thus $(\eta \otimes \iota_n)(A) \geq 0$ when $A \geq 0$; and η is a completely positive mapping in this case.

(iii) If $\eta = \eta_1 \circ \eta_2$, then, employing the $n \times n$ matrix representation of $\mathfrak{A} \otimes M_n(\mathbb{C})$ as in (ii), we see that for each positive integer n,

$$\eta \otimes \iota_n = (\eta_1 \otimes \iota_n) \circ (\eta_2 \otimes \iota_n).$$

Since the composition of positive linear mappings is a positive linear mapping, $\eta \otimes \iota_n$ is positive when each of η_1 and η_2 is completely positive.

(iv) From (i) and (ii), η is the composition of completely positive mappings. Hence, from (iii), η is completely positive.

(v) The mapping η described in the preceding example is a positive linear mapping η of a C*-algebra such that η is not completely positive. □

THEOREM 2.4. *Let η be a completely positive mapping of a C^*-algebra $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let $\{e_a\}_{a\in\mathbb{A}}$ be an orthonormal basis for $\mathcal{H}$. Denote by $\tilde{\mathfrak{A}}$ the linear space of functions from $\mathbb{A}$ to $\mathfrak{A}$ that take the value 0 at all but a finite number of elements of $\mathbb{A}$, where $\tilde{\mathfrak{A}}$ is provided with pointwise addition and scalar multiplication (so that $\tilde{\mathfrak{A}}$ is the restricted direct sum of $\mathfrak{A}$ with itself over the index set $\mathbb{A}$). Then*

(i) $\langle \tilde{A}, \tilde{A}' \rangle = \sum_{a,a'\in\mathbb{A}} \langle \eta(A'^*_{a'} A_a)e_a, e_{a'} \rangle$ *defines an inner product on $\tilde{\mathfrak{A}}$, where $\tilde{A} = \{A_a\}_{a\in\mathbb{A}}$ and $\tilde{A}' = \{A'_{a'}\}_{a'\in\mathbb{A}}$;*

(ii) $0 = \langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{B}, \tilde{A}$ *for each $\tilde{B}$ in $\tilde{\mathfrak{A}}$, when $\langle \tilde{A}, \tilde{A} \rangle = 0$, $\tilde{\mathfrak{L}}$ is a linear space, where $\tilde{\mathfrak{L}} = \{\tilde{A} \in \tilde{\mathfrak{A}} : \langle \tilde{A}, \tilde{A} \rangle = 0\}$, and $\langle \tilde{A} + \tilde{\mathfrak{L}}, \tilde{B} + \tilde{\mathfrak{L}} \rangle_0 = \langle \tilde{A}, \tilde{B} \rangle$ defines a definite inner product on $\mathcal{K}_0$, the quotient space $\tilde{\mathfrak{A}}/\tilde{\mathfrak{L}}$;*

(iii) $0 \leq \langle \tilde{B}, \tilde{B} \rangle \leq \|A\|^2 \langle \tilde{A}, \tilde{A} \rangle$, *where $\tilde{A} = \{A_a\}_{a\in\mathbb{A}}$ and $\tilde{B} = \{AA_a\}_{a\in\mathbb{A}}$; $\varphi_0(A)$ is a bounded linear mapping of $\mathcal{K}_0$ into $\mathcal{K}_0$, where $\varphi_0(A)(\tilde{A} + \tilde{\mathfrak{L}}) = \tilde{B} + \tilde{\mathfrak{L}}$;*

(iv) φ *is a representation of $\mathfrak{A}$ on $\mathcal{K}$, where $\varphi(A)$ is the (unique) bounded extension of $\varphi_0(A)$ from $\mathcal{K}_0$ to $\mathcal{K}$, and $\mathcal{K}$ is the completion of $\mathcal{K}_0$ relative to $\langle\ ,\ \rangle_0$;*

(v) $\{\tilde{I}_a + \tilde{\mathfrak{L}}\}_{a\in\mathbb{A}}$ *is an orthonormal set in $\mathcal{K}$, when $\eta(I) = I$, where $\tilde{I}_a$ is the element of $\tilde{\mathfrak{A}}$ with I at the a coordinate and 0 at all others;*

(vi) $V^*\varphi(A)V = \eta(A)\,(A \in \mathfrak{A})$, *when $\eta(I) = I$, where V is the (unique) isometry of $\mathcal{H}$ into $\mathcal{K}$ such that $Ve_a = \tilde{I}_a + \tilde{\mathfrak{L}}$ for each a in $\mathbb{A}$.*

PROOF. (i) With $\tilde{A}$ and $\tilde{A}'$ in $\tilde{\mathfrak{A}}$, there are at most a finite number of indices in $\mathbb{A}$, say, $1, \ldots, n$, at which $\tilde{A}$ or $\tilde{A}'$ have non-zero coordinates. Thus

$$\langle \tilde{A}, \tilde{A}' \rangle = \sum_{j,k=1}^{n} \langle \eta(A_k'^* A_j) e_j, e_k \rangle,$$

and the sum defining $\langle \tilde{A}, \tilde{A}' \rangle$ converges. With $\tilde{B}$ $(= \{B_a\}_{a\in\mathbb{A}})$ in $\tilde{\mathfrak{A}}$ and b a scalar,

$$\begin{aligned}\langle \tilde{A} + b\tilde{B}, \tilde{A}' \rangle &= \sum_{a,a'\in\mathbb{A}} \langle \eta(A_{a'}'^*(A_a + bB_a)) e_a, e_{a'} \rangle \\ &= \langle \tilde{A}, \tilde{A}' \rangle + b\langle \tilde{B}, \tilde{A}' \rangle.\end{aligned}$$

Since η is a positive linear mapping, it is hermitian. Hence $\eta(A^*) = \eta(A)^*$ for each A in $\mathfrak{A}$, and

$$\begin{aligned}\langle \tilde{A}, \tilde{A}' \rangle &= \sum_{a,a'\in\mathbb{A}} \langle \eta(A_{a'}'^* A_a) e_a, e_{a'} \rangle \\ &= \sum_{a,a'\in\mathbb{A}} \langle \eta(A_a^* A_{a'}')^* e_a, e_{a'} \rangle \\ &= \sum_{a,a'\in\mathbb{A}} \overline{\langle \eta(A_a^* A_{a'}') e_{a'}, e_a \rangle} \\ &= \overline{\langle \tilde{A}', \tilde{A} \rangle}.\end{aligned}$$

Let T be the $n \times n$ matrix whose non-zero entries are in the first row, and this first row consists of the non-zero coordinates of $\tilde{A}$. (We assume that these non-zero coordinates have the indices $\{1, \ldots, n\}$). Then T^*T is the $n \times n$ matrix whose (j, k) entry is $A_j^* A_k$. Since η is assumed to be completely positive, the matrix S whose (j, k) entry is $\eta(A_j^* A_k)$ is positive. With e the vector $\{e_1, \ldots, e_n\}$ in $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$,

$$0 \le \langle Se, e \rangle = \sum_{j,k=1}^{n} \langle \eta(A_j^* A_k) e_k, e_j \rangle = \langle \tilde{A}, \tilde{A} \rangle.$$

Hence $(\tilde{A}, \tilde{A}') \to \langle \tilde{A}, \tilde{A}' \rangle$ is an inner product on $\tilde{\mathfrak{A}}$.

(ii) By applying the Cauchy–Schwarz inequality to the inner product defined in (i), we see that

$$0 = \langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{B}, \tilde{A} \rangle,$$

when $\langle A, \tilde{A} \rangle = 0$ and $\tilde{B} \in \tilde{\mathfrak{A}}$. Hence $\tilde{\mathfrak{L}}$ is a linear space. At the same time, the equation

$$\langle \tilde{A} + \tilde{\mathfrak{L}}, \tilde{B} + \tilde{\mathfrak{L}} \rangle_0 = \langle \tilde{A}, \tilde{B} \rangle$$

defines (unambiguously) an inner product on $\mathcal{K}_0$. If $\langle \tilde{B} + \tilde{\mathfrak{L}}, \tilde{B} + \tilde{\mathfrak{L}} \rangle_0 = 0$, then $\langle \tilde{B}, \tilde{B} \rangle = 0$ and $\tilde{B} \in \tilde{\mathfrak{L}}$, whence $\tilde{B} + \tilde{\mathfrak{L}} = 0 + \tilde{\mathfrak{L}}$. Hence $\langle\ ,\ \rangle_0$ is a definite inner product on $\mathcal{K}_0$.

(iii) Let T be the $n \times n$ matrix whose non-zero entries are in the first row, and this first row consists of the non-zero coordinates of $\tilde{A}$. Let R and S be the $n \times n$ matrices whose only non-zero entries are their (1,1) entries, and these are A^*A and $\| A \|^2 I$, respectively. Since $A^*A \le \| A \|^2 I$, $R \le S$ and $T^*RT \le T^*ST$. With

$A_1, \ldots, A_n$ the non-zero coordinates of $\tilde{A}$, the (j,k) entries of T^*RT and T^*ST are, respectively, $A_j^*A^*AA_k$ and $\|A\|^2A_j^*A_k$. Since η is completely positive, the matrix whose (j,k) entry is $\eta(A_j^*A^*AA_k)$ is less than or equal to the matrix whose (j,k) entry is $\|A\|^2\eta(A_j^*A_k)$. Applying the vector state corresponding to the vector $\{e_1, \ldots, e_n\}$ to these matrices, we have

$$\begin{aligned} 0 \le \langle \tilde{B}, \tilde{B} \rangle &= \sum_{j,k=1}^{n} \langle \eta(A_j^*A^*AA_k)e_k, e_j \rangle \\ &\le \sum_{j,k=1}^{n} \|A\|^2 \langle \eta(A_j^*A_k)e_k, e_j \rangle \\ &= \|A\|^2 \langle \tilde{A}, \tilde{A} \rangle. \end{aligned}$$

It follows that $\varphi_0(A)(\tilde{A}+\tilde{\mathfrak{L}}) = 0$ if $\tilde{A} \in \tilde{\mathfrak{L}}$, so that $\varphi_0(A)$ is a well-defined linear mapping of $\mathcal{K}_0$ into itself. From the same inequality, we have that $\|\varphi_0(A)\| \le \|A\|$; hence $\varphi_0(A)$ has a (unique) bounded extension $\varphi(A)$ mapping the completion $\mathcal{K}$ of $\mathcal{K}_0$ into itself.

(iv) To see that φ is linear, note that, with A and B in $\mathfrak{A}$, b a scalar, and $\{A_a\}_{a \in \mathbb{A}}$ $(= \tilde{A})$ in $\tilde{\mathfrak{A}}$,

$$\varphi(A + bB)(\tilde{A} + \tilde{\mathfrak{L}}) = \tilde{B} + \tilde{\mathfrak{L}},$$

where $\tilde{B} = \{(A+bB)A_a\}_{a\in\mathbb{A}} = \{AA_a\}_{a\in\mathbb{A}} + b\{BA_a\}_{a\in\mathbb{A}}$. Hence

$$\varphi(A + bB)(\tilde{A} + \tilde{\mathfrak{L}}) = \varphi(A)(\tilde{A} + \tilde{\mathfrak{L}}) + b\varphi(B)(\tilde{A} + \tilde{\mathfrak{L}}),$$

and the bounded operators $\varphi(A + bB)$ and $\varphi(A) + b\varphi(B)$ agree on the dense subset $\mathcal{K}_0$ of $\mathcal{K}$. Hence $\varphi(A + bB) = \varphi(A) + b\varphi(B)$. Note, too, that

$$\varphi(AB)(\tilde{A} + \tilde{\mathfrak{L}}) = \tilde{C} + \tilde{\mathfrak{L}},$$

where $\tilde{C} = \{ABA_a\}_{a\in\mathbb{A}}$. Thus

$$\varphi(AB)(\tilde{A} + \tilde{\mathfrak{L}}) = \varphi(A)\varphi(B)(\tilde{A} + \tilde{\mathfrak{L}}),$$

and the bounded operators $\varphi(AB)$ and $\varphi(A)\varphi(B)$ agree on the dense subset $\mathcal{K}_0$ of $\mathcal{K}$. It follows that $\varphi(AB) = \varphi(A)\varphi(B)$. Finally, when $A = \{A_a\}_{a\in\mathbb{A}}$ and $\tilde{B} = \{B_a\}_{a\in\mathbb{A}}$,

$$\begin{aligned} \langle \varphi(A)(\tilde{A} + \tilde{\mathfrak{L}}), \tilde{B} + \tilde{\mathfrak{L}} \rangle_0 &= \sum_{a,a' \in \mathbb{A}} \langle \eta(B_{a'}^*AA_a)e_a, e_{a'} \rangle \\ &= \sum_{a,a' \in \mathbb{A}} \langle \eta((A^*B_{a'})^*A_a)e_a, e_{a'} \rangle \\ &= \langle \tilde{A} + \tilde{\mathfrak{L}}, \varphi(A^*)(\tilde{B} + \tilde{\mathfrak{L}}) \rangle_0, \end{aligned}$$

so that $\varphi(A)^* = \varphi(A^*)$. Hence φ is a representation of $\mathfrak{A}$ on $\mathcal{K}$.

(v) Under the assumption that $\eta(I) = I$, we have

$$\langle \tilde{I}_a + \tilde{\mathfrak{L}}, \tilde{I}_{a'} + \tilde{\mathfrak{L}} \rangle_0 = \langle \tilde{I}_a, \tilde{I}_{a'} \rangle = \langle \eta(I)e_a, e_{a'} \rangle = \langle e_a, e_{a'} \rangle,$$

so that $\{\tilde{I}_a + \tilde{\mathcal{L}}\}_{a \in \mathbb{A}}$ is an orthonormal set in $\mathcal{K}$.

(vi) Since $\eta(I) = I$, $\{\tilde{I}_a + \tilde{\mathcal{L}}\}_{a \in \mathbb{A}}$ is an orthonormal set in $\mathcal{K}$ from (v), and there is a unique isometry V mapping $\mathcal{H}$ into $\mathcal{K}$ such that $Ve_a = \tilde{I}_a + \tilde{\mathcal{L}}$ for all a in $\mathbb{A}$. For all a and a' in $\mathbb{A}$ and A in $\mathfrak{A}$, we have

$$\begin{aligned}\langle V^*\varphi(A)Ve_a, e_{a'}\rangle &= \langle \varphi(A)(\tilde{I}_a + \tilde{\mathcal{L}}), \tilde{I}_{a'} + \tilde{\mathcal{L}}\rangle_0 \\ &= \langle \eta(I^*A)e_a, e_{a'}\rangle.\end{aligned}$$

Thus $V^*\varphi(A)V = \eta(A)$ for all A in $\mathfrak{A}$. □

PROPOSITION 2.5. *Adopt the notation and assumptions of the preceding theorem (exclusive of the assumption that $\eta(I) = I$). Let $\mathcal{H}_0$ be the dense linear manifold in $\mathcal{H}$ consisting of finite linear combinations of $\{e_a\}_{a \in \mathbb{A}}$, and let $T_0(\sum_{a \in \mathbb{A}_0} r_a e_a)$ be $\sum_{a \in \mathbb{A}_0} r_a(\tilde{I}_a + \tilde{\mathcal{L}})$ for each finite subset $\mathbb{A}_0$ of $\mathbb{A}$. Then*

(i) *T_0 is a bounded linear transformation;*

(ii) *$T^*\varphi(A)T = \eta(A)$ ($A \in \mathfrak{A}$), where T is the (unique) bounded extension of T_0 from $\mathcal{H}_0$ to $\mathcal{H}$;*

(iii) *when $\eta(I) = I$, there is a Hilbert space $\mathcal{H}'$ containing $\mathcal{H}$ and a representation φ' of $\mathfrak{A}$ on $\mathcal{H}'$ such that $E\varphi'(A)E = \eta(A)E$ for each A in $\mathfrak{A}$, where E is the projection of $\mathcal{H}'$ on $\mathcal{H}$.*

PROOF. (i) As in the proof of (v) of the preceding theorem,

$$\langle \tilde{I}_a + \tilde{\mathcal{L}}, \tilde{I}_{a'} + \tilde{\mathcal{L}}\rangle_0 = \langle \eta(I)e_a, e_{a'}\rangle,$$

whence

$$\begin{aligned}\Big\|T_0\Big(\sum_{a \in \mathbb{A}_0} r_a e_a\Big)\Big\|^2 &= \sum_{a,a' \in \mathbb{A}_0} r_a \bar{r}_{a'} \langle \eta(I)e_a, e_{a'}\rangle \\ &= \Big\langle \eta(I)\Big(\sum_{a \in \mathbb{A}_0} r_a e_a\Big), \sum_{a' \in \mathbb{A}_0} r_{a'} e_{a'}\Big\rangle \\ &\leq \|\eta(I)\| \,\Big\|\sum_{a \in \mathbb{A}_0} r_a e_a\Big\|^2.\end{aligned}$$

Thus $\|T_0\| \leq \|\eta(I)\|^{\frac{1}{2}}$.

(ii) For all a and a' in $\mathbb{A}$ and all A in $\mathfrak{A}$,

$$\begin{aligned}\langle T^*\varphi(A)Te_a, e_{a'}\rangle &= \langle \varphi(A)(\tilde{I}_a + \tilde{\mathcal{L}}), \tilde{I}_{a'} + \tilde{\mathcal{L}}\rangle_0 \\ &= \langle \eta(A)e_a, e_{a'}\rangle.\end{aligned}$$

Hence $T^*\varphi(A)T = \eta(A)$ for all A in $\mathfrak{A}$.

(iii) From (vi) of the preceding theorem, V is a unitary transformation of $\mathcal{H}$ onto some subspace $\mathcal{K}_1$ of $\mathcal{K}$ and $V^*\varphi(A)V = \eta(A)$ for all A in $\mathfrak{A}$. Let $\mathcal{K}_2$ be the orthogonal complement of $\mathcal{K}_1$ in $\mathcal{K}, \mathcal{H}'$ be $\mathcal{H} \oplus \mathcal{K}_2$, and $U(x, y)$ be $Vx + y$ for x in $\mathcal{H}$ and y in $\mathcal{K}_2$. Identify $\mathcal{H}$ with $\{(x, 0) : x \in \mathcal{H}\}$ and $\eta(A)(x, 0)$ with $(\eta(A)x, 0)$. Define $\varphi'(A)$ to be $U^*\varphi(A)U$. Then U is a unitary transformation of $\mathcal{H}'$ onto $\mathcal{K}$, and φ' is a representation of $\mathfrak{A}$ on $\mathcal{H}'$. With A in $\mathfrak{A}$, x in $\mathcal{H}$, and y in

$\mathcal{K}_2$, let $\varphi(A)Vx$ be $u+v$, where $u \in \mathcal{K}_1$ and $v \in \mathcal{K}_2$. We have, by an application of [**K-R I; Proposition 2.5.13**], that $V^*v = 0$, and

$$\begin{aligned} E\varphi'(A)E(x,y) &= E\varphi'(A)(x,0) = EU^*\varphi(A)U(x,0) \\ &= EU^*\varphi(A)Vx = EU^*(u+v) \\ &= E(V^*u, v) = (V^*(u+v), 0) \\ &= (V^*\varphi(A)Vx, 0) = (\eta(A)x, 0) \\ &= \eta(A)E(x,y). \end{aligned}$$

Hence $E\varphi'(A)E = \eta(A)E$. □

THEOREM 2.6. *Let η be a linear mapping of a C^*-algebra $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$; let φ and φ' be representations of $\mathfrak{A}$ on Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, respectively; and let T and T' be bounded linear transformations of $\mathcal{H}$ into $\mathcal{K}$ and $\mathcal{K}'$, respectively, such that $T^*\varphi(A)T = \eta(A)$ and $T'^*\varphi'(A)T' = \eta(A)$ for each A in $\mathfrak{A}$. Let $\mathcal{K}_0$ and $\mathcal{K}'_0$ be the closure of the ranges of T and T', respectively, and let E and E' be the projections of $\mathcal{K}$ and $\mathcal{K}'$ onto $\mathcal{K}_0$ and $\mathcal{K}'_0$, respectively. Then there is a unitary transformation U of $\mathcal{K}_0$ onto $\mathcal{K}'_0$ such that, for each A in $\mathfrak{A}$,*

$$T' = UT, \quad E\varphi(A)E|\mathcal{K}_0 = U^*E'\varphi'(A)E'U.$$

PROOF. Define U_0Tx to be $T'x$ for each x in $\mathcal{H}$. Then $T' = U_0T$. With A in $\mathfrak{A}$,

$$\begin{aligned} \langle \varphi(A)Tx, Ty \rangle &= \langle T^*\varphi(A)Tx, y \rangle = \langle \eta(A)x, y \rangle \\ &= \langle T'^*\varphi'(A)T'x, y \rangle = \langle \varphi'(A)T'x, T'y \rangle \end{aligned}$$

for all x and y in $\mathcal{H}$. Letting A be I, we conclude that

$$\langle Tx, Ty \rangle = \langle T'x, T'y \rangle = \langle U_0Tx, U_0Ty \rangle.$$

Thus U_0 is well defined, linear, and extends (uniquely) to a unitary transformation U mapping $\mathcal{K}_0$ onto $\mathcal{K}'_0$, and $T' = UT$. At the same time,

$$\begin{aligned} \langle E\varphi(A)ETx, Ty \rangle &= \langle \varphi(A)Tx, Ty \rangle \\ &= \langle \varphi'(A)T'x, T'y \rangle \\ &= \langle U^*E'\varphi'(A)E'UTx, Ty \rangle \end{aligned}$$

for all x and y in $\mathcal{H}$. Since the range of T is dense in $\mathcal{K}_0$ and the operators $E\varphi(A)E|\mathcal{K}_0$ and $U^*E'\varphi'(A)E'U$ are bounded operators on $\mathcal{K}_0$, we have that

$$E\varphi(A)E|\mathcal{K}_0 = U^*E'\varphi'(A)E'U. \qquad \square$$

To complete the circle with Neumark's theorem, Stinespring shows [**St**] that when $\mathfrak{A}$ is abelian η is automatically completely positive and, independently, Størmer [**Sr; Lemma 6.1**] and Arveson [**A; Proposition 1.2.2**] showed that the same is true, no matter what $\mathfrak{A}$ is, when η maps into an abelian subalgebra of $\mathcal{B}(\mathcal{H})$. These last results are not obvious. They can be proved by the technique of *multi-states*. We discuss and apply that technique.

LEMMA 2.7. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$, $M_n(\mathfrak{A})$ be the C^*-algebra of $n \times n$ matrices with entries from $\mathfrak{A}$, and $M_n(\mathfrak{A})^+$ be the cone of positive elements in $M_n(\mathfrak{A})$. Then*

(i) *the matrix $[\langle x_k, x_j\rangle]$ with (j,k) entry $\langle x_k, x_j\rangle$ is positive, where $x_1, \dots, x_n$ are vectors in $\mathcal{H}$;*
(ii) *the matrix $[A_j^* A_k]$ with (j,k) entry $A_j^* A_k$ is in $M_n(\mathfrak{A})^+$ for each set of n elements $\{A_1, \dots, A_n\}$ in $\mathfrak{A}$, and the matrix all of whose entries are a given positive A in $\mathfrak{A}$ is in $M_n(\mathfrak{A})^+$;*
(iii) *each positive element of $M_n(\mathfrak{A})$ is a sum of matrices of the form $[A_j^* A_k]$;*
(iv) *$[A_{jk}] \to [\langle A_{jk} x_k, x_j\rangle]\colon M_n(\mathfrak{A}) \to M_n(\mathbb{C})$ is a positive linear mapping for each set of n vectors $\{x_1, \dots, x_n\}$ in $\mathcal{H}$, and $[\langle A_{jk} x, x\rangle] \geq 0$ for each x in $\mathcal{H}$ when $[A_{jk}] \geq 0$;*
(v) *if $[A_{jk}]$ is in $M_n(\mathfrak{A})^+$ and $[B_{jk}]$ is in $M_n(\mathfrak{A}')^+$, then $[A_{jk} B_{jk}] \geq 0$.*

PROOF. (i) Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for an n-dimensional subspace of $\mathcal{H}$ containing $x_1, \dots, x_n$. Note that $[\langle x_k, x_j\rangle]$ is the matrix of T^*T relative to $\{e_1, \dots, e_n\}$, where $Te_j = x_j$. The (j,k) entry of the matrix for T^*T relative to the basis $\{e_1, \dots, e_n\}$ is $\langle T^*Te_k, e_j\rangle = \langle Te_k, Te_j\rangle = \langle x_k, x_j\rangle$. Thus $[\langle x_k, x_j\rangle] \geq 0$.

(ii) If $\tilde{A}$ is the element of $M_n(\mathfrak{A})$ whose first row is $A_1, \dots, A_n$ and all of whose other entries are 0, then $\tilde{A}^*\tilde{A} = [A_j^* A_k] \in M_n(\mathfrak{A})^+$. If $A \in \mathfrak{A}^+$ and each A_j is $A^{\frac{1}{2}}$, then each $A_j^* A_k$ is A. Hence the $n \times n$ matrix all of whose entries are A is in $M_n(\mathfrak{A})^+$.

(iii) Each element of $M_n(\mathfrak{A})^+$ has the form

$$[B_{jk}]^*[B_{jk}] = \left[\sum_{r=1}^{n} B_{rj}^* B_{rk}\right] = \sum_{r=1}^{n} [B_{rj}^* B_{rk}],$$

where $[B_{jk}] \in M_n(\mathfrak{A})$. If $A_j = B_{rj}$, then $[B_{rj}^* B_{rk}] = [A_j^* A_k]$.

(iv) The mapping described is clearly linear; hence, from (iii), it suffices to show that the value of this mapping at each matrix of the form $[A_j^* A_k]$ is an element of $M_n(\mathbb{C})^+$. But, from (i),

$$[\langle A_j^* A_k x_k, x_j\rangle] = [\langle A_k x_k, A_j x_j\rangle] \geq 0.$$

If each of the vectors x_j is the same vector x in $\mathcal{H}$, then our mapping becomes $[A_{jk}] \to [\langle A_{jk} x, x\rangle]$. Hence $[\langle A_{jk} x, x\rangle] \geq 0$ when $[A_{jk}] \geq 0$.

(v) From (iii), $[A_{jk}]$ is a sum $\sum_{r=1}^{m} [A_j^{(r)*} A_k^{(r)}]$ with each $A_j^{(r)}$ in $\mathfrak{A}$. Thus $A_{jk} = \sum_{r=1}^{m} A_j^{(r)*} A_k^{(r)}$ and

$$[A_{jk} B_{jk}] = \sum_{r=1}^{m} [A_j^{(r)*} A_k^{(r)} B_{jk}].$$

Hence, it suffices to show that $[A_j^* A_k B_{jk}] \geq 0$ for each subset $\{A_1, \dots, A_n\}$ of $\mathfrak{A}$. By the same argument, it now suffices to show that $[A_j^* A_k B_j^* B_k] \geq 0$ for each subset $\{A_1, \dots, A_n\}$ of $\mathfrak{A}$ and each subset $\{B_1, \dots, B_n\}$ of $\mathfrak{A}'$. As A_j and A_j^*

commute with B_k and B_k^* for all j and k, $A_j^* A_k B_j^* B_k = (A_j B_j)^*(A_k B_k)$. Thus, from (ii),

$$[A_j^* A_k B_j^* B_k] = [(A_j B_j)^*(A_k B_k)] \geq 0,$$

and $[A_{jk} B_{jk}] \geq 0$. □

DEFINITION 2.8. An *n-positive functional* on (*n-state* of) a C*-algebra $\mathfrak{A}$ is a matrix $[\rho_{jk}]$ of linear functionals on $\mathfrak{A}$ such that $[\rho_{jk}(A_{jk})] \geq 0$ when $[A_{jk}] \in M_n(\mathfrak{A})^+$ (and $\rho_{jj}(I) = 1$ for j in $\{1, \ldots, n\}$). We speak of a *multi-state* when there is no need to specify n.

THEOREM 2.9. *(i) If $\mathfrak{A}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$ and $\{x_1, \ldots, x_n\}$ is a set of n (unit) vectors in $\mathcal{H}$, then $[\omega_{x_k,x_j}|\mathfrak{A}]$ is an n-positive functional on (n-state of) $\mathfrak{A}$.*

(ii) A linear mapping η of $\mathfrak{A}$ into a C-algebra $\mathfrak{B}$, such that $\eta(I) = I$, is completely positive, if and only if $[\rho_{jk} \circ \eta]$ is n-positive on $\mathfrak{A}$ for each n-state $[\rho_{jk}]$ of $\mathfrak{B}$.*

(iii) $[\rho_{jk}]$ is n-positive on (an n-state of) $\mathfrak{A}$ when each ρ_{jk} is the same positive functional (state) ρ on $\mathfrak{A}$.

(iv) a positive linear mapping of $\mathfrak{A}$ into an abelian C-algebra $\mathfrak{B}$ is completely positive.*

PROOF. (i) If $[A_{jk}] \in M_n(\mathfrak{A})^+$, then

$$0 \leq [\langle A_{jk} x_k, x_j \rangle] = [\omega_{x_k,x_j}(A_{jk})]$$

from (iv) of the preceding lemma, so that $[\omega_{x_k,x_j}|\mathfrak{A}]$ is n-positive on $\mathfrak{A}$. (If x_j is a unit vector, $\omega_{x_j,x_j}(I) = 1$ and $[\omega_{x_k,x_j}|\mathfrak{A}]$ is an n-state of $\mathfrak{A}$.)

(ii) Suppose η is completely positive and $[A_{jk}] \in M_n(\mathfrak{A})^+$. Then $[\eta(A_{jk})] \in M_n(\mathfrak{B})^+$ and $[(\rho_{jk} \circ \eta)(A_{jk})] \in M_n(\mathbb{C})^+$ for each n-positive functional $[\rho_{jk}]$ of $\mathfrak{B}$. Thus $[\rho_{jk} \circ \eta]$ is n-positive on $\mathfrak{A}$ when $[\rho_{jk}]$ is n-positive on $\mathfrak{B}$.

Suppose, now, that $[\rho_{jk} \circ \eta]$ is an n-state of $\mathfrak{A}$ for each n-state $[\rho_{jk}]$ of $\mathfrak{B}$. For this part, n-states will suffice in place of the stronger assumption on n-positive functionals. Suppose $\mathfrak{B}$ acts on a Hilbert space $\mathcal{K}$ and $\{x_1, \ldots, x_n\}$ is a set of n vectors in $\mathcal{K}$. Choose unit vectors $y_1, \ldots, y_n$ in $\mathcal{K}$ and non-negative (real) scalars $a_1, \ldots, a_n$ such that $a_j y_j = x_j$ for each j. From (i), $[\omega_{y_k,y_j}|\mathfrak{B}]$ is an n-state of $\mathfrak{B}$, so that $[\omega_{y_k,y_j} \circ \eta]$ is an n-state of $\mathfrak{A}$ by hypothesis. Thus, with $[A_{jk}]$ in $M_n(\mathfrak{A})^+$,

$$[\langle \eta(A_{jk}) y_k, y_j \rangle] \geq 0.$$

From (ii) and (v) of the preceding lemma, $[a_j a_k] \in M_n(\mathbb{C})^+$ and

$$[\langle \eta(A_{jk}) x_k, x_j \rangle] = [\langle \eta(A_{jk}) y_k, y_j \rangle a_j a_k] \geq 0.$$

Note that $\langle [B_{jk}]\tilde{x}, \tilde{x} \rangle = \langle [\langle B_{jk} x_k, x_j \rangle]\tilde{a}, \tilde{a} \rangle$, where $\tilde{x} = \{x_1, \ldots, x_n\}$ and $\tilde{a} = \{1, 1, \ldots 1\}$. With $\eta(A_{jk})$ in place of B_{jk}, we have that

$$\langle [\eta(A_{jk})]\tilde{x}, \tilde{x} \rangle = \langle [\langle \eta(A_{jk}) x_k, x_j \rangle]\tilde{a}, \tilde{a} \rangle \geq 0.$$

Thus $[\eta(A_{jk})] \in M_n(\mathfrak{B})^+$ and η is completely positive.

(iii) Let π be the GNS representation of $\mathfrak{A}$ corresponding to the positive functional (state) ρ. From (i) of the first proposition of this section, π is completely positive so that $[\pi(A_{jk})] \geq 0$ when $[A_{jk}] \in M_n(\mathfrak{A})^+$. In this case, $[\langle\pi(A_{jk})x, x\rangle] \geq 0$ for each vector x in the representation space for π, from (i), and in particular, for a generating (unit) vector x_0 for $\pi(\mathfrak{A})$ such that $\omega_{x_0} \circ \pi = \rho$. Thus

$$[\rho_{jk}(A_{jk})] = [\langle\pi(A_{jk})x_0, x_0\rangle] \geq 0,$$

and $[\rho_{jk}]$ is an n-positive functional on (n-state of) $\mathfrak{A}$, when $\rho_{jk} = \rho$ for all j and k in $\{1, \dots, n\}$.

(iv) Suppose η is a positive linear mapping of $\mathfrak{A}$ into an abelian C*-algebra $\mathfrak{B}$. Let $[A_{jk}]$ be an element of $M_n(\mathfrak{A})^+$. Since $\mathfrak{B}$ is abelian, $\mathfrak{B} \cong C(X)$. Using the identification of $M_n(\mathfrak{B})$ with the C*-algebra of continuous mappings from X to $M_n(\mathbb{C})$, and the fact that positive operators and matrices have positive square roots, we have that $[\eta(A_{jk}] \in M_n(\mathfrak{B})^+$ if and only if $[(\rho \circ \eta)(A_{jk})] \geq 0$ for each pure state ρ of $\mathfrak{B}$. Now $\rho \circ \eta$ is a positive linear functional on $\mathfrak{A}$, so that $[(\rho \circ \eta)(A_{jk})] \geq 0$ from (iii). Thus $[\eta(A_{jk})]$ is in $M_n(\mathfrak{B})^+$ and η is completely positive. □

THEOREM 2.10. *Let $\mathfrak{A}$ be an abelian C*-algebra and $[\rho_{jk}]$ be an $n \times n$ matrix of bounded linear functionals on $\mathfrak{A}$.*

(i) There is a representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ with a cyclic vector u and a matrix of operators H_{jk} in $\pi(\mathfrak{A})^-$ such that $\rho_{jk}(A) = \langle\pi(A)H_{jk}u, u\rangle$ for each A in $\mathfrak{A}$.

(ii) Suppose $[\rho_{jk}(A)] \geq 0$ for each A in $\mathfrak{A}^+$. Then $[H_{jk}(p)] \geq 0$ for each p in X, where $\pi(\mathfrak{A})^- \cong C(X)$, and we denote by the same symbol an element of $\pi(\mathfrak{A})^-$ and the function representing it. In addition, $[H_{jk}] \geq 0$.

(iii) $[\rho_{jk}]$ is an n-positive functional (n-state of) $\mathfrak{A}$ if and only if $[\rho_{jk}(A)] \geq 0$ for each A in $\mathfrak{A}^+$ (and $\rho_{jj}(I) = 1$).

(iv) Each positive linear mapping of $\mathfrak{A}$ is completely positive.

PROOF. (i) Express each ρ_{jk} as $\eta_{jk} + i\tau_{jk}$ with η_{jk} and τ_{jk} hermitian, and let ρ be $\sum_{j,k=1}^{n}(\eta_{jk}^+ + \eta_{jk}^- + \tau_{jk}^+ + \tau_{jk}^-)$. Let π be the GNS representation corresponding to ρ. Let u_0 be a unit generating vector for $\pi(\mathfrak{A})$. Since $\pi(\mathfrak{A})^-$ is abelian with a generating vector, $\pi(\mathfrak{A})^-$ is maximal abelian by **[K-R II; Corollary 7.2.16]**. Choose u_0 and a multiple u of u_0 so that $\rho(A) = \langle\pi(A)u, u\rangle$ for each A in $\mathfrak{A}$. As $\eta_{jk}^+, \eta_{jk}^-, \tau_{jk}^+, \tau_{jk}^-$ are positive linear functionals on $\mathfrak{A}$ dominated by ρ, they induce positive linear functionals on the image $\pi(\mathfrak{A})$ dominated by $\omega_u|\pi(\mathfrak{A})$ from **[K-R III; Exercise 4.6.23(ii)]**. Thus, from **[K-R II; Proposition 7.3.5]**, there are operators $A_{jk}^+, A_{jk}^-, B_{jk}^+, B_{jk}^-$ in $\pi(\mathfrak{A})^-$ such that

$$\eta_{jk}^+(A) = \langle\pi(A)A_{jk}^+u, u\rangle, \quad \eta_{jk}^-(A) = \langle\pi(A)A_{jk}^-u, u\rangle$$
$$\tau_{jk}^+(A) = \langle\pi(A)B_{jk}^+u, u\rangle, \quad \tau_{jk}^-(A) = \langle\pi(A)B_{jk}^-u, u\rangle$$

for each A in $\mathfrak{A}$. It follows that, for each A in $\mathfrak{A}$,

$$\rho_{jk}(A) = \langle\pi(A)H_{jk}u, u\rangle,$$

where $H_{jk} = A_{jk}^{+} - A_{jk}^{-} + i(B_{jk}^{+} - B_{jk}^{-})(\in \pi(\mathfrak{A})^{-})$.

(ii) From **[K-R I; Theorem 5.2.1]**, $\pi(\mathfrak{A})^{-} \cong C(X)$ for some extremely disconnected compact Hausdorff space X. For each T in $\mathfrak{A}$,

$$\begin{aligned}\langle H_{jk}\pi(T)u, \pi(T)u\rangle &= \langle \pi(T^*T)H_{jk}u, u\rangle = \rho_{jk}(T^*T)\\ &= \overline{\rho_{kj}(T^*T)} = \langle \pi(T)u, H_{kj}\pi(T)u\rangle.\end{aligned}$$

Since u is generating for $\mathfrak{A}$, $H_{jk} = H_{kj}^*$ and $[H_{jk}]$ is self-adjoint. If $[H_{jk}(p_0)] \not\geq 0$, there is some $\{a_1, \ldots, a_n\}$ $(= \tilde{a})$ in $\mathbb{C}^n$ such that $\langle [H_{jk}(p_0)]\tilde{a}, \tilde{a}\rangle < 0$. By continuity of all H_{jk}, there is a clopen subset X_0 of X containing p_0 such that $\langle [H_{jk}(p)]\tilde{a}, \tilde{a}\rangle < 0$ for each p in X_0. Let E be the characteristic function of X_0. With p in X_0,

$$0 > \sum_{j,k=1}^{n} E(p)H_{jk}(p)a_k\bar{a}_j,$$

whence

$$0 > \sum_{j,k=1}^{n} EH_{jk}\bar{a}_j a_k.$$

Since u is separating for $\pi(\mathfrak{A})^{-}$,

$$0 > \sum_{j,k=1}^{n} \bar{a}_j a_k \langle EH_{jk}u, u\rangle = \langle [\langle EH_{jk}u, u\rangle]\tilde{a}, \tilde{a}\rangle,$$

and $[\langle EH_{jk}u, u\rangle] \not\geq 0$. But for each A in $\mathfrak{A}^{+}$,

$$0 \leq [\rho_{jk}(A)] = [\langle \pi(A)H_{jk}u, u\rangle],$$

by assumption. Hence, by strong-operator continuity and density,

$$0 \leq [\langle EH_{jk}u, u\rangle]$$

— a contradiction. Thus $[H_{jk}(p)] \geq 0$ for each p in X and $[H_{jk}] \geq 0$.

(iii) If $[\rho_{jk}]$ is an n-positive functional on (an n-state of) $\mathfrak{A}$, then $[\rho_{jk}(A)] \geq 0$ for each A in $\mathfrak{A}^{+}$ since the $n \times n$ matrix with each entry equal to A is in $M_n(\mathfrak{A})^{+}$ from (ii) of the preceding lemma.

Suppose $[\rho_{jk}(A)] \geq 0$ for each A in $\mathfrak{A}^{+}$. Then with H_{jk} as in (ii), $[H_{jk}] \in M_n(\pi(\mathfrak{A})^{-})^{+}$. If $[A_{jk}] \in M_n(\mathfrak{A})^{+}$, then $[\pi(A_{jk})]$ is in $M_n(\pi(\mathfrak{A})^{-})^{+}$ from (i) of the first proposition of this section. Thus, from (v) of the preceding lemma, $[\pi(A_{jk})H_{jk}] \geq 0$. Consequently,

$$0 \leq [\langle \pi(A_{jk})H_{jk}u, u\rangle] = [\rho_{jk}(A_{jk})]$$

from (iv) of the preceding lemma and by choice of H_{jk}. Hence $[\rho_{jk}]$ is n-positive on (an n-state of) $\mathfrak{A}$ in this case.

(iv) From (ii) of the preceding theorem, the assumption that $\mathfrak{A}$ is abelian, and (iii), it will suffice to show that $[(\rho_{jk} \circ \eta)(A)] \geq 0$ for each A in $\mathfrak{A}^{+}$ and each n-state $[\rho_{jk}]$ of $\mathfrak{B}$, the C*-algebra into which η maps. But with A in $\mathfrak{A}^{+}$,

$\eta(A) \in \mathfrak{B}^+$ and the matrix all of whose entries are $\eta(A)$ is in $M_n(\mathfrak{B})^+$. Thus $[(\rho_{jk} \circ \eta)(A)] \geq 0$ for each n-state $[\rho_{jk}]$ of $\mathfrak{B}$, and η is completely positive. □

THEOREM 2.11. *Let η be a positive linear mapping of a C^*-algebra $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that $\eta(I) = I$. Then*

(i) $ET^*ETE \leq ET^*TE$ when $E, T \in \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ and E is a projection;

(ii) $\eta(A)^*\eta(A) \leq \eta(A^*A)$ for each normal operator A in $\mathfrak{A}$.

PROOF. (i) For each x in $\mathcal{K}$,

$$\langle ET^*ETEx, x\rangle = \|ETEx\|^2 \leq \|TEx\|^2 = \langle ET^*TEx, x\rangle.$$

Hence $ET^*ETE \leq ET^*TE$.

(ii) Let η_0 be $\eta|\mathfrak{A}_0$, where $\mathfrak{A}_0$ is the C*-subalgebra of $\mathfrak{A}$ generated by A, A^*, and I. Then η_0 is a positive linear mapping, $\eta_0(I) = I$, and $\mathfrak{A}_0$ is abelian. From (iv) of the preceding theorem, η_0 is completely positive. Suppose $\mathfrak{B}$ acts on a Hilbert space $\mathcal{H}$. From (iii) of Proposition 2.5, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a representation φ of $\mathfrak{A}_0$ in $\mathcal{B}(\mathcal{K})$ such that

$$E\varphi(B)E = \eta_0(B)E \quad (= E\eta_0(B)E)$$

for each B in $\mathfrak{A}_0$, where E is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Thus, from (i), we have that

$$\begin{aligned}\eta(A)^*\eta(A)E &= \eta_0(A)^*E\eta_0(A)E = E\varphi(A)^*E\varphi(A)E \\ &\leq E\varphi(A)^*\varphi(A)E = E\varphi(A^*A)E \\ &= \eta_0(A^*A)E = \eta(A^*A)E.\end{aligned}$$

It follows that

$$\eta(A)^*\eta(A) \leq \eta(A^*A). \qquad \square$$

COROLLARY 2.12. *Let $\{A_n\}$ be a sequence of positive operators on a Hilbert space $\mathcal{H}$ with $\sum\limits_{n=1}^{\infty} A_n$ weak-operator convergent to I.*

(i) There is a positive linear mapping η of $C(X)$ into $\mathcal{B}(\mathcal{H})$ such that $\eta(1) = I$ and $\eta(f_n) = A_n$, where X is the compact subset $\{0, \frac{1}{n}: n = 1, 2, \dots\}$ of $\mathbb{R}$ and f_n takes the value 1 at $\frac{1}{n}$ and 0 at other points.

(ii) There is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a sequence $\{E_n\}$ of projections on $\mathcal{K}$ with sum I such that $EE_nE = A_nE$ for each n, where E is the projection in $\mathcal{B}(\mathcal{K})$ with range $\mathcal{H}$.

PROOF. (i) Since $\sum_{n=1}^{\infty}\langle A_nx, x\rangle$ converges for each x in $\mathcal{H}$, with f in $C(X)$,

$$\sum_{n=1}^{\infty} |\langle f(\tfrac{1}{n})A_nx, x\rangle| \leq \|f\| \sum_{n=1}^{\infty} |\langle A_nx, x\rangle| = \|f\| \sum_{n=1}^{\infty}\langle A_nx, x\rangle < \infty.$$

Hence, by polarization, $\sum_{n=1}^{\infty} f(\frac{1}{n})A_n$ converges in the weak-operator topology to some $\eta(f)$, and η is a positive linear mapping with the desired properties.

(ii) From (iv) of Theorem 2.10, the mapping η, constructed in (i), is completely positive and $\eta(1) = I$. Thus, from (iii) of Proposition 2.5, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a representation φ of $C(X)$ on $\mathcal{K}$ such that $E\varphi(f)E = \eta(f)E$, where E is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Since $\{f_n\}$ (as in (i)) is an orthogonal family of idempotents in $C(X)$, $\{\varphi(f_n)\}$ is an orthogonal family $\{F_n\}$ of projections on $\mathcal{K}$ and $EF_nE = A_nE$ for all n. If $F = \sum_{n=1}^{\infty} F_n$, then

$$EFE = \sum_{n=1}^{\infty} EF_nE = \sum_{n=1}^{\infty} A_nE = \Big(\sum_{n=1}^{\infty} A_n\Big)E = E.$$

Thus $E \leq F$ and $I - F \leq I - E$. Let E_1 be $F_1 + I - F$ and E_n be F_n for n in $\{2, 3, \dots\}$. Then $\sum_{n=1}^{\infty} E_n = I$ and $EE_nE = A_nE$ for all n. □

3. Multi-states and Kaplan's representation

In [**Ka1**], Kaplan studies multi-states and constructs a representation associated with an n-positive functional akin to the GNS representation. In this section, we present some of Kaplan's results as well as some further information about multi-states.

We begin by recognizing that the linear space $\mathcal{M}$ of n-linear functionals $[\rho_{jk}]$ (matrices of bounded linear functionals ρ_{jk} on a C*-algebra $\mathfrak{A}$ with entrywise addition and multiplication by scalars) endowed with the structure of a partially-ordered vector space whose positive cone is the family of n-positive functionals on $\mathfrak{A}$ is order isomorphic to the dual of $M_n(\mathfrak{A})$ with its dual order.

PROPOSITION 3.1. *The mapping η that assigns $\tilde{\rho}$ to $[\rho_{jk}]$, where*

$$\tilde{\rho}([A_{jk}]) = \sum_{j,k=1}^{n} \rho_{jk}(A_{jk}),$$

is a linear order isomorphism of $\mathcal{M}$ onto $M_n(\mathfrak{A})^{\#}$.

PROOF. Of course, η is linear. Given a $\tilde{\sigma}$ in $M_n(\mathfrak{A})^{\#}$, let $\sigma_{jk}(A)$, for A in $\mathfrak{A}$, be $\tilde{\sigma}(\tilde{A}_{jk})$, where $\tilde{A}_{jk}$ is the $n \times n$ matrix whose (j,k) entry is A and all of whose other entries are 0. Then $\tilde{\sigma} = \eta([\sigma_{jk}])$. Thus η is onto. If $\eta([\rho_{jk}]) = 0$, then $0 = \tilde{\rho}(\tilde{A}_{jk}) = \rho_{jk}(A)$ for each A in $\mathfrak{A}$. Hence $\rho_{jk} = 0$ for all j and k. It follows that η is a linear isomorphism of $\mathcal{M}$ onto $M_n(\mathfrak{A})^{\#}$.

It remains to prove that η is an order isomorphism. Suppose that $[\rho_{jk}]$ is n-positive and $[A_{jk}] \geq 0$. Then

$$\tilde{\rho}([A_{jk}]) = \sum_{j,k=1}^{n} \rho_{jk}(A_{jk}) = \langle [\rho_{jk}(A_{jk})]\tilde{a}, \tilde{a}\rangle \geq 0,$$

where $\tilde{a}$ is the vector $(1, \dots, 1)$ in $\mathbb{C}^n$. Thus $\tilde{\rho}$ is a positive element of $M_n(\mathfrak{A})^{\#}$.

Suppose $\tilde{\rho}$ is positive and $[A_{jk}] \geq 0$. Let $\tilde{a}'$ be $(a_1, \dots, a_n)$ in $\mathbb{C}^n$. Then

$$\begin{aligned}\langle[\rho_{jk}(A_{jk})]\tilde{a}', \tilde{a}'\rangle &= \sum_{j,k=1}^{n} \rho_{jk}(A_{jk})\bar{a}_j a_k \\ &= \sum_{j,k=1}^{n} \rho_{jk}(A_{jk}\bar{a}_j a_k) = \tilde{\rho}([\bar{a}_j a_k A_{jk}]) \geq 0,\end{aligned}$$

since $[\bar{a}_j a_k A_{jk}] \geq 0$ from (i) and (v) of Lemma 2.7. It follows that $[\rho_{jk}(A_{jk})] \geq 0$ and that $[\rho_{jk}]$ is n-positive. Thus η is a linear order isomorphism of $\mathcal{M}$ onto $M_n(\mathfrak{A})^{\#}$. □

Kaplan's theorem (with an added uniqueness condition) follows.

THEOREM 3.2. *If $[\rho_{jk}]$ is an n-positive functional on a C^*-algebra $\mathfrak{A}$, there is a representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ and n vectors $x, \dots, x_n$ in $\mathcal{H}$ such that $\{x_1, \dots, x_n\}$ is generating for $\pi(\mathfrak{A})$ and*

$$\rho_{jk} = \omega_{x_k, x_j} \circ \pi \quad (j, k \in \{1, \dots, n\}).$$

If π' is another representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and $x'_1, \dots, x'_n$ are n vectors in $\mathcal{H}'$ such that $\{x'_1, \dots, x'_n\}$ is generating for $\pi'(\mathfrak{A})$ and $\rho_{jk} = \omega_{x'_k, x'_j} \circ \pi'$ for all j, k in $\{1, \dots, n\}$, then there is a unitary transformation U of $\mathcal{H}$ onto $\mathcal{H}'$ such that $Ux_j = x'_j$ for all j in $\{1, \dots, n\}$ and $\pi(A) = U^*\pi'(A)U$ for all A in $\mathfrak{A}$.

PROOF. Let $\tilde{\mathfrak{A}}$ be the n-fold direct sum of $\mathfrak{A}$ with itself (as a linear space) and let $\langle \tilde{A}, \tilde{B}\rangle$, for elements $(A_1, \dots, A_n)$ $(= \tilde{A})$ and $(B_1, \dots, B_n)$ $(= \tilde{B})$ be $\sum_{j,k=1}^{n} \rho_{jk}(B_j^* A_k)$. The conjugate bilinearity of $\langle\,,\,\rangle$ is established by routine computation. We show that $\langle\,,\,\rangle$ is positive semi-definite on $\mathfrak{A}$. For this, note that $[A_j^* A_k] \geq 0$ and that $\tilde{\rho}$ $(= \eta([\rho_{jk}]))$ is a positive functional on $M_n(\mathfrak{A})$ from Proposition 3.1. Thus

$$\langle \tilde{A}, \tilde{A}\rangle = \sum_{j,k=1}^{n} \rho_{jk}(A_j^* A_k) = \tilde{\rho}([A_j^* A_k]) \geq 0.$$

It follows, now, from polarization (cf. **[K-R I, Proposition 2.1.7]**) that $\langle \tilde{A}, \tilde{B}\rangle = \overline{\langle \tilde{B}, \tilde{A}\rangle}$.

Let $\tilde{B}$ be $\langle AA_1, \dots, AA_n\rangle$. Then

$$\begin{aligned}\langle \tilde{B}, \tilde{B}\rangle &= \sum_{j,k=1}^{n} \rho_{jk}(A_j^* A^* A A_k) = \tilde{\rho}([A_j^* A^* A A_k]) \\ &\leq \|A\|^2 \tilde{\rho}([A_j^* A_k]) = \|A\|^2 \langle \tilde{A}, \tilde{A}\rangle.\end{aligned} \tag{3}$$

For the last inequality, note that $A^*A \leq \|A\|^2 I$ and that $[A_j^* H A_k] \geq 0$ when $H \geq 0$. (View $A_j^* H A_k$ as $(H^{\frac{1}{2}} A_j)^* H^{1/2} A_k$.) The family $\tilde{\mathfrak{A}}_0$ of null vectors in $\tilde{\mathfrak{A}}$ is a subspace (from the Cauchy-Schwarz inequality) and a submodule of $\tilde{\mathfrak{A}}$ as a left $\mathfrak{A}$-module (from (3)). If we define $\langle \tilde{A} + \tilde{\mathfrak{A}}_0, \tilde{B} + \tilde{\mathfrak{A}}_0\rangle$ to be $\langle \tilde{A}, \tilde{B}\rangle$, then

$\langle\ ,\ \rangle$ is a positive definite inner product on the quotient vector space $\tilde{\mathfrak{A}}/\tilde{\mathfrak{A}}_0$. Let $\mathcal{H}$ be the completion of this quotient vector space and let $\pi(A)$ be the (unique) bounded extension to $\mathcal{H}$ of the linear operator on the quotient vector space induced by the left-module action of A in $\mathfrak{A}$ on $\tilde{\mathfrak{A}}$. From (3), $\|\pi(A)\| \le \|A\|$. Routine computation shows that π is a homomorphism of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$. As noted, $\|\pi\| \le 1$. Since $\pi(I)$ is the identity operator on $\mathcal{H}$, $\omega_x \circ \pi$ is a functional of norm 1 whose value at I in $\mathfrak{A}$ is 1 for each unit vector x in $\mathcal{H}$. Thus $\omega_x \circ \pi$ is a state of $\mathfrak{A}$, and $\langle \pi(A)x, x\rangle$ is real when A is self-adjoint. It follows that $\pi(A)$ is self-adjoint when A is, and $\pi(T^*) = \pi(T)^*$ for each T in $\mathfrak{A}$.

Denote by $\tilde{I}_j$ the element of $\tilde{\mathfrak{A}}$ whose only non-zero coordinate is I at the j th coordinate. Let x_j be $\tilde{I}_j + \tilde{\mathfrak{A}}_0$. Note that if $[\rho_{jk}]$ is an n-state, each ρ_{jj} is a state of $\mathfrak{A}$ and $\langle x_j, x_j\rangle = \rho_{jj}(I) = 1$. In this case, each x_j is a unit vector. In any event,

$$(\omega_{x_k,x_j} \circ \pi)(A) = \langle \pi(A)x_k, x_j\rangle = \rho_{jk}(A)$$

for all j, k in $\{1, \dots, n\}$. Moreover, the linear span of the set of vectors $\{\pi(A)x_j : A \in \mathfrak{A}, j \in \{1, \dots, n\}\}$ is $\tilde{\mathfrak{A}}/\tilde{\mathfrak{A}}_0$, which is dense in $\mathcal{H}$.

With π' and $x'_1, \dots, x'_n$ as in the statement, note that

$$\begin{aligned}\Big\|\sum_{j=1}^n \pi(A_j)x_j\Big\|^2 &= \sum_{j,k=1}^n \langle \pi(A_j^* A_k)x_k, x_j\rangle \\ &= \sum_{j,k=1}^n \rho_{jk}(A_j^* A_k) = \Big\|\sum_{j=1}^n \pi'(A_j)x'_j\Big\|^2,\end{aligned}$$

for all $A_1, \dots, A_n$ in $\mathfrak{A}$. Thus, the mapping that assigns the vector $\sum_{j=1}^n \pi'(A_j)x'_j$ in $\mathcal{H}'$ to the vector $\sum_{j=1}^n \pi(A_j)x_j$ in $\mathcal{H}$ is a well-defined linear isometry of a dense submanifold of $\mathcal{H}$ onto a dense submanifold of $\mathcal{H}'$. This mapping has a unique extension to a unitary transformation U of $\mathcal{H}$ onto $\mathcal{H}'$. For U, as defined, we have that

$$\begin{aligned}U^*\pi'(A)U\Big(\sum_{j=1}^n \pi(A_j)x_j\Big) &= U^*\pi'(A)\Big(\sum_{j=1}^n \pi'(A_j)x'_j\Big) = U^*\Big(\sum_{j=1}^n \pi'(AA_j)x'_j\Big) \\ &= \sum_{j=1}^n \pi(AA_j)x_j = \pi(A)\Big(\sum_{j=1}^n \pi(A_j)x_j\Big).\end{aligned}$$

Since $U^*\pi'(A)U - \pi(A)$ is bounded and $\{x_1, \dots, x_n\}$ is generating for $\pi(\mathfrak{A})$, $\pi(A) = U^*\pi'(A)U$ for all A in $\mathfrak{A}$. □

Kaplan [**Ka1; Proposition 2.6**] identifies the n-positive functionals on a C*-algebra with the completely positive mappings of that algebra into $M_n(\mathbb{C})$.

THEOREM 3.3. *If $[\rho_{jk}]$ $(= \bar{\rho})$ is an n-positive functional on a C*-algebra $\mathfrak{A}$, the mapping $\Psi_{\bar{\rho}}$ that assigns to A in $\mathfrak{A}$ the matrix $[\rho_{jk}(A)]$ is a completely positive mapping of $\mathfrak{A}$ into $M_n(\mathbb{C})$. If Ψ is a completely positive mapping of $\mathfrak{A}$*

into $M_n(\mathbb{C})$ *and* $\rho_{jk}(A)$ *is the* (j,k) *entry of* $\Psi(A)$*, then* $[\rho_{jk}]$ $(=\bar{\rho})$ *is n-positive on* $\mathfrak{A}$ *and* $\Psi_{\bar{\rho}} = \Psi$.

PROOF. Let $[\rho_{jk}]$ $(=\bar{\rho})$ be an n-positive functional on $\mathfrak{A}$. We shall express $\Psi_{\bar{\rho}}$ as a composition of completely positive mappings. From Kaplan's representation (Theorem 3.2), there is a Hilbert space $\mathcal{H}$, n vectors $x_1, \dots, x_n$ in $\mathcal{H}$, and a * homomorphism π of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ such that $\{x_1, \dots, x_n\}$ is generating for $\pi(\mathfrak{A})$ and $\rho_{jk} = \omega_{x_k,x_j} \circ \pi$. Let $\{y_1, \dots, y_n\}$ be an orthonormal basis for a space $\mathcal{H}_0$ containing $\{x_1, \dots, x_n\}$ and let T be the (unique) bounded operator that maps y_j to x_j for each j in $\{1, \dots, n\}$ and annihilates the orthogonal complement of $[y_1, \dots, y_n]$. Let φ be the * isomorphism of $\mathcal{B}(\mathcal{H}_0)$ with $M_n(\mathbb{C})$ that assigns to each S in $\mathcal{B}(\mathcal{H}_0)$ its matrix relative to the basis $\{y_1, \dots, y_n\}$. The (j,k) entry of $\varphi(T^*\pi(A)T)$ is

$$\langle T^*\pi(A)Ty_k, y_j\rangle = \langle \pi(A)x_k, x_j\rangle = \rho_{jk}(A).$$

Thus $\Psi_{\bar{\rho}}(A) = \varphi(T^*\pi(A)T)$ for each A in $\mathfrak{A}$, and $\Psi_{\bar{\rho}}$ is a composition of completely positive mappings (cf. Proposition 2.3 (i) and (ii)).

Suppose, now, that Ψ is a completely positive mapping of $\mathfrak{A}$ into $M_n(\mathbb{C})$, and $\rho_{jk}(A)$ is the (j,k) entry of $\Psi(A)$. From Theorem 2.9 (ii), $[\sigma_{jk} \circ \Psi]$ is n-positive on $\mathfrak{A}$ for each n-positive functional $[\sigma_{jk}]$ on $M_n(\mathbb{C})$. Let e_j be the vector in $\mathbb{C}^n$ with 1 at the jth coordinate and 0 at all others. Then $[\omega_{e_k,e_j}]$ is an n-state of $M_n(\mathbb{C})$ and ω_{e_k,e_j} assigns to each matrix its (j,k) entry. Thus $[\omega_{e_k,e_j} \circ \Psi]$ is n-positive on $\mathfrak{A}$, and $\omega_{e_k,e_j} \circ \Psi = \rho_{jk}$. By definition, the (j,k) entry of $\Psi_{\bar{\rho}}(A)$ is $\rho_{jk}(A)$. Thus $\Psi = \Psi_{\bar{\rho}}$. $\square$

4. Diagonalizing n-states

If $\{\rho_1, \dots, \rho_n\}$ is a set of states of (or positive linear functionals on) a C*-algebra $\mathfrak{A}$, the n-functional $[\rho_{jk}]$ all of whose off-diagonal entries are 0 and whose diagonal entry ρ_{jj} is $\rho_j (j = 1, \dots, n)$ is an n-state (positive functional) on $\mathfrak{A}$. In this case, $[\rho_{jk}]$ is said to be in *diagonal form*.

In this section, we consider the possibility of diagonalizing n-states (positive functionals) $\bar{\rho}$ $(= [\rho_{jk}])$ on $\mathfrak{A}$. We mean by this that we try to locate a unitary element $\bar{U}$ in $M_n(\mathfrak{A})$ such that the state $\tilde{\rho}_{\bar{U}}$ of (positive functional on) $M_n(\mathfrak{A})$ is in diagonal form, where $\tilde{\rho}_{\bar{U}}([A_{jk}]) = \tilde{\rho}(\bar{U}[A_{jk}]\bar{U}^*)$ and $\tilde{\rho}$ is the state of (positive functional) on $M_n(\mathfrak{A})$ that is the image of $\bar{\rho}$ under the order isomorphism η of Proposition 3.1. To say that a state $\tilde{\sigma}$ of (positive functional on) $M_n(\mathfrak{A})$ is in diagonal form is to say that $\tilde{\sigma}(\bar{A}_{jk}) = 0$ when $j \neq k$, where $\bar{A}_{jk}$ is the matrix with A in $\mathfrak{A}$ at the (j,k) entry.

If $\tilde{\sigma}$ is in diagonal form and $\bar{E}_r$ is $\bar{I}_{rr}$, then

$$\tilde{\sigma}(\bar{E}_r[A_{jk}]) = \sigma_r(A_{rr}) = \tilde{\sigma}([A_{jk}]\bar{E}_r) \quad ([A_{jk}] \in M_n(\mathfrak{A})).$$

Thus $\bar{E}_r$ is in the centralizer $\tilde{\sigma}$. Of course, $\bar{E}_r \cong \bar{E}_s$ for all r and s (since $\bar{I}_{rs}^*\bar{I}_{rs} = \bar{I}_{ss} = \bar{E}_s$ and $\bar{I}_{rs}\bar{I}_{rs}^* = \bar{E}_r$). Suppose, conversely, that the centralizer of $\tilde{\sigma}$ contains projections $\bar{F}_1, \dots, \bar{F}_n$ having sum $\bar{I}$, each equivalent to $\bar{E}_1$, and

$\bar{E}_j = \bar{V}_j^* \bar{V}_j$, $\bar{F}_j = \bar{V}_j \bar{V}_j^*$ for some $\bar{V}_j$ in $M_n(\mathfrak{A})$. We shall see that $\tilde{\sigma}_{\bar{U}}$ is in diagonal form, where $\bar{U}$ is the unitary element $\bar{V}_1 + \cdots + \bar{V}_n$. For this, note that $\bar{E}_j \bar{A}_{jk} = \bar{A}_{jk}, \bar{A}_{jk} \bar{E}_j = 0$, and $\bar{U} \bar{E}_j \bar{U}_j^* = \bar{F}_j$, when $j \neq k$. Thus, when $j \neq k$,

$$\begin{aligned}\tilde{\sigma}_{\bar{U}}(\bar{A}_{jk}) &= \tilde{\sigma}(\bar{U}\bar{A}_{jk}\bar{U}^*) = \tilde{\sigma}(\bar{U}\bar{E}_j\bar{A}_{jk}\bar{U}^*)\\ &= \tilde{\sigma}(\bar{U}\bar{E}_j\bar{U}^*\bar{U}\bar{A}_{jk}\bar{U}^*) = \tilde{\sigma}(\bar{F}_j\bar{U}\bar{A}_{jk}\bar{U}^*)\\ &= \tilde{\sigma}(\bar{U}\bar{A}_{jk}\bar{U}^*\bar{F}_j) = \tilde{\sigma}(\bar{U}\bar{A}_{jk}\bar{U}^*\bar{F}_j\bar{U}\bar{U}^*)\\ &= \tilde{\sigma}(\bar{U}\bar{A}_{jk}\bar{E}_j\bar{U}^*) = \tilde{\sigma}_{\bar{U}}(0)\\ &= 0.\end{aligned}$$

Hence $\tilde{\sigma}_{\bar{U}}$ is in diagonal form.

From the preceding discussion, we see that the problem of diagonalizing an n-state (positive functional) depends, largely, on finding appropriate projections in its centralizer. In a more general context (infinite and not necessarily discrete), diagonalizing a state amounts to finding a maximal abelian self-adjoint subalgebra in its centralizer. At the very least, $\tilde{\sigma}$ must have n orthogonal equivalent projections with sum $\bar{I}$ in its centralizer if it can be diagonalized.

In [**H-T**], Herman and Takesaki answer a question raised by Glimm (at the 1967 Baton Rouge conference) by producing a factor $\mathcal{M}$ of type III and a (faithful normal) state of it whose centralizer contains only scalar multiples of the identity. Since $M_n(\mathcal{M})$ is * isomorphic to $\mathcal{M}$, there is a state of $M_n(\mathcal{M})$ whose centralizer consists of scalar multiples. This state and its associated n-state are not diagonalizable (in our present sense).

If $\mathcal{R}$ is a semi-finite von Neumann algebra, the situation changes significantly. We prove that the centralizer of each normal state of $\mathcal{R}$ contains a maximal abelian self-adjoint subalgebra of $\mathcal{R}$. This proof is effected with the aid of tracial-weight, Radon-Nikodým techniques. We begin with a sequence of preparatory results.

LEMMA 4.1. *The support of a normal state of a von Neumann algebra lies in the center of the centralizer of that state.*

PROOF. Let $\mathcal{R}$ be a von Neumann algebra, ω be a normal state of $\mathcal{R}$, E be the support of ω, and A be an element of the centralizer of ω. Since $\omega(I-E) = 0$ and $0 \leq I - E$, $I - E$ and E are in the centralizer of ω (for $0 = \omega((I-E)B) = \omega(B(I-E))$ when $B \in \mathcal{R}$). Hence $EA(I-E)$ is in the centralizer of ω, and

$$0 = \omega((I-E)A^*EA(I-E)) = \omega(EA(I-E)A^*E).$$

Since E is the support of ω and $0 \leq EA(I-E)A^*E$, we have that

$$EA(I-E)A^*E = 0.$$

Hence $EA(I-E) = 0$. As A^* is also in the centralizer of ω, $EA^*(I-E) = 0$ and $(I-E)AE = 0$. It follows that

$$A = EAE + (I-E)A(I-E),$$

whence

$$EA = EAE = AE. \qquad \square$$

LEMMA 4.2. *If ω is a normal state of a von Neumann algebra $\mathcal{R}$, E is the support of ω, and $\mathcal{R}_\omega$ is the centralizer of ω, then $\mathcal{R}_\omega$ is the direct sum of $(I-E)\mathcal{R}(I-E)$ and $\mathcal{R}_\omega E$.*

PROOF. From Lemma 4.1., E is in the center of $\mathcal{R}_\omega$. Thus $\mathcal{R}_\omega$ is (isomorphic to) the direct sum of $\mathcal{R}_\omega(I-E)$ and $\mathcal{R}_\omega E$. We complete the proof by showing that $\mathcal{R}_\omega(I-E) = (I-E)\mathcal{R}(I-E)$. Since $I-E$ is in the center of $\mathcal{R}_\omega$,

$$\mathcal{R}_\omega(I-E) = (I-E)\mathcal{R}_\omega(I-E) \subseteq (I-E)\mathcal{R}(I-E).$$

Suppose S and T are in $\mathcal{R}$. Since $\omega(I-E) = 0$, $I-E$ is in the left and right kernels of ω. Thus

$$0 = \omega(S(I-E)T(I-E)) = \omega((I-E)T(I-E)S).$$

In particular, $(I-E)T(I-E) \in \mathcal{R}_\omega$, whence $(I-E)T(I-E) \in \mathcal{R}_\omega(I-E)$. It follows that

$$(I-E)\mathcal{R}(I-E) \subseteq \mathcal{R}_\omega(I-E).$$

Combining this with the reverse inclusion, noted above, we conclude that $\mathcal{R}_\omega(I-E) = (I-E)\mathcal{R}(I-E)$. $\square$

LEMMA 4.3. *If $\mathcal{R}$ is a von Neumann algebra, ω is a normal state of $\mathcal{R}$, and E is the support of ω, then the centralizer of $\omega|E\mathcal{R}E$ is $\mathcal{R}_\omega E$.*

PROOF. From Lemma 4.1, E is in the center of the centralizer of ω so that

$$E\mathcal{R}_\omega E = \mathcal{R}_\omega E \subseteq \mathcal{R}_\omega, \quad \mathcal{R}_\omega E \subseteq E\mathcal{R}E.$$

Hence $\mathcal{R}_\omega E$ is contained in the centralizer of $\omega|E\mathcal{R}E$ $(=\omega_0)$.

Suppose T in $\mathcal{R}$ is such that ETE is in the centralizer of ω_0. With S in $\mathcal{R}$, we have that

$$\begin{aligned}\omega(SETE) &= \omega((I-E)SETE) + \omega(ESETE) = \omega(ESETE)\\ &= \omega(ETESE) = \omega(ETESE) + \omega(ETES(I-E))\\ &= \omega(ETES).\end{aligned}$$

Thus $ETE \in \mathcal{R}_\omega$ and $ETE \in \mathcal{R}_\omega E$. It follows that the centralizer of ω_0 is contained in $\mathcal{R}_\omega E$. From these inclusions, we have that the centralizer of ω_0 is $\mathcal{R}_\omega E$. $\square$

LEMMA 4.4. *Let $\mathcal{H}$ be a Hilbert space and K be an operator on $\mathcal{H}$ such that $0 < K < I$. Suppose that K and $I-K$ have null space (0). If $H = (I-K)K^{-1}$, then K and $\{H^{it} : t \in \mathbb{R}\}$ generate the same von Neumann algebra.*

PROOF. Let $\mathcal{A}$ be the von Neumann algebra generated by K. Then H is a positive self-adjoint operator affiliated with $\mathcal{A}$, and H^{it} $(= \exp it \log H) \in \mathcal{A}$ for each real t. It remains to show that the von Neumann algebra $\mathcal{A}_0$ $(\subseteq \mathcal{A})$

generated by $\{H^{it}: t \in \mathbb{R}\}$ coincides with $\mathcal{A}$. From [**K-R I; Theorem 5.2.1**], $\mathcal{A}$ is isomorphic to $C(X)$ for some extremely disconnected compact Hausdorff space X. Suppose K corresponds to k in $C(X)$ via this isomorphism and H in $\mathcal{S}(\mathcal{A})$ corresponds to h in $\mathcal{S}(X)$ via the extension of this isomorphism. (See [**K-R I; Theorem 5.6.19**].) Then H^{it} corresponds to $\exp(it\log h)$ in $C(X)$. Let X_n be the closure of the set of points in X at which $\log h$ takes values in $(-\log n, \log n)$ with n an integer greater than 1. Then X_n is the closure of the set of points where h takes values in (n^{-1}, n) and where k takes values in $((n+1)^{-1}, n(n+1)^{-1})$. Let e_n be the characteristic function of X_n and E_n be the projection in $\mathcal{A}$ corresponding to e_n. Suppose $0 < t < \pi(2\log n)^{-1}$. The shorter open arc α_t of the unit circle in $\mathbb{C}$ with boundary points $\exp(-it\log n)$ and $\exp(it\log n)$ lies in the right half plane. The set of points x of X such that $\exp(it\log h(x)) \in \alpha_t$ is the union of open sets $O_{m,t}$ consisting of those points x in X where

$$\log h(x) \in (2\pi m t^{-1} - \log n, 2\pi m t^{-1} + \log n)$$

and m is some integer. Under the assumption that $t < \pi(2\log n)^{-1}$, the intervals $(2\pi m t^{-1} - \log n, 2\pi m t^{-1} + \log n)$ are disjoint for distinct m so that the $O_{m,t}$ are disjoint sets. Since X is extremely disconnected, the sets $O_{m,t}$ have disjoint closures, and the characteristic functions $e_{m,t}$ of these closures correspond to orthogonal projections $E_{m,t}$ in $\mathcal{A}$. Let F_t be $\sum_{m=-\infty}^{\infty} E_{m,t}$. Then F_t is a spectral projection for $\exp(it\log H)$ (corresponding to α_t) so that $F_t \in \mathcal{A}_0$. Thus

$$\bigwedge_{r=5}^{\infty} F_{2\pi(r\log n)^{-1}} = \bigwedge_{r=5}^{\infty}\Big(\sum_{m=-\infty}^{\infty} E_{m,2\pi(r\log n)^{-1}}\Big) = F \in \mathcal{A}_0.$$

Now $E_{0,t} = E_n$ for each real t, whence $E_n \le F$. Suppose that $F - E_n \neq 0$. We are going to derive a contradiction from this assumption. Let Y be the non-null clopen subset of X corresponding to $F - E_n$. Since $F - E_n \le \sum_{m=-\infty}^{\infty} E_{m,2\pi(r\log n)^{-1}}$ for all integers r greater than 4, Y is a subset of the closure of $\bigcup_{m=-\infty}^{\infty} O_{m,2\pi(r\log n)^{-1}}$ for each such r. Now $Y \cap (\bigcup_{m=-\infty}^{\infty} O_{m,2\pi(r\log n)^{-1}})$ is an open subset Y_r of Y, and $Y\backslash Y_r^-$ is a clopen subset of Y that does not meet the closure of $\bigcup_{m=-\infty}^{\infty} O_{m,2\pi(r\log n)^{-1}}$ and, at the same time, is contained in this closure. It follows that $Y\backslash Y_r^- = \emptyset$, whence Y_r is an open dense subset of Y. Applying the Baire Category theorem to the (non-null) compact Hausdorff space Y, we have that $\bigcap_{r=5}^{\infty} Y_r \neq \emptyset$. Let y be a point of this intersection. Then for each integer r greater than 4, there is an integer m such that $y \in O_{m,2\pi(r\log n)^{-1}}$, whence

$$\log h(y) \in ((mr-1)\log n, (mr+1)\log n).$$

If this m were 0 for some r, then y would lie in X_n and $Y \cap X_n$ would be non-empty, from which we would have the contradiction, $(F - E_n)E_n \neq 0$. Thus, for each integer r greater than 4,

$$\log h(y) \in ((mr-1)\log n, (mr+1)\log n)$$

for some non-zero integer m. Suppose $\log h(y) \geq \log n$. We can choose an integer r greater than 4 such that $\log h(y) < (r-1)\log n$. Then $\log h(y)$ lies in none of the intervals $((mr-1)\log n, (mr+1)\log n)$ with m an integer. Similarly, if $\log h(y) \leq -\log n$, we choose r, an integer greater than 4, such than $-\log h(y) < (r-1)\log n$, whence $(-r+1)\log n < \log h(y)$. Again, $\log h(y)$ lies in no interval $((rm-1)\log n, (rm+1)\log n)$ with m an integer. From this contradiction, we conclude that $E_n = F \in \mathcal{A}_0$.

Let g be the function on $\{0\} \cup \{e^{i\theta}: -\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi\}$ that assigns 0 to 0 and θ to $e^{i\theta}$. Then g is a continuous function on $\mathrm{sp}(H^{it}E_n)$, where $t = \pi(2\log n)^{-1}$, whence $g(H^{it}E_n) \in \mathcal{A}_0$. Now $g(H^{it}E_n) = t(\log H)E_n$. Thus $(\log H)E_n \in \mathcal{A}_0$. Again, $E_n \exp((\log H)E_n) = HE_n \in \mathcal{A}_0$. Let f be the function defined on $\{0\} \cup [n^{-1}, n]$ that assigns $(1+s)^{-1}$ to s in $[n^{-1}, n]$ and 0 to 0. Then f is a continuous function on $\mathrm{sp}(HE_n)$, and $E_n f(HE_n) = KE_n \in \mathcal{A}_0$. Since E_n corresponds to e_n, the characteristic function of the closure of the set of points at which k takes values in $((n+1)^{-1}, n(n+1)^{-1})$, and k takes the values 0 and 1 at nowhere-dense subsets of X, we have that $\vee_{n=1}^{\infty} E_n = I$. It follows that $K \in \mathcal{A}_0$. Thus $\mathcal{A} = \mathcal{A}_0$. $\square$

THEOREM 4.5. *If ρ is a faithful normal semi-finite tracial weight on a von Neumann algebra $\mathcal{R}$, ω is a normal state of $\mathcal{R}$, and K is a positive operator in the unit ball of $\mathcal{R}$ such that K and $I-K$ are one-to-one, $I-K \in F_\rho$ (the set of positive A in $\mathcal{R}$ such that $\rho(A) < \infty$), and $\rho((I-K)A) = \omega(KA) = \omega(AK)$, then the centralizer of ω consists of those operators in $\mathcal{R}$ that commute with K. The centralizer of ω contains a maximal abelian self-adjoint subalgebra of $\mathcal{R}$.*

PROOF. We note that, under the given conditions, ω is faithful. To see this, suppose $A > 0$ and $\omega(A) = 0$. Then A is in the left kernel of ω, and

$$0 = \omega(KA) = \rho((I-K)A) = \rho(A^{\frac{1}{2}}(I-K)A^{\frac{1}{2}}),$$

from **[K-R II; Proposition 8.5.1]**. Thus $A^{\frac{1}{2}}(I-K)A^{\frac{1}{2}} = 0$ since ρ is faithful. It follows that $(I-K)^{\frac{1}{2}}A^{\frac{1}{2}} = 0$ and $(I-K)A = 0$. As $I-K$ is one-to-one, $A = 0$. Thus ω is faithful.

The hypotheses of **[K-R II; Lemma 9.2.20]** are satisfied and the modular group $\{\sigma_t\}$ corresponding to ω is implemented by the unitary group, $t \to H^{it}$ with t in $\mathbb{R}$, where $H = K^{-1}(I-K)\ \eta\ \mathcal{R}$. From **[K-R II; Proposition 9.2.14]**, A is in the centralizer of ω if and only if $H^{it}AH^{-it} = A$ for each real t. From Lemma 4.4, $\{H^{it}: t \in \mathbb{R}\}$ and K generate the same von Neumann algebra. Thus A commutes with each H^{it} if and only if A commutes with K. It follows that the centralizer of ω contains each maximal abelian subalgebra of $\mathcal{R}$ to which K belongs. In particular, using Zorn's lemma, there is a maximal abelian self-adjoint subalgebra of $\mathcal{R}$ contained in the centralizer of ω. $\square$

As noted at the beginning of this section, the problem of diagonalizing an n-state (positive functional) amounts to finding n orthogonal projections in the centralizer of the n-state (positive functional) each equivalent to the main matrix units. We can now use Theorem 4.5 and the results of **[Kad2]** to locate the required projections. We need some minor preparation before we can apply **[Kad2]**.

LEMMA 4.6. *Suppose that E is a countably decomposable projection in a von Neumann algebra $\mathcal{S}$. If A and B are in $\mathcal{S}$, then $R(AEB)$ is countably decomposable in $\mathcal{S}$.*

PROOF. Since

$$R(AEB) \leq R(AE) \sim R(EA^*) \leq E$$

(from [**K-R II; Proposition 6.1.6**]), and a projection equivalent to a countably decomposable projection in $\mathcal{S}$ is countably decomposable in $\mathcal{S}$, we have that $R(AEB)$ is countably decomposable in $\mathcal{S}$. $\square$

From [**K-R IV; Exercise 7.6.13**], the support of a normal state of a von Neumann algebra is a countably decomposable projection in the algebra. Let $\mathcal{R}_0$ be a von Neumann algebra, $\mathcal{R}$ be $M_n(\mathcal{R}_0)$, and E_{jk} be the element of $\mathcal{R}$ whose (j,k) entry is I and all of whose other entries are zero. Let E_0 be the support of the normal state ρ of $\mathcal{R}$ and let $[A_{jk}]$ be the matrix representation of E_0. From Lemma 4.6, $R(E_{jj}E_0E_{kk})$ is countably decomposable, and $E_{jj}E_0E_{kk}$ has all its entries 0 with the possible exception of its (j,k) entry, which is A_{jk}. Moreover, $E_{jj}E_0E_{kk}E_0E_{jj}$ has all its entries 0 with the possible exception of its (j,j) entry, which is $A_{jk}A_{jk}^*$. Now $R(E_{jj}E_0E_{kk}E_0E_{jj}) = R(E_{jj}E_0E_{kk})$ so that $R(E_{jj}E_0E_{kk}E_0E_{jj})$ is countably decomposable. Thus $R(A_{jk}A_{jk}^*)$ $(= R(A_{jk}))$ is countably decomposable in $\mathcal{R}_0$. It follows that $\vee_{j,k=1}^n R(A_{jk})$ $(= F)$ is a countably decomposable projection in $\mathcal{R}_0$. Let F_0 be the projection in $\mathcal{R}$ with F at each diagonal entry and 0 at all others. Since $FA_{jk} = A_{jk}$ for all j and k in $\{1, \ldots, n\}$, we have that $F_0E_0 = E_0$ and $\rho(A) = \rho(F_0AF_0)$ for each A in $\mathcal{R}$. Now $F_0\mathcal{R}F_0$ is the algebra of $n \times n$ matrices with entries in $F\mathcal{R}_0F$ $(= \mathcal{R}_1)$, a countably decomposable von Neumann algebra. Suppose that we can find a unitary operator V in $M_n(\mathcal{R}_1)$ that diagonalizes the restriction of ρ to $F_0\mathcal{R}F_0$ $(= M_n(\mathcal{R}_1))$. Let W be the diagonal matrix in $\mathcal{R}$ with $I - F$ at each diagonal entry. Then $V + W$ is a unitary operator in $\mathcal{R}$ $(= M_n(\mathcal{R}_0))$ that diagonalizes ρ.

The theorem that follows is proved in [**Ka2; Proposition 3.18**], for the case where $\mathcal{R}_0$ is countably decomposable and ρ is faithful, by making use of the results [**K-R II; Lemma 9.2.19**] and [**Kad2 Theorem 3.18**]. That proof proceeds by a limiting state argument and some detailed spectral theoretic considerations. The simplification and extension contained in the next theorem results from a systematic use of information about the centralizers of normal states. The basic ingredients of the argument, namely, the results [**K-R II; Lemma 9.2.19**] and [**Kad2; Theorem 3.18**], remain the same however.

THEOREM 4.7. *Each normal state ρ of the von Neumann algebra $\mathcal{R}$ of $n \times n$ matrices over a semi-finite von Neumann algebra $\mathcal{R}_0$ is diagonalizable.*

PROOF. In the notation of the discussion preceding this theorem, $\mathcal{R}_1$ is countably decomposable and semi-finite [**K-R IV; Exercise 6.9.16**]. From that discussion, it will suffice to prove that the restriction of ρ to $M_n(\mathcal{R}_1)$ is diagonalizable. By applying [**K-R II; Proposition 11.2.21**], we see that $M_n(\mathcal{R}_1)$ is semi-finite since it is the tensor product of $\mathcal{R}_1$ with the algebra of (bounded) operators on an n-dimensional Hilbert space. At the same time, $M_n(\mathcal{R}_1)$ is, with $\mathcal{R}_1$, countably decomposable. Thus we may assume that $\mathcal{R}_0$ and $\mathcal{R}$ are countably decomposable as well as semi-finite.

From Theorem 4.5, the centralizer $\mathcal{R}_\rho$ of ρ contains a maximal abelian self-adjoint subalgebra $\mathcal{A}$ of $\mathcal{R}$. From [**Kad2; Theorem 3.18**], $\mathcal{A}$ contains n orthogonal equivalent projections $\{F_1, \dots, F_n\}$ with sum I. Let $\{E_{jk}\}$ be the matrix unit system for $\mathcal{R}$ described earlier in this section. As established at the beginning of the proof of [**Kad2; Theorem 3.19**], F_j and E_{kk} are equivalent in $\mathcal{R}$ for each j and k in $\{1, \dots, n\}$. If V_j is a partial isometry in $\mathcal{R}$ with initial projection F_j and final projection E_{jj}, then $\sum_{j=1}^n V_j$ $(= U)$ is a unitary operator in $\mathcal{R}$ such that $UF_jU^* = E_{jj}$. Let σ be the state of $\mathcal{R}$ defined by

$$\sigma(A) = \rho(U^*AU) \quad (A \in \mathcal{R}).$$

Then, since $F_j \in \mathcal{R}_\rho$, for each j in $\{1, \dots, n\}$,

$$\begin{aligned}\sigma(E_{jj}A) &= \rho(U^*UF_jU^*AU) = \rho(U^*AUF_j)\\ &= \rho(U^*AUF_jU^*U) = \rho(U^*AE_{jj}U)\\ &= \sigma(AE_{jj}).\end{aligned}$$

It follows that each E_{jj} is in the centralizer of σ. From the discussion at the beginning of this section, σ is in diagonal form, whence ρ is diagonalizable. □

To apply Theorem 4.7 to an n-positive functional $\bar{\sigma}$ $(= [\sigma_{jk}])$ on the von Neumann algebra $\mathcal{R}_0$, we must have that its image $\tilde{\sigma}$ is a normal positive functional on $\mathcal{R}$. The natural condition that yields this is the requirement that each σ_{jk} be normal. In this case, we refer to $\bar{\sigma}$ as a *normal* n-state (positive functional). It follows from Theorem 4.7 that $\bar{\sigma}$ is diagonalizable when it is normal and $\mathcal{R}_0$ is semi-finite.

5. Automatic continuity

We defined an n-positive functional on a C*-algebra $\mathfrak{A}$ as a matrix $[\rho_{jk}]$ $(= \bar{\rho})$ of linear functionals satisfying the indicated positivity condition (Definition 2.8). When n is 1, an n-positive functional is a positive functional in the usual sense, and therefore, norm continuous (cf. [**K-R I; Theorem 4.3.2**]). Is it true, for arbitrary n, that each ρ_{jk} is norm continuous?

In Theorem 5.3, we answer this question affirmatively. Some preparation is needed.

DEFINITION 5.1. With $\bar{\rho}$ $(= [\rho_{jk}])$ an n-functional on the C*-algebra $\mathfrak{A}$ and $\bar{A}$ $([A_{jk}])$ in $M_n(\mathfrak{A})$, we write $\bar{\rho}(\bar{A})$ for the matrix $[\rho_{jk}(A_{jk})]$ in $M_n(\mathbb{C})$ and refer to the associated linear mapping of $M_n(\mathfrak{A})$ into $M_n(\mathbb{C})$ as the *Hadamard mapping* (corresponding to $\bar{\rho}$).

If $\bar{\rho}(\bar{A}^*) = \bar{\rho}(\bar{A})^*$, we say that $\bar{\rho}$ is *hermitian*. When $\bar{\rho}$ is n-positive, it is hermitian since each element of $M_n(\mathfrak{A})$ is a linear combination of (at most four) positive elements. With $\bar{A}$ the element of $M_n(\mathfrak{A})$ whose (j,k) and (k,j) entries are the elements A and A^*, respectively, in $\mathfrak{A}$ and all of whose other entries are 0, if $\bar{\rho}$ is hermitian, then $\bar{\rho}(\bar{A})$ has $\rho_{jk}(A)$ and $\rho_{kj}(A^*)$ as its (j,k) and (k,j) entries, respectively. Since $\bar{\rho}(\bar{A})$ is hermitian, we have that $\rho_{jk}(A) = \overline{\rho_{kj}(A^*)} = \rho^*_{kj}(A)$, here σ^*, for a functional σ on $\mathfrak{A}$, is defined by $\sigma^*(A) = \overline{\sigma(A^*)}$. It follows that $\rho_{jk} = \rho^*_{kj}$ for all j and k when $[\rho_{jk}]$ is a hermitian n-functional on $\mathfrak{A}$ and, in particular, when $[\rho_{jk}]$ is n-positive.

To prove our automatic continuity result for n-positive functionals on $\mathfrak{A}$, we shall make use of the following elementary lemma (whose proof we include for the reader's convenience).

LEMMA 5.2. *If $\bar{A}$ is the $n \times n$ matrix whose (j,j) and (k,k) entries are aI and bI, respectively, whose (j,k) and (k,j) entries are A and A^*, respectively, where A is in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and all of whose other entries are 0, then $\bar{A}$ is positive if and only if a and b are real and non-negative and $\|A\|^2 \le ab$.*

PROOF. Let $\bar{z}$ be a vector in the n-fold direct sum of $\mathcal{H}$ with itself. Suppose that x and y are the j and k coordinates of $\bar{z}$, respectively. Then

$$\begin{aligned} \langle \bar{A}\bar{z}, \bar{z}\rangle &= \langle ax + Ay, x\rangle + \langle A^*x + by, y\rangle \\ &= a\|x\|^2 + b\|y\|^2 + 2\mathbf{Re}\langle Ay, x\rangle. \end{aligned} \tag{4}$$

If $\|A\|^2 \le ab$ with a and b real and non-negative, then, using the inequality $(a^{\frac{1}{2}}\|x\| - b^{\frac{1}{2}}\|y\|)^2 \ge 0$, we have that

$$\begin{aligned} -2\mathbf{Re}\langle Ay, x\rangle &\le 2|\langle Ay, x\rangle| \le 2\|Ay\|\,\|x\| \\ &\le 2(ab)^{\frac{1}{2}}\|y\|\,\|x\| \le a\|x\|^2 + b\|y\|^2. \end{aligned}$$

If follows that $\langle \bar{A}\bar{z}, \bar{z}\rangle \ge 0$ and that $\bar{A} \ge 0$.

Suppose, conversely, that $\langle \bar{A}\bar{z}, \bar{z}\rangle \ge 0$ for all $\bar{z}$. Choosing for $\bar{z}$, first, a vector whose only non-zero entry is a unit vector in the j th coordinate and, then, a vector whose only non-zero entry is a unit vector in the k th coordinate, we see that $a \ge 0$ and $b \ge 0$. If $a = 0$, then $-2\mathbf{Re}\langle Ay, x\rangle \le b\|y\|^2$ for all vectors x and y in $\mathcal{H}$ (from (4)). Choosing x to be $-tAy$, with t a positive real number, we see that $Ay = 0$ for all y. Thus $A = 0$ in this case, and $\|A\|^2 \le ab$. If $a > 0$, choosing $-a^{-1}Ay$ for x in (4), we have that

$$\|Ay\|^2 \le ab\|y\|^2$$

for all y in $\mathcal{H}$. Thus $\|A\|^2 \le ab$. $\square$

LEMMA 5.3. *If $\bar{\rho}$ $(= [\rho_{jk}])$ is an n-positive functional on a C^*-algebra $\mathfrak{A}$, then each ρ_{jj} is positive, $\rho_{jk} = \rho^*_{kj}$ for all j and k, and each ρ_{jk} is bounded with bound not exceeding $(\|\rho_{jj}\|\,\|\rho_{kk}\|)^{\frac{1}{2}}$.*

PROOF. If $\bar{A}$ is the $n \times n$ matrix whose only non-zero entry is the positive element A of $\mathfrak{A}$ at the (j,j) entry, then $\bar{A} \ge 0$. Thus $\bar{\rho}(\bar{A}) \ge 0$, whence $\rho_{jj}(A) \ge 0$. It follows that each $\rho_{jj} \ge 0$ and that $\|\rho_{jj}\| = \rho_{jj}(I)$. We noted earlier that $\bar{\rho}$ is hermitian and that $\rho_{jk} = \rho^*_{kj}$ for all j and k.

Let $\bar{A}$ be the matrix whose (j,k) and (k,j) entries are A in $\mathfrak{A}$ and A^*, respectively, whose (j,j) and (k,k) entries are I and all of whose other entries are 0.

Suppose, further, that $j \neq k$ and $\|A\| \leq 1$. Then $\bar{A} \geq 0$ from Lemma 5.2. Thus $\bar{\rho}(\bar{A}) \geq 0$. Now $\rho_{jk}(A) = \rho^*_{kj}(A) = \overline{\rho_{kj}(A^*)}$. Again, from Lemma 5.2, where $\mathcal{H}$ is one dimensional and $\mathcal{B}(\mathcal{H})$ is $\mathbb{C}$, $|\rho_{jk}(A)|^2 \leq \rho_{jj}(I)\rho_{kk}(I)$. It follows that $\|\rho_{jk}\| \leq (\|\rho_{jj}\| \, \|\rho_{kk}\|)^{\frac{1}{2}}$. □

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