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Operator algebras in dynamical systems, by Shôichirô Sakai. Encyclopedia of Mathematics and its Applications, vol. 41, Cambridge University Press, Cambridge, 219 + 12 pp., $59.50. ISBN 0-521-40096

The central topic of the treatise under review is the theory of derivations of operator algebras (of which, the author is the leading developer) and the interpretation of that theory in the framework of quantum physics. This theory is, in the reviewer's opinion, destined to become one of the most important research areas in analysis serving, as it does, as the basis of the theory of noncommutative differential equations. The principal mathematical construct dealt with in this book is the C*-algebra ('operator algebra'). It is most easily viewed as a family $\mathscr{A}$ of bounded operators on a Hilbert space $\mathscr{H}$ to which the adjoint operator A^* belongs when A belongs to it. This family is assumed to be an algebra under the usual addition and multiplication of transformations of a vector space into itself and to contain all limits of sequences of elements in it relative to the metric induced on all bounded operators by the operator bound (norm). Such algebras have an "algebraic" existence (independent of an underlying Hilbert space) by virtue of the Gelfand-Neumark theorem [GN] that characterizes the C*-algebras among the Banach algebras as those with an involution $(A \to A^*)$ imitating the algebraic properties of the adjoint operation on Hilbert space transformations and satisfying $\|AA^*\| = \|A\|\|A^*\|$. Sakai proceeds from the abstract characterization. It is best to do this, with models for physical systems in mind; it provides the possibility of focusing on those families of 'states' appropriate to the intended model. In mathematical terms, a *state* of a C*-algebra $\mathscr{A}$ is a linear functional ρ on $\mathscr{A}$ (to the complex numbers $\mathbb{C}$) that takes the value 1 at I (the unit of the algebra—if there is no unit, we assume that ρ has bound 1) and assumes nonnegative real values on positive (selfadjoint) operators in $\mathscr{A}$.

In the same vein, Sakai proceeds from his characterization of von Neumann algebras (he uses the older terminology 'W*-algebras' but not the original terminology of von Neumann [vN], 'rings of operators', when operator algebras were first introduced) as those C*-algebras that are the (continuous) duals of Banach spaces. In represented form, the von Neumann algebras are those C*-algebras $\mathscr{R}$ acting on a Hilbert space $\mathscr{H}$ that contain the 'strong-operator' limits of nets of operators in $\mathscr{R}$ (if $\{A_a\}_{a\in\mathbb{A}}$ is a net of operators A_a in $\mathscr{R}$ and A is a (bounded) operator such that $A_a x \to Ax$ for each x in $\mathscr{H}$, then A is in $\mathscr{R}$).

In the first chapter (pp. 1–15), Sakai defines the objects and states the basic results he needs (largely without proofs). Among these are a few of the main facts about C*- and von Neumann algebras. He provides some of the harmonic and complex-analytic function theory results to be used as well as some perturbation theory. There is a brief discussion of the group Aut($\mathscr{A}$) of (adjoint-preserving) automorphisms of $\mathscr{A}$ and of representations α of a locally compact group G by such automorphisms of $\mathscr{A}$ (α is a homomorphism, satisfying appropriate continuity conditions, of G into Aut($\mathscr{A}$)). The triple (A, G, α) is a C*-dynamical system or W*-dynamical system when $\mathscr{A}$ is a

W*-algebra (and α satisfies the continuity suitable for that case).

Among the preparatory topics in the first chapter is the theorem that makes clear the underlying rationale for studying operator algebras and the basis for their fundamental role as models for quantum physical systems. This theorem states that each commutative C*-algebra (with unit) is isomorphic to the algebra $C(X)$ (under pointwise operations) of complex-valued, continuous functions on a compact (locally compact when there is no unit in $\mathscr{A}$) Hausdorff space. The isomorphism preserves all discernible structure. Each $C(X)$ is (isomorphic to) a commutative C*-algebra. Thus the classes of abelian C*-algebras and of function algebras are coextensive. The noncommutative C*-algebras are, then, the right model for "noncommutative function algebras" and the natural framework for "noncommutative analysis", precisely the analysis needed for quantum mechanics. The possibility of (a "small amount" of) noncommutativity ($QP - PQ = \imath\hbar I$) provides the soil for growing the "uncertainty" and "indeterminacy" that are the basic characteristics of quantum mechanics.

From the inspired insight of Bohr's *ad hoc* quantum rules to the initial search of Heisenberg ("matrix mechanics") and Schrödinger ("wave mechanics") for a suitable mathematical model and then to Dirac's brilliant volume in which the C*-algebra is all but defined (much in the way that the fiber bundle is all but defined in, and is basic to, E. Cartan's work on global differential geometry), the formal mathematical model for quantum physics that emerges is the operator algebra. In classical mechanics, with X the phase space of the system, the real functions in $C(X)$ are the (bounded) observables of the system. The selfadjoint elements in the operator algebra $\mathscr{A}$ play this role for quantum mechanical systems. The dynamical (time) evolution of a system can be modeled as a one-parameter group of automorphisms (the Heisenberg picture of "moving observables") or as a one-parameter family of transformations of the space of (pure) states (the Schrödinger picture of "moving states"). The space of pure states corresponds to the phase space in the classical case.

In loose terms, the natural way to view the one-parameter group $t \to \alpha_t$ of (time-evolution) automorphisms of the operator algebra is as $t \to e^{t\delta}$, where δ is a linear mapping of $\mathscr{A}$ into $\mathscr{A}$ satisfying Leibniz's rule, $\delta(AB) = \delta(A)B + A\delta(B)$. Such mappings δ are, of course, the *derivations* of $\mathscr{A}$ (into $\mathscr{A}$). A special class of these derivations δ_T arises from elements T in $\mathscr{A}$, where $\delta_T(A) = AT - TA$. Such derivations are said to be *inner*. It can happen that some operator H not in $\mathscr{A}$ has the property that $AH - HA$ lies in $\mathscr{A}$ whenever A is in $\mathscr{A}$, in which case, $A \to AH - HA$ is a (not necessarily inner) derivation of $\mathscr{A}$. The operator H that generates the group of time-evolution automorphisms of $\mathscr{A}$ in this manner corresponds to the Hamiltonian of the physical system being modeled (and to the total energy "observable"). "Bracketing" with the "moving" observable ($[\alpha_t(A), H] = \alpha_t(A)H - H\alpha_t(A)$) amounts to time differentiation of $\alpha_t(A)$. If we interpret the group property of $t \to \alpha_t$ in terms of the differential information encoded in H, we arrive at Schrödinger's equation $d\psi(t)/dt = \imath H\psi(t)$.

These considerations underscore a key part for the theory of derivations in the relation between operator algebras and quantum physics. Answering a question of Kaplansky [K], Sakai [S1] showed that each (everywhere-defined) derivation of a C*-algebra $\mathscr{A}$ into itself is necessarily bounded. This would result, in applications to physical systems, in the Hamiltonian being bounded, which is too restrictive. The theory must encompass unbounded (not-everywhere-defined) derivations. In the commutative case, a result of Singer states that the only everywhere-defined derivation of $C(X)$ into itself is 0. In the case of a noncommutative $C(X)$, a C*-algebra $\mathscr{A}$ acting on a Hilbert space $\mathscr{H}$, each everywhere-defined derivation δ of $\mathscr{A}$ into itself has an automatic special continuity property that permits its extension to a derivation $\overline{\delta}$ of the smallest von Neumann algebra $\overline{\mathscr{A}}$ containing $\mathscr{A}$ ($\overline{\mathscr{A}}$ is the *strong-operator closure* of $\mathscr{A}$). The main theorem in the theory of bounded derivations states that each bounded derivation of a von Neumann algebra is inner. (This is the "derivation theorem".) Sakai's own contribution to the proof of this theorem was crucial (and consummately ingenious).

The theory of bounded derivations is deftly and elegantly presented in Chapter 2 (pp. 16–54). This chapter includes a thumbnail description of the Haag-Kastler-Araki formulation of quantum field theory, the Araki and Borchers theorems on "observability" of the energy, uniformly continuous dynamical systems and ground states, and a discussion of extending the "positive energy" or "spectrum" condition from real n-space to more general Lie groups.

With the bounded derivation and norm-continuous, automorphism group theory firmly in place, Sakai turns, in Chapter 3 (pp. 55–100), to a development of the theory of unbounded derivations. Here the domain of the derivation is assumed to be a dense subalgebra of the algebra. The emphasis is on *-derivations (those that preserve adjoints). Sakai begins by noting six specific examples, the last three directly from physics (the Ising models, the Heisenberg models, and the algebra of the (infinite) anticommutation relations on Fock space). He goes on to discuss, in detail, the closability of derivations, pointing out that, while everywhere-defined derivations are automatically bounded, the unbounded derivation need not even be closable (that is, have graph with closure the graph of a linear transformation). He develops the results on "well-behaved" elements in the domain and well-behaved derivations (related to vanishing of the derivation at maxima) and the notion of "approximate innerness" (there is a sequence H_n in $\mathscr{A}$ such that $\delta(A) = \lim \imath[H_n, A]$ for each A in the domain of δ). The next topic in the chapter deals with the domain of a closed *-derivation and is focused on the theorem (3.3.7) that if A is a selfadjoint element in the domain, then $f(A)$ is in the domain for each twice continuously differentiable function f. In the commutative case, the same is true for a C_1-function f, but this is *not* valid in the general noncommutative C*-algebra. The next section of the chapter deals with the generator problem: when the closure of a closable *-derivation of $\mathscr{A}$ generates a (strongly continuous) one-parameter group of *-automorphisms of $\mathscr{A}$. Various conditions involving being well behaved are proved for a *-derivation to have a generating closure. The sets of analytic, entire, and geometric elements in the domain are studied. In the section that follows that, results on unbounded derivations of commutative C*-algebras are noted. The chapter ends with a section on transformation groups and unbounded derivations.

The fourth and final chapter (pp. 101–206) deals with C*-dynamical systems. The concept of *approximately inner* C*-dynamics is defined (loosely, a one-parameter group of *-automorphisms that is the limit, in an appropriate sense, of a sequence of such groups generated by inner derivations). The (general) normal *-derivation of $\mathscr{A}$ is defined as one with domain the union of an ascending sequence of C*-subalgebras, each containing the unit of $\mathscr{A}$ and such that the restriction of the derivation to each subalgebra is inner. If such a sequence (or *nest* for $\mathscr{A}$) can be found for which each subalgebra is a full matrix algebra over $\mathbb{C}$, then $\mathscr{A}$ is a UHF algebra (these were studied by Glimm [G]). They provide the algebras (especially one of them) for a large class of physical systems (among them, quantum lattice and spin systems). The normal *-derivations apply to these algebras and such nests. They were introduced by Sakai [S2] in a paper in which he proves his nice theorem (appearing in this chapter as Corollary 4.5.3) to the effect that each closed *-derivation of a UHF algebra has a (matrix) nest for the algebra in its domain (each normal *-derivation has the union of such a nest as its precise domain). It was a question by N. M. Hugenholtz on the possibility of realizing a dynamical evolution group on a UHF algebra as an approximately inner evolution group that started Sakai toward this line of investigation. Much of what is being done in this chapter is inspired by the C*-algebra approach to quantum statistical mechanics. Some of the important physical concepts—ground state, equilibrium state, surface energy, phase transition—appear in strict mathematical form in this chapter; numerous detailed and interesting results are proved about them. The developments centering around the Powers-Sakai conjecture (Is each C*-dynamical system based on a UHF algebra approximately inner?) and the "core problem" (Is a generator of such a dynamics the closure of a normal *-derivation?) are another main focus of this chapter.

Sakai's book makes for pleasant browsing and really worthwhile reading (but be prepared to concentrate, Sakai does not waste words!). It is evident that the book has been carefully proofread. Inevitably, typos remain (e.g., some occurrences of A at the bottom of p. 104 and the top of p. 105 should be $\mathscr{A}$) but they are scarce. If we must have something to quibble over, there is no index of notation (which causes no trouble even when browsing quickly). This volume is a gem!

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