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Triangular Algebras–Another Chapter

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ABSTRACT. Basic constructions involving triangular operator algebras are studied.

Introduction.

In [**K-S**], the subject of triangular operator algebras is introduced and the basic theory of these algebras is developed. Along with the basic theory, a detailed account of the special class of (maximal) triangular algebras called *ordered bases* (also, *hyperreducible algebras*) concludes that article. At the time that article was prepared, I. M. Singer and this author developed further results involving some constructions with triangular algebras. These results were formulated as a fourth chapter with the intention of including it in [**K-S**]. Before [**K-S**] appeared, it was decided to remove chapter 4 from [**K-S**] and to include it in a later article with further results on constructions. That later article did not materialize.

In the intervening years, a number of people have had access to that chapter. This area of mathematics, non-self-adjoint operator algebras, has flourished; the chapter has been quoted. At the conference which was the occasion for the present Proceedings, several people inquired about that chapter. These proceedings provide an especially appropriate means for making the chapter generally available. The chapter appears in the remainder of this article exactly as it was formulated in the latter part of 1958.

Reference

[K-S] R. Kadison and I. M. Singer *Triangular operator algebras* Amer. J. Math. **82** 1960 227-259.

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Chapter IV. Operations on Triangular Algebras

We study operations on families of triangular algebras which yield triangular algebras again. For the most part, the difficulties we shall encounter stem from the problem of proving maximality in those circumstances where it is to be reasonably suspected that the triangular algebra constructed is maximal.

4.1. Restrictions of triangular algebras. We begin by studying restrictions of triangular algebras to projections. For this purpose, we define "an interval" in a triangular algebra to be a projection which is the difference of two hulls. Thus, an interval is an element of the core.

LEMMA 4.1.1. *If $\mathcal{T}$ is a maximal triangular algebra with diagonal $\mathcal{A}$, core $\mathcal{C}$, and hulls $\{E_\alpha\}$, then each cyclic projection in the core is the intersection of projections in $\mathcal{C}$ which are the unions of intervals, and each projection in $\mathcal{C}$ is a strong limit point of finite unions of intervals.*

PROOF. We shall not be concerned with $\mathcal{T}$, but just with $\mathcal{C}$ and the fact that $\{E_\alpha\}$ is totally ordered, closed under unions and intersections, generates $\mathcal{C}$, and each minimal projection in $\mathcal{C}$ is an interval (cf. Lemma 2.3.4). We note that all these properties are preserved under restriction to a projection in $\mathcal{C}$ (i.e., $\{E_\alpha P\}$ is totally ordered, etc.). If E is a projection in $\mathcal{C}$, and M is the union of the minimal projections in $\mathcal{C}$, then M is a union of intervals; so that E is the intersection of unions of intervals provided $E(I - M)$ is such an intersection in $\mathcal{C}(I - M)$. We may assume that $\mathcal{C}$ has no minimal projections. Suppose that we have our result in the case where I is a cyclic projection for $\mathcal{C}$. Then, for arbitrary $\mathcal{C}$, if E is a cyclic projection, G is the intersection of the unions of intervals containing it, and $G \neq E$, let F be a cyclic projection in $G - E$; so that $E + F$ is cyclic. By assumption, E in $\mathcal{C}(E + F)$ is the intersection of unions of intervals in $\mathcal{C}(E+F)$. Thus, if x is a generator for F, there is a union of intervals in $\mathcal{C}(E + F)$ containing E but not x. This union has the form $F'(E + F)$, with F' a union of intervals in $\mathcal{C}$. It follows that F' contains E and does not contain x. However, F' contains G (by definition of G) and G contains x. Thus $G = E$. We may assume that $\mathcal{C}$ has a generating vector.

Each union of intervals is a strong limit point of finite unions of intervals, so that it suffices to approximate projections strongly with such unions. Moreover, since $\mathcal{C}$ is abelian, finite sums of orthogonal cyclic projections are cyclic, whence each projection in $\mathcal{C}$ is a strong limit point of cyclic projections in $\mathcal{C}$. It remains to establish that each projection, E, in $\mathcal{C}$ is the intersection of the unions of intervals containing it, under the assumption that $\mathcal{C}$ has a generating vector.

If x is a unit vector generating I (i.e., $I = [\mathcal{C}'x]$, where $\mathcal{C}'$ is the commutant of $\mathcal{C}$), then x is a separating vector for $\mathcal{C}$. Let Q be $[\mathcal{C}x]$, so that Q lies in $\mathcal{C}'$. Since x is separating for $\mathcal{C}$, the mapping, $\mathcal{C} \to \mathcal{C}Q$, of $\mathcal{C}$ onto $\mathcal{C}Q$ is a *-isomorphism, and $\mathcal{C}Q$ is maximal abelian on $[\mathcal{C}x]$. Since the properties of $\mathcal{C}$ and $\{E_\alpha\}$ are preserved under *-isomorphisms, we may assume that $\mathcal{C}$ is maximal abelian (has no minimal projections, and x is a generating and separating unit vector for $\mathcal{C}$). From Theorem 3.3.1, we may assume, in addition, that $\mathcal{C}$ is the multiplication algebra of $L_2(0,1)$ under Lebesgue measure, $\{E_\alpha\}$ is indexed by points of $[0,1]$ so that E_α is multiplication by the characteristic function of $[0,\alpha]$. Then E is multiplication by the characteristic function of some measurable set, S. Regularity of the measure allows us to conclude that S is the difference of an intersection of open sets and a set of measure 0; so that E is the intersection of the projections corresponding to these open sets. Now, each open set is a union of (disjoint) open intervals, and each open interval corresponds to a projection which is an interval (relative to $\{E_\alpha\}$). The proof is complete.

Note that if two intervals have a non-zero intersection their union is an interval (immediate, from the definitions of "hull" and "interval"). In the preceding lemma, therefore, we may speak of "sums of orthogonal intervals" in place of "unions of intervals." Moreover, the intersection of a finite family of orthogonal sums of intervals is again such a sum, so that the intersection of the preceding lemma may be taken over a totally-ordered family of unions of intervals.

THEOREM 4.1.2. *If $\mathcal{T}$ is a triangular algebra with diagonal $\mathcal{A}$, core $\mathcal{C}$, hulls $\{E_\alpha\}$, and P is a projection in $\mathcal{A}$, then $P\mathcal{T}P$ is triangular with diagonal $\mathcal{A}P$, core $\mathcal{C}P$ and hulls $\{E_\alpha P\}$. If, in addition, $\mathcal{T}$ is maximal triangular and P is a finite sum of intervals in $\mathcal{C}$, then $P\mathcal{T}P$ is maximal triangular.*

PROOF. It is clear that $P\mathcal{T}P$ is an algebra. If A in $P\mathcal{T}P$ is self-adjoint, then A lies in $\mathcal{A}$, since $\mathcal{T}$ is triangular with diagonal $\mathcal{A}$ and contains A. Thus A lies in $\mathcal{A}P$. Now $\mathcal{A}P$ is maximal abelian on $P(\mathcal{H})$ and is contained in $P\mathcal{T}P$; from which it follows that $P\mathcal{T}P$ is triangular with diagonal $\mathcal{A}P$.

That $E_\alpha P$ is a hull in $P\mathcal{T}P$ follows from:

$$PTPPE_\alpha = PTPE_\alpha = E_\alpha PTPE_\alpha = PE_\alpha PTPPE_\alpha,$$

for each T in $\mathcal{T}$. Moreover, if F is a hull of $P\mathcal{T}P$, then $F = h(F)P$, which lies in $\{E_\alpha P\}$. In fact, clearly $F \le h(F)P$; and $h(F) = [\mathcal{T}F(\mathcal{H})]$, so that $Ph(F) = [P\mathcal{T}PF(\mathcal{H})] \le F$. The core of $P\mathcal{T}P$ is the von Neumann subalgebra of $\mathcal{A}P$ generated by $\{E_\alpha P\}$, which is $\mathcal{C}P$. Thus, in particular, $P\mathcal{T}P$ is hyperreducible or irreducible if the same is true for $\mathcal{T}$.

We suppose, now, that $\mathcal{T}$ is maximal and P lies in $\mathcal{C}$. To demonstrate the maximality of $P\mathcal{T}P$, we shall show that if T is an operator which generates with $P\mathcal{T}P$ a triangular algebra, $\mathcal{T}_0$, and $T = PTP$, then T and $\mathcal{T}$ generate a triangular algebra, $\mathcal{T}_1$. Hence, T lies in $\mathcal{T}$, by maximality of $\mathcal{T}$, so that T lies in $P\mathcal{T}P$. We begin by showing that T leaves each hull of $\mathcal{T}$ invariant, so that the same is true for each operator in $\mathcal{T}_1$.

If E is a hull in $\mathcal{T}$, then $ET^*(I-E)$ lies in $\mathcal{T}$, by Lemma 2.3.2. Now, $ET^*(I-E) = EPT^*P(I-E) = PET^*(I-E)P$ lies in $P\mathcal{T}P$, and $(I-E)TE = P(I-E)PTPEP$ lies in $\mathcal{T}_0$, a triangular algebra. Thus, the self-adjoint operator,

$ET^*(I-E)+(I-E)TE$ lies in $\mathcal{T}_0$ and hence in $\mathcal{A}$. In particular, it commutes with E, so that

$$0=(I-E)[ET^*(I-E)+(I-E)TE]E=(I-E)TE,$$

and $TE=ETE$. Thus, T leaves E invariant.

If S is a self-adjoint operator in $\mathcal{T}_1$ and F and G are orthogonal intervals in $\mathcal{T}$, then one of $h(F)G, h(G)F$ is 0; so that in either case, $FSG=(GSF)^*=0$, by invariance of the hulls of $\mathcal{T}$ under S. With Q a finite sum of orthogonal intervals, the same is true for $I-Q$, whence $0=QS(I-Q)=(I-Q)SQ$.

According to Lemma 4.1.1, P_0 in $\mathcal{C}$ is a strong limit point of a family, $\{Q\gamma\}$, of finite sums of orthogonal intervals. Thus $P_0S(I-P_0)$ and $(I-P_0)SP_0$ are weak limit points of $\{Q_\gamma S(I-Q_\gamma)=0\}$ and $\{(I-Q_\gamma)SQ_\gamma=0\}$, respectively. Both $P_0S(I-P_0)$ and $(I-P_0)SP_0$ are, therefore, 0.

Observe, next, that, with S in $\mathcal{T}_0$,

$$S=T_0+\sum T_1T^{k_1}T_2T^{k_2}T_3\ldots T_{n-1}T^{k_{n-1}}T_n,$$

where $T_0,\ldots,T_n$ are operators in $\mathcal{T}$ and $k_1,\ldots,k_{n-1}$ are positive integers (the sum is finite). Now,

$$PSP=PT_0P+\sum PT_1PT^{k_1}P\ldots PT^{k_{n-1}}PT_nP,$$

which lies in $\mathcal{T}_0$ (recall that $T=PTP$), and PSP is self-adjoint; whence PSP lies in $\mathcal{A}$. Thus

$$\begin{aligned}S&=(I-P)S(I-P)+(I-P)SP+PS(I-P)+PSP\\&=(I-P)S(I-P)+PSP\end{aligned}$$

lies in $\mathcal{A}$ if $(I-P)S(I-P)$ does.

If P is an interval, then $I-P=P_1+P_2$, where $P_1(=h(P)-P)$ is a hull orthogonal to the interval, $P_2(=I-h(P))$. In this case we have

$$\begin{aligned}(I-P)S(I-P)&=(I-P)T_0(I-P)+\sum_{j,h=1,2}\sum P_jT_1PT^{k_1}\ldots T^{k_{n-1}}PT_nP_h\\&=(I-P)T_0(I-P)+\sum P_1T_1PT^{k_1}\ldots T^{k_{n-1}}PT_nP_2,\end{aligned}$$

where the second equality follows from the fact that P_1 is a hull, since P is an interval, so that $PT_nP_1=0$; and $P_2T_1P=0$. Since S commutes with $\mathcal{C}$,

$$\begin{aligned}0&=P_1(I-P)S(I-P)P_2\\&=P_1(I-P)T_0(I-P)T_0(I-P)P_2+\sum P_1T_1PT^{k_1}\ldots T^{k_{n-1}}PT_nP_2.\end{aligned}$$

Thus

$$\sum P_1T_1PT^{k_1}\ldots T^{k_{n-1}}PT_nP_2(=-P_1(I-P)T_0(I-P)P_2)$$

lies in $\mathcal{T}$; and $(I-P)S(I-P)$ is a self-adjoint operator in $\mathcal{T}$, hence in $\mathcal{A}$. It follows that PTP is maximal when P is an interval.

Let $\{Q_1, \ldots, Q_n\}$ be a finite set of orthogonal intervals in $\mathcal{T}$ so indexed that $h(Q_j)Q_k = 0$ if and only if $j < k$; and let P be $\sum Q_j$. If S is an element of a triangular algebra, $\mathcal{S}$, on $P(\mathcal{H})$ containing $P\mathcal{T}P$, then $Q_jSQ_k = 0$ if $k < j$, for $Q_kS^*Q_j$ is in $\mathcal{T}$ (from Lemma 2.3.2, since $\mathcal{T}$ is maximal), so that $Q_kS^*Q_j$ is in $P\mathcal{T}P$ which is contained in $\mathcal{S}$. Hence $Q_jSQ_k + Q_kS^*Q_j$ lies in $\mathcal{A}$ and $Q_jSQ_k = 0$, as asserted. Moreover, Q_kSQ_j lies in $P\mathcal{T}P$, if $k < j$, from what we have just noted. Since $Q_j\mathcal{T}Q_j$ is maximal, $Q_j\mathcal{T}Q_j = Q_j\mathcal{S}Q_j$ which contains Q_jSQ_j. Thus, $S(= \sum Q_kSQ_j)$ lies in $P\mathcal{T}P$, and $P\mathcal{T}P$ is maximal triangular.

In the maximality proof of the preceding theorem, it would not be a difficult technical matter to pass from the finite sum of orthogonal intervals to arbitrary projections in the core, with the aid of Lemma 4.1.1 if we knew that $\mathcal{T}$ were strongly closed. Making use of the "triangular direct product", we describe in Section 4.3 (cf. Question 10 and the remarks following) a class of maximal triangular algebras which illustrate the fact that such algebras need not be strongly closed. This does not imply however that maximality can fail upon restriction to arbitrary projections in the core, for a very special strong limit is necessary in the general case. The transition from sums of intervals to arbitrary core projections by strong limit methods seems to hinge upon subtle questions involving the multiplicity decomposition of the core and its relation to the triangular algebra. We have placed such matters outside the scope of the present investigation.

REMARK 4.1.3. Despite the difficulties which arise in the case of the general core, it is a simple matter to show that the restriction of a hyperreducible maximal triangular algebra, $\mathcal{T}$, to a core (diagonal) projection, P, is maximal (and hyperreducible, in view of Theorem 4.1.2). In fact, if $\mathcal{A}$ is the diagonal and $\{E_\alpha\}$ the hulls of $\mathcal{T}$, then $\{E_\alpha P\}$ are the hulls of the triangular algebra $P\mathcal{T}P$ with diagonal $\mathcal{A}P$, by Theorem 4.1.2. If $T = PTP$ and T leaves each $E_\alpha P$ invariant, then

$$TE_\alpha = TPE_\alpha = PE_\alpha TPE_\alpha = E_\alpha PTPE_\alpha = E_\alpha TE_\alpha.$$

Thus T lies in $\mathcal{T}$, and hence in $P\mathcal{T}P$. Since $\mathcal{A}P$ is maximal abelian and generated by the totally-ordered family, $\{E_\alpha P\}$, the set, $P\mathcal{T}P$, of all operators leaving each $E_\alpha P$ invariant is maximal triangular (cf. Theorem 3.3.1).

If the maximal triangular algebra is irreducible instead of hyperreducible, then restriction to core projections (0 and I) preserves maximality by default.

REMARK 4.1.4. For the most part, the maximality of a triangular algebra appears in a proof to assure us that the hulls form a totally-ordered family or to guarantee the conclusion of Lemma 2.3.2 (viz. if $T = ET(I - E)$, with E a hull, then T lies in the algebra). It seems worth noting that if $\mathcal{T}$ is a triangular algebra which has either of these properties and P is a projection in the diagonal of $\mathcal{T}$, then $P\mathcal{T}P$ has the corresponding properties. Indeed, if $\{E_\alpha\}$, the hulls of $\mathcal{T}$, form a totally-ordered family, then so do $\{E_\alpha P\}$, the hulls of $P\mathcal{T}P$. If $\mathcal{T}$ has the property of Lemma 2.3.2, and $T = EPT(P - EP)$, with E a hull in $\mathcal{T}$, then $T = PTP = ET(I - E)$, so that T lies in $\mathcal{T}$ and hence in $P\mathcal{T}P$.

4.2. Triangular direct sums. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are triangular algebras with diagonals $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; the direct sum, $\mathcal{T}_1 \oplus \mathcal{T}_2$, of $\mathcal{T}_1$ and $\mathcal{T}_2$ is triangular, for

$$\begin{aligned}(\mathcal{T}_1 \oplus \mathcal{T}_2)^* \cap (\mathcal{T}_1 \oplus \mathcal{T}_2) &= (\mathcal{T}_1^* \oplus \mathcal{T}_2^*) \cap (\mathcal{T}_1 \oplus \mathcal{T}_2)\\ &= (\mathcal{T}_1^* \cap \mathcal{T}_1) \oplus (\mathcal{T}_2^* \cap \mathcal{T}_2)\\ &= \mathcal{A}_1 \oplus \mathcal{A}_2,\end{aligned}$$

and $\mathcal{A}_1 \oplus \mathcal{A}_2$ is maximal abelian. However, $\mathcal{T}_1 \oplus \mathcal{T}_2$ cannot be maximal triangular even if $\mathcal{T}_1$ and $\mathcal{T}_2$ are, for the algebra generated by $\mathcal{T}_1 \oplus \mathcal{T}_2$ and an arbitrary operator mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ and annihilating $\mathcal{H}_1$ is triangular. (Without checking this, we can conclude that $\mathcal{T}_1 \oplus \mathcal{T}_2$ is not maximal triangular by noting that the projections of $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$ are non-zero, orthogonal hulls for $\mathcal{T}_1 \oplus \mathcal{T}_2$, in contrast with the conclusion of Lemma 2.3.3.) The fact that such operators may be added to $\mathcal{T}_1 \oplus \mathcal{T}_2$ with the resulting algebra still triangular indicates the route we must take in forming a "triangular direct sum" of $\mathcal{T}_1$ and $\mathcal{T}_2$ which will be maximal when $\mathcal{T}_1$ and $\mathcal{T}_2$ are. Of course, we can adjoin the operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$ instead of those mapping $\mathcal{H}_2$ into $\mathcal{H}_1$; so that we must give preference to one of $\mathcal{H}_1$ or $\mathcal{H}_2$. Interpreted in terms of "triangular direct sums" of arbitrary families of triangular algebras, this means that the indexing set should be totally ordered.

DEFINITION 4.2.1. If Γ is a totally-ordered set and $\{\mathcal{T}_\gamma\}_{\gamma\in\Gamma}$ is a family of triangular algebras, $\mathcal{T}_\gamma$, acting on Hilbert spaces, $\mathcal{H}_\gamma$; we denote by "$\sum \oplus_t \mathcal{T}_\gamma$" the set of all bounded operators, T, on $\mathcal{H}$, the direct sum of $\{\mathcal{H}_\gamma\}_{\gamma\in\Gamma}$, such that $P_\gamma T P_{\gamma'} = 0$ if $\gamma' < \gamma$, and $P_\gamma T P_{\gamma'} \in \mathcal{T}_\gamma$, if $\gamma = \gamma'$ where P_γ is the projection of $\mathcal{H}$ on $\mathcal{H}_\gamma$. We refer to $\sum \oplus_t \mathcal{T}_\gamma$ as "the triangular direct sum of $\{\mathcal{T}_\gamma\}_{\gamma\in\Gamma}$."

With the notation of this definition, we prove:

THEOREM 4.2.2. *The set, $\sum \oplus_t \mathcal{T}_\gamma (= \mathcal{T})$, is a triangular algebra with diagonal, $\sum \oplus \mathcal{A}_\gamma (= \mathcal{A})$, and core, $\sum \oplus \mathcal{C}_\gamma (= \mathcal{C})$, if and only if each $\mathcal{T}_\gamma$ is triangular with diagonal, $\mathcal{A}_\gamma$, and core, $\mathcal{C}_\gamma$; so that $\mathcal{T}$ is irreducible if and only if each $\mathcal{T}_\gamma$ is, and $\mathcal{T}$ is hyperreducible if and only if each $\mathcal{T}_\gamma$ is. The hulls of $\mathcal{T}$ are either I or projections of the form $\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}$, where $E_{\gamma'}$ is a hull of $\mathcal{T}_{\gamma'}$ and P_γ is the identity of $\mathcal{T}_\gamma$. Moreover, $\mathcal{T}$ is maximal triangular if and only if each $\mathcal{T}_\gamma$ is.*

PROOF. If $\mathcal{T}$ is triangular with core, $\sum \oplus \mathcal{C}_\gamma$, then $P_{\gamma'}$ is in this core, so that $P_{\gamma'} \mathcal{T} P_{\gamma'} (= \mathcal{T}_{\gamma'})$ is triangular with core, $P_{\gamma'} \mathcal{C} P_{\gamma'} (= \mathcal{C}_{\gamma'})$, and diagonal, $P_{\gamma'} \mathcal{A} P_{\gamma'} (= \mathcal{A}_{\gamma'})$, from Theorem 4.1.2. On the other hand, if each $\mathcal{T}_\gamma$ is triangular with diagonal $\mathcal{A}_\gamma$, and T is a self-adjoint operator in $\mathcal{T}$, then

$$0 = P_\gamma T P_{\gamma'} = (P_\gamma T P_{\gamma'})^* = P_{\gamma'} T P_\gamma,$$

when $\gamma' < \gamma$, and $P_\gamma T P_\gamma$ is a self- adjoint operator in $\mathcal{T}_\gamma$, hence in $\mathcal{A}_\gamma$, for each γ. Thus $T = \sum P_\gamma T P_\gamma \in \mathcal{A}$; and since $\mathcal{T}$ contains $\sum \oplus \mathcal{A}_\gamma (= \mathcal{A})$, $\mathcal{T}$ is triangular with diagonal $\mathcal{A}$.

If we establish that the projections described in the statement of this theorem, are the hulls of $\mathcal{T}$, it will follow at once that $\sum \oplus \mathcal{C}_\gamma (= \mathcal{C})$ is the core of $\mathcal{T}$. Now, if $E = \sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}$, then

$$\begin{aligned} TE &= (\sum_{\gamma''} P_{\gamma''})T(\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}) \\ &= (\sum_{\gamma''\le\gamma'} P_{\gamma''})T(\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}) \\ &= (\sum_{\gamma''<\gamma'} P_{\gamma''})T(\sum_{\gamma<\gamma'} P_\gamma) + (\sum_{\gamma''<\gamma'} P_{\gamma''})TE_{\gamma'} + P_{\gamma'}TP_{\gamma'}E_{\gamma'} \\ &= (\sum_{\gamma''<\gamma'} P_{\gamma''})T(\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}) + E_{\gamma'}P_{\gamma'}TP_{\gamma'}E_{\gamma'} \\ &= (\sum_{\gamma''<\gamma'} P_{\gamma''})T(\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}) + E_{\gamma'}P_{\gamma'}T(\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}) \\ &= (\sum_{\gamma''<\gamma'} P_{\gamma''} + E_{\gamma'})T(\sum_{\gamma<\gamma'} P_\gamma + E_{\gamma'}) = ETE, \end{aligned}$$

for T in $\mathcal{T}$. Thus E is a hull in $\mathcal{T}$.

If F is a hull in $\mathcal{T}$, then F lies in $\mathcal{A}$, so that $F = \sum F_\gamma$, with F_γ in $\mathcal{A}_\gamma$. Since $\mathcal{T}$ contains $\sum \oplus \mathcal{T}_\gamma$ and F is invariant under $\mathcal{T}$, each F_γ is a hull in $\mathcal{T}_\gamma$. If some $F_{\gamma'} \neq 0$, $\gamma < \gamma'$, x is a unit vector in the range of $F_{\gamma'}$, and y is an arbitrary vector in the range of P_γ; the transformation, $z \to (z,x)y$, is bounded and lies in $\mathcal{T}$. Since F is invariant under $\mathcal{T}$, $F_\gamma = P_\gamma$, for $\gamma < \gamma'$. It follows that $F = I$ or $F = \sum_{\gamma<\gamma'} P_\gamma + F_{\gamma'}$, for some γ'.

If some $\mathcal{T}_{\gamma'}$ is not maximal and $\mathcal{S}_{\gamma'}$ is a maximal triangular algebra containing $\mathcal{T}_{\gamma'}$, then $(\sum_{\gamma\neq\gamma'} \oplus_t \mathcal{T}_\gamma) \oplus_t \mathcal{S}_{\gamma'}$ is a triangular algebra containing $\mathcal{T}$ properly. Suppose, now, that each $\mathcal{T}_\gamma$ is maximal triangular. If $B \notin \mathcal{T}$, then $P_{\gamma'}BP_{\gamma'} \notin \mathcal{T}_{\gamma'}$, for some γ', or $P_{\gamma'}BP_\gamma \neq 0$, for some γ', γ, with $\gamma < \gamma'$. Let $\mathcal{S}$ be the algebra generated by $\mathcal{T}$ and B, so that $\mathcal{S}$ contains the algebra generated by $\mathcal{T}_{\gamma'}$ and $P_{\gamma'}BP_{\gamma'}$. If $P_{\gamma'}BP_{\gamma'} \notin \mathcal{T}_{\gamma'}$, this last algebra contains a self-adjoint operator not in $\mathcal{A}_{\gamma'}$, hence not in $\mathcal{A}$, since $\mathcal{T}_{\gamma'}$ is maximal triangular. If $P_{\gamma'}BP_\gamma \neq 0$, with $\gamma < \gamma'$, then $P_\gamma B^* P_{\gamma'}$ lies in $\mathcal{T}$, so that $P_{\gamma'}BP_\gamma + P_\gamma B^* P_{\gamma'}$, a self-adjoint operator, lies in $\mathcal{S}$ but not in $\mathcal{A}$ (e.g., does not commute with P_γ). Thus, in either case, $\mathcal{S}$ is not triangular, and $\mathcal{T}$ is maximal.

REMARK 4.2.3. An intrinsic characterization of a triangular direct sum is easily had: If the diagonal, $\mathcal{A}$, of a triangular algebra, $\mathcal{T}$, contains an orthogonal family of projections, $\{P_\gamma\}$, indexed by a totally-ordered set, such that $\mathcal{T}$ contains each bounded operator B for which $P_\gamma BP_{\gamma'} = 0$ when $\gamma' < \gamma$ and $P_\gamma BP_\gamma \in \mathcal{T}$, then $\mathcal{T}$ is isomorphic to $\sum_\gamma \oplus_t P_\gamma \mathcal{T} P_\gamma$.

4.3. Tensor products of triangular algebras. The theory which we develop in this section bears very little resemblance to the standard tensor product theories for algebraic structures. Each step uncovers new pathological phenomena which require special consideration. There is still much to be desired in the way of generality–our more incisive results impose the restriction that one of the algebras of the product be an integer-ordered basis. The list of questions at the end of this section gives some indication of the points which still need clarification.

The constructions we shall describe enable us to give an example of a maximal triangular algebra which is not strongly closed (see Question 10). For this purpose, we shall need some information concerning ideals in hyperreducible algebras. In the following lemma, we make reference to "a diagonal process relative to a maximal abelian algebra."This subject is treated in detail in [**5**].

LEMMA 4.3.1. *If $\mathcal{T}$ is a hyperreducible maximal triangular algebra with diagonal $\mathcal{A}$ then each two-sided ideal in $\mathcal{T}$ whose intersection with $\mathcal{A}$ is (0) is contained in one maximal with respect to this property. Moreover, if $\mathcal{D}$ is a diagonal process relative to $\mathcal{A}$, each two-sided ideal of $\mathcal{T}$ which is annihilated by $\mathcal{D}$ is contained in one which is maximal with respect to this property—there is just one such maximal ideal and it has intersection (0) with $\mathcal{A}$. If E and $I - F$ are hulls in $\mathcal{T}$, then $E\mathcal{T}F$ is a two-sided ideal in $\mathcal{T}$ and when $EF = 0$, is contained in each one of the two types of maximal ideals just mentioned.*

PROOF. Since the union of an ascending chain of two-sided ideals in $\mathcal{T}$ having intersection (0) with $\mathcal{A}$ or annihilated by $\mathcal{D}$ has, itself, intersection (0) with $\mathcal{A}$ or is annihilated by $\mathcal{D}$, respectively, an application of Zorn's lemma establishes the existence of the maximal ideals in question. A diagonal process is additive, so that the sum of two ideals annihilated by $\mathcal{D}$ is annihilated by $\mathcal{D}$. Thus, there is a unique two-sided ideal maximal with respect to annihilation by $\mathcal{D}$ which contains all other such ideals.

Clearly $E\mathcal{T}F$ is a linear subspace of $\mathcal{T}$. With B and T operators in $\mathcal{T}$,

$$BETF = EBETF \in E\mathcal{T}F,$$

since E is a hull; while

$$ETFB = ETFBF \in E\mathcal{T}F,$$

for, with $I - F$ a hull, $B(I - F) = (I - F)B(I - F)$, whence $FB = FBF$. Thus $E\mathcal{T}F$ is a two-sided ideal of $\mathcal{T}$. Now $\mathcal{D}(ETF) = E\mathcal{D}(T)F = EF\mathcal{D}(T) = 0$, when $EF = 0$; so that $E\mathcal{T}F$ is contained in the maximal $0 - \mathcal{D}$ diagonal ideal, in this case (cf. [**5, Lemma 2**]).

Suppose, now, that $\mathcal{I}$ is a two-sided ideal of $\mathcal{T}$ maximal with respect to the property of having intersection (0) with $\mathcal{A}$, and that $EF = 0$. If $B + ETF = A$, with B in $\mathcal{I}$ and A in $\mathcal{A}$, then

$$EBE + ETFE = EBE = EAE = AE \in \mathcal{I}$$

and

$$\begin{aligned}(I-E)B(I-E)+(I-E)ETF(I-E) &= (I-E)B(I-E)\\ &= (I-E)A(I-E)\\ &= A(I-E)\in\mathcal{I}.\end{aligned}$$

Thus $A = AE + A(I-E)$ is in $\mathcal{I}$, and since $\mathcal{I}\cap\mathcal{A} = (0)$, $A = 0$. Hence $(\mathcal{I}+ET F)\cap\mathcal{A} = (0)$ and $\mathcal{I}$ contains $ET F$.

Finally, we note that if $\mathcal{I}$ is a two-sided ideal in $\mathcal{T}$ annihilated by $\mathcal{D}$ then $0 = \mathcal{D}(A) = A$, for each A in $\mathcal{I}\cap\mathcal{A}$, whence $\mathcal{I}\cap\mathcal{A} = (0)$.

We shall refer to the ideals having intersection (0) with $\mathcal{A}$ as "A–disjoint ideals"and those annihilated by $\mathcal{D}$ as "diagonal 0 ideals."

LEMMA 4.3.2. *If $\mathcal{T}$ is a hyperreducible maximal triangular algebra with diagonal $\mathcal{A}$, then all maximal diagonal 0, and $\mathcal{A}$-disjoint ideals in $\mathcal{T}$ coincide with the strongly (and weakly) closed ideal consisting of those operators in $\mathcal{T}$ annihilated by $\mathcal{D}$, the unique diagonal process relative to $\mathcal{A}$, in the case where $\mathcal{A}$ is totally atomic. If $\mathcal{A}$ is non-atomic, each such ideal has strong closure equal to $\mathcal{T}$, in the separable case.*

PROOF. If $\mathcal{A}$ is totally atomic and $\{F_\gamma\}$ is its (generating) set of minimal projections, then the mapping $\mathcal{D}$ defined by

$$\mathcal{D}(T) = \sum_\gamma F_\gamma T F_\gamma$$

is the unique diagonal process relative to $\mathcal{A}$ (cf. [**5, Theorem 1**]). If $T\in\mathcal{T}$ and $D(T) = 0$, then $F_\gamma T F_\gamma = 0$, for each γ; and with S in $\mathcal{T}$,

$$\begin{aligned}F_\gamma STF_\gamma &= F_\gamma STh(F_\gamma)F_\gamma = F_\gamma Sh(F_\gamma)TF_\gamma\\ &= F_\gamma S[h(F_\gamma)-F_\gamma+F_\gamma]TF_\gamma = F_\gamma S[h(F_\gamma)-F_\gamma]TF_\gamma\\ &= F_\gamma[h(F_\gamma)-F_\gamma]S[h(F_\gamma)-F_\gamma]TF_\gamma = 0.\end{aligned}$$

(Recall that, with F_γ a minimal projection in $\mathcal{A}$, $h(F_\gamma)-F_\gamma$ is a hull as well as $h(F_\gamma)$.) Thus $\mathcal{D}(ST) = 0$. Similarly, $\mathcal{D}(TS) = 0$; and the set, $\mathcal{I}$, of elements in $\mathcal{T}$ annihilated by $\mathcal{D}$ is a two-sided ideal in $\mathcal{T}$ (disjoint from $\mathcal{A}$). Suppose, now, that $\mathcal{I}'$ is a two-sided ideal in $\mathcal{T}$ disjoint from $\mathcal{A}$. Then, for each T in $\mathcal{I}'$ and each γ, $F_\gamma TF_\gamma\in\mathcal{A}\cap\mathcal{I}'$; so that $F_\gamma TF_\gamma = 0$. Thus, $\mathcal{D}(T) = 0$, and $\mathcal{I}'\subseteq\mathcal{I}$.

If $\mathcal{A}$ is non-atomic and the underlying Hilbert space is separable, then $\mathcal{T}$ is unitarily equivalent to the algebra of bounded operators on $L_2(0,1)$ with Lebesgue measure, leaving each of the projections E_λ, corresponding to multiplication by the characteristic function of $[0,\lambda]$, invariant. We may assume that $\mathcal{T}$ is this algebra. Define V_a, "translation by a," for $a\geq 0$, as follows: $(V_af)(\gamma) = f(\gamma+a)$ if $\gamma+a\leq 1$, and $(V_af)(\gamma) = 0$ if $\gamma+a > 1$. A simple computation shows that V_a has bound less than or equal to 1 and leaves each E_λ invariant, so that V_a lies in $\mathcal{T}$. Choose n so that $1/n < a$; let F_j be the projection corresponding to multiplication by the characteristic function, χ_j, of $[\frac{j-1}{n}, \frac{j}{n}]$; and note that $F_j V_a F_j = 0$. In fact, if $a + \gamma \leq 1$, then

$$(F_j V_a F_j f)(\gamma) = \chi_j(\gamma)\chi_j(\gamma + a) f(a + \gamma) = 0;$$

since γ and $\gamma + a$ cannot both lie in $[\frac{j-1}{n}, \frac{j}{n}]$; and if $a + \gamma > 1$, $(F_j V_a F_j f)(\gamma)$ is again 0. Thus, since

$$\sum_{j=1}^{n} F_j = I, V_a = (\sum_{j=1}^{n} F_j) V_a (\sum_{k=1}^{n} F_k) = \sum_{k>j} F_j V_a F_k.$$

Now $F_j V_a F_k = E F_j V_a F_k F$, where $E = \sum_{h=1}^{j} F_h$ and $F = \sum_{h=k}^{n} F_h$. Note that E and $I - F$ are hulls and that $EF = 0$, when $j < k$; so that $F_j V_a F_k$ and, hence V_a, lie in each maximal $\mathcal{A}$-disjoint and 0-diagonal ideal of $\mathcal{T}$, for $a > 0$. Since left and right multiplication by a fixed operator are strongly continuous, the strong closure of such an ideal is a two-sided ideal in $\mathcal{T}$. We conclude by noting that V_a tends strongly to I. In fact, since $\|V_a\| \leq 1$, we need establish this limit relation only for the set of generators of some dense linear manifold, and it is obvious for the characteristic functions of intervals.

We proceed to our tensor product considerations. With $\mathcal{C}_1$ and $\mathcal{C}_2$ algebras of operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, we write $\mathcal{C}_1 \otimes'' \mathcal{C}_2$ for the algebra of operators on the tensor product, $\mathcal{H}_1 \otimes \mathcal{H}_2$, generated by operators of the form $C_1 \otimes I$ and $I \otimes C_2$, with C_1 in $\mathcal{C}_1$ and C_2 in $\mathcal{C}_2$. For von Neumann algebras, $\mathcal{R}_1$ and $\mathcal{R}_2$, we shall adhere to the customary notation, $\mathcal{R}_1 \otimes \mathcal{R}_2$, for the von Neumann algebra generated by $\mathcal{R}_1 \otimes'' \mathcal{R}_2$. By $\mathcal{C}_1 \otimes \mathcal{B}_2$, we shall mean the algebra of all infinite matrices with entries from $\mathcal{C}_1$ which give bounded operators by action on $\mathcal{H}_1 \otimes \mathcal{H}_2$ in its representation as a direct sum of copies of $\mathcal{H}_1$ (a number of times equal to the dimension of $\mathcal{H}_2$). If $\mathcal{S}$ is a set of operators on $\mathcal{H}_1$ and $\mathcal{A}_2$ is a totally-atomic maximal abelian algebra on $\mathcal{H}_2$, we shall denote by $\mathcal{S} \otimes_{\mathcal{A}_2} \mathcal{B}_2$ the set of those matrices with a finite number of non-zero entries from $\mathcal{S}$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ relative to a diagonalizing basis for $\mathcal{A}_2$. This notation adopted, we can state:

DEFINITION 4.3.3. With $\mathcal{T}_1$ and $\mathcal{T}_2$ triangular algebras whose diagonals are $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, we denote by $\mathcal{T}_1 \otimes' \mathcal{T}_2$ the algebra generated by $\mathcal{T}_1 \otimes'' \mathcal{T}_2$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are hyperreducible maximal triangular algebras with hulls $\{E_\alpha\}$ and $\{F_\beta\}$, respectively, we write $\mathcal{T}_1 \otimes \mathcal{T}_2$ for the set of operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ leaving each projection in $\{I \otimes F_\beta\}$ and $\{E_\alpha \otimes I\}$ invariant. If, in addition, $\mathcal{T}_2$ is an integer-ordered basis and $\mathcal{I}_1$ is an $\mathcal{A}_1$-disjoint two-sided ideal in $\mathcal{T}_1$, we denote by $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ the linear space generated by $\mathcal{T}_1 \otimes \mathcal{T}_2$ and $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$.

With the notation of this definition:

THEOREM 4.3.4. *The algebra, $\mathcal{T}_1 \otimes' \mathcal{T}_2$, is triangular with diagonal $\mathcal{A}_1 \otimes \mathcal{A}_2$. The algebra, $\mathcal{T}_1 \otimes \mathcal{T}_2$, is hyperreducible and contains $\mathcal{T}_1 \otimes' \mathcal{T}_2$. The algebra, $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$, is triangular with diagonal $\mathcal{A}_1 \otimes \mathcal{A}_2$. If $\mathcal{I}_1$ is a maximal $\mathcal{A}_1$-disjoint ideal in $\mathcal{T}_1$ and $\mathcal{T}$ is a maximal triangular extension of $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$, then relative to* *the ordered basis, $\mathcal{T}_2$, each operator of $\mathcal{T}$ has a matrix representation with entries from $\mathcal{I}_1$ below the diagonal and from $\mathcal{T}_1$ elsewhere. If $\mathcal{T}_2$ is finite dimensional, $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ is maximal. The hulls of $\mathcal{T}$ are $\{E_\alpha \otimes I\}$, for those hulls, E_α, of $\mathcal{T}_1$ which are not the hull of a minimal projection in $\mathcal{A}_1$, and $\{E_\alpha \otimes I - G \otimes (I - F_\beta)\}$, for those projections, E_α, which are the hull of some minimal projection, G, in $\mathcal{A}_1$, and each hull, F_β, in $\mathcal{T}_2$. The core of $\mathcal{T}$ is $(\mathcal{A}_{1c} \otimes I) \oplus (\mathcal{A}_{1d} \otimes \mathcal{A}_2)$, where $\mathcal{A}_{1d} = \mathcal{A}_1 E$ and $\mathcal{A}_{1c} = \mathcal{A}_1(I - E)$, with E the sum of the minimal projections in $\mathcal{A}_1$.*

PROOF. From [**1; Sect. 6, Prop.14, p.102**], $(\mathcal{A}_1 \otimes \mathcal{A}_2)' = \mathcal{A}_1' \otimes \mathcal{A}_2' = \mathcal{A}_1 \otimes \mathcal{A}_2$; whence $\mathcal{A}_1 \otimes \mathcal{A}_2$ is maximal abelian. An operator, T, in $\mathcal{T}_1 \otimes' \mathcal{T}_2$ lies in $\mathcal{T}_1 \otimes \mathcal{B}_2$. If T is self-adjoint, then each entry of a matrix representation for T has its adjoint some other entry, and since all such entries lie in $\mathcal{T}_1$, a triangular algebra with diagonal $\mathcal{A}_1$, each entry of T lies in $\mathcal{A}_1$. Thus T lies in $\mathcal{A}_1 \otimes \mathcal{B}_2$. Symmetrically, T lies in $\mathcal{B}_1 \otimes \mathcal{A}_2$. Now,

$$\begin{aligned}(\mathcal{A}_1 \otimes \mathcal{B}_2) \cap (\mathcal{B}_1 \otimes \mathcal{A}_2) &= (\mathcal{A}_1' \otimes I)' \cap (I \otimes \mathcal{A}_2')' \\ &\subseteq \mathcal{R}(\mathcal{A}_1 \otimes I, I \otimes \mathcal{A}_2)' \\ &= (\mathcal{A}_1 \otimes \mathcal{A}_2)' = \mathcal{A}_1 \otimes \mathcal{A}_2.\end{aligned}$$

Thus T lies in $\mathcal{A}_1 \otimes \mathcal{A}_2$, and $\mathcal{T}_1 \otimes' \mathcal{T}_2$ is triangular.

With T_1 in $\mathcal{T}_1$ and T_2 in $\mathcal{T}_2$, it is clear that $T_1 \otimes T_2$ leaves $E_\alpha \otimes I$ and $I \otimes F_\beta$ invariant; whence $\mathcal{T}_1 \otimes' \mathcal{T}_2$ is a subset of $\mathcal{T}_1 \otimes \mathcal{T}_2$. As the set of operators which leave each of a fixed set of projections invariant, $\mathcal{T}_1 \otimes \mathcal{T}_2$ is an algebra. Now $\{E_\alpha \otimes I\}$ and $\{I \otimes F_\beta\}$ generate $\mathcal{A}_1 \otimes I$ and $I \otimes \mathcal{A}_2$, respectively, and these last generate $\mathcal{A}_1 \otimes \mathcal{A}_2$, a maximal abelian algebra. Each self-adjoint operator in $\mathcal{T}_1 \otimes \mathcal{T}_2$ leaves each $E_\alpha \otimes I$ and $I \otimes F_\beta$ invariant and so commutes with them and lies in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Thus $\mathcal{T}_1 \otimes \mathcal{T}_2$ is triangular and hyperreducible.

With $\mathcal{T}_2$ an integer-ordered basis, $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ is triangular. For the remainder of the proof, each matrix representation to which we refer will be understood relative to a diagonalizing basis for $\mathcal{A}_2$ with its $\mathcal{T}_2$ order. In this representation the elements of $\mathcal{T}_1 \otimes \mathcal{T}_2$ are precisely those bounded operators whose matrices have all entries in $\mathcal{T}_1$ and 0 entries below the diagonal. Note that $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$ is a linear space, so that $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ consists of operators which are sums of an operator in $\mathcal{T}_1 \otimes \mathcal{T}_2$ and one in $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$. To show that $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ is an algebra it will suffice to show that the product, $(T+S)(T'+S')$, of two such sums is again such a sum. Clearly TT' lies in $\mathcal{T}_1 \otimes \mathcal{T}_2$ and SS' in $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$. We show that TS' lies in $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ (a similar proof holds for ST'). Since S' is in $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$ and $\mathcal{T}_2$ is integer ordered, there is an interval, F, in $\mathcal{T}_2$ which is the sum of a finite number of atoms, $F_{\gamma_1}, \ldots, F_{\gamma_n}$, in $\mathcal{T}_2$, such that $S' = (I \otimes F)S'(I \otimes F)$. Let $\{F_\gamma\}$ be the set of atoms in $\mathcal{T}_2$. We write $\gamma < \gamma'$ when $h(F_\gamma) < h(F_{\gamma'})$. With $\gamma_1 < \gamma_2 < \cdots < \gamma_n$, note that if $\gamma_1 \leq \gamma \leq \gamma_n$ then $\gamma \in \{\gamma_1, \ldots, \gamma_n\}$, since F is an interval. For each $\gamma, \delta, (I \otimes F_\delta)TS'(I \otimes F_\gamma)$ is 0 unless $\gamma \in \{\gamma_1, \ldots, \gamma_n\}$. We examine the entries of TS' below the diagonal. If $\gamma_n < \delta$ then $\delta \notin \{\gamma_1, \ldots, \gamma_n\}$

and

$$\begin{aligned}(I \otimes F_\delta)TS'(I \otimes F_{\gamma_j}) &= (I \otimes F_\delta)T(I \otimes F)S'(I \otimes F_{\gamma_j}) \\ &= (I \otimes F_\delta)T(I \otimes Fh(F))S'(I \otimes F_{\gamma_j}) \\ &= (I \otimes F_\delta h(F))T(I \otimes F)S'(I \otimes F_{\gamma_j}) = 0\end{aligned}$$

since $h(F) = h(F_{\gamma_n}), F_\delta h(F_{\gamma_n}) = 0$, and T (in $\mathcal{T}_1 \otimes \mathcal{T}_2$) leaves $I \otimes h(F)$ invariant. Thus $(I \otimes (I - h(F)))TS' = 0$ and

$$\begin{aligned}TS' &= [(I \otimes F) + (I \otimes (h(F) - F)) + (I \otimes (I - h(F)))]TS'(I \otimes F) \\ &= (I \otimes F)TS'(I \otimes F) + (I \otimes (h(F) - F))TS'(I \otimes F).\end{aligned}$$

The first term lies in $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$, since F is the sum of a finite number of atoms, the entries of T lie in $\mathcal{T}_1$, and those of S' in $\mathcal{I}_1$ (an ideal in $\mathcal{T}_1$). The second term lies in $\mathcal{T}_1 \otimes \mathcal{T}_2$, since, with F an interval, all its entries below the diagonal are 0 and all entries lie in $\mathcal{T}_1$. Thus TS' lies in $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$.

If H is a self-adjoint operator in $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$, then as we have seen (all entries lie in $\mathcal{T}_1$), H lies in $\mathcal{A}_1 \otimes \mathcal{B}_2$. Since all entries of H below the diagonal lie in $\mathcal{I}_1$ and $\mathcal{I}_1 \cap \mathcal{A}_1 = (0)$, all such entries are 0, and with H self-adjoint, all entries above the diagonal are 0. The diagonal entries lie in $\mathcal{A}_1$, so that H lies in $\mathcal{A}_1 \otimes \mathcal{A}_2$; and $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$ is triangular.

We assume, now, that $\mathcal{I}_1$ is a maximal $\mathcal{A}_1$-disjoint ideal in $\mathcal{T}_1$ and that $\mathcal{T}$ is a maximal triangular extension of $\mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2$. If $B \in \mathcal{T}$, then

$$((I - E_\alpha) \otimes F_\gamma)B(E_\alpha \otimes F_\delta)(= B_1)$$

lies in $\mathcal{T}$ and has 0 entries with the possible exception of the γ, δ position where the entry has the form $(I - E_\alpha)CE_\alpha$, C a bounded operator on $\mathcal{H}_1$. Thus B_1^* has at most one non-zero entry, $E_\alpha C^*(I - E_\alpha)$, an element of $\mathcal{I}_1$, if E_α is a hull in $\mathcal{T}_1$, by Lemma 4.3.1. Thus $B_1^* \in \mathcal{T}_{1_{\mathcal{I}_1}} \otimes \mathcal{T}_2 \subseteq \mathcal{T}$. Since $\mathcal{T}$ is triangular, $B_1 \in \mathcal{A}_1 \otimes \mathcal{A}_2$; so that $(I - E_\alpha)CE_\alpha \in \mathcal{A}_1$, and $(I - E_\alpha)CE_\alpha = 0$. Hence

$$((I - E_\alpha) \otimes F_\gamma)B(E_\alpha \otimes F_\delta) = 0,$$

holds for all γ, δ; whence

$$((I - E_\alpha) \otimes I)B(E_\alpha \otimes I) = 0,$$

and B leaves $E_\alpha \otimes I$ invariant. Thus B lies in $\mathcal{T}_1 \otimes \mathcal{B}_2$. We write $B_{\gamma\delta}$ for the γ, δ entry of B. The only non-zero entry of $(T \otimes F_\gamma)B(T' \otimes F_\delta)$ is $TB_{\gamma\delta}T'$ in position γ, δ; and with T, T' in $\mathcal{T}_1, (T \otimes F_\gamma)B(T' \otimes F_\delta)$ lies in $\mathcal{T}$. Since the operator whose matrix has only one non-zero entry from $\mathcal{I}_1$ in position γ, δ lies in $\mathcal{I}_1 \otimes_{\mathcal{A}_2} \mathcal{B}_2$, it follows that T contains all operators whose only non-zero entry is an element of the ideal generated by $B_{\gamma\delta}$ and $\mathcal{I}_1$ at position γ, δ, with $\delta < \gamma$; for $(I \otimes F_\gamma)B(I \otimes F_\delta)T$ has $B_{\gamma\delta}$ as its only non-zero entry in position γ, γ, where T, the operator with I as its only non-zero entry in position δ, γ lies in $\mathcal{T}$, so that the operator whose only non-zero entry is $T_1 B_{\gamma\delta} \cdots T_{n-1} B_{\gamma\delta} T_n$, with T_j in $\mathcal{T}_1$, at position γ, δ lies in $\mathcal{T}$. In particular, if $B_{\gamma\delta}$ is not in $\mathcal{I}_1$, there is such an operator with entry a non-zero element of $\mathcal{A}_1$. The adjoint of this operator lies in $\mathcal{T}_1 \otimes \mathcal{T}_2$, and $\mathcal{T}$ would contain a self- adjoint operator not in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Thus $B_{\gamma\delta} \in \mathcal{I}_1$ when $\delta < \gamma$.

If A in $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a hull for $\mathcal{T}$, then each non-zero entry lies on the diagonal (since $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$), is a projection in $\mathcal{A}_1$ (since A is a projection), and is a hull for $\mathcal{T}_1$ (since $(T_1 \otimes I)A = A(T_1 \otimes I)A$, for each T_1 in $\mathcal{T}_1$). Let the γ, γ and δ, δ entries be E_α and $E_{\alpha'}$, respectively. The operator V and V' whose matrices have I and S at the γ, δ and δ, γ entries, respectively, and zeros elsewhere lie in $\mathcal{T}_{1\mathcal{I}_1} \otimes \mathcal{T}_2$ and hence in $\mathcal{T}$, where $\gamma < \delta$ and S is an arbitrary element of $\mathcal{I}_1$. Since $VA = AVA, E_\alpha E_{\alpha'} = E_{\alpha'}$, and since $V'A = AV'A, SE_\alpha = E_{\alpha'}SE_\alpha$. Thus $E_{\alpha'} \leq E_\alpha$ and $[\mathcal{I}_1 E_\alpha] \subseteq E_{\alpha'}$. If E is a hull in $\mathcal{T}_1$ such that $E < E_\alpha$, the operator mapping some non-zero vector of $E_\alpha - E$ onto an arbitrary vector of E and annihilating the orthogonal complement of this vector lies in $\mathcal{I}_1$, by Lemma 4.3.1; whence $E \leq E_{\alpha'}$. Thus $E_{\alpha'}$ is the hull immediately preceding E_α or else $E_{\alpha'} = E_\alpha$. The first case obtains only if $E_\alpha = h(G)$ with G a minimal projection in $\mathcal{A}_1$, from Lemma 2.3.4, and in this case, an element, S, of $\mathcal{I}_1$ will not map a non-zero vector of G onto another such vector (i.e., a scalar multiple of itself), for then $0 \neq GSG \in \mathcal{I}_1 \cap \mathcal{A}_1$, so that $[\mathcal{I}_1 E_\alpha] = h(G) - G$. Thus we distinguish two cases: if $E_\alpha = h(G)$, with G minimal in $\mathcal{A}_1, E_{\alpha'} = E_\alpha$ or $E_\alpha - G$; and if E_α is not the hull of a minimal projection, then $E_{\alpha'} = E_\alpha$. We conclude that the hulls of $\mathcal{T}$ have the form $E_\alpha \otimes I$ for those hulls E_α in $\mathcal{T}_1$ which are not the hull of some minimal projection in $\mathcal{A}_1$ and

$$E_\alpha \otimes F_\beta + (E_\alpha - G) \otimes (I - F_\beta)(= E_\alpha \otimes I - G \otimes (I - F_\beta))$$

for those projections $E_\alpha = h(G)$, with G minimal in $\mathcal{A}_1$. (It is easy to see that such projections are hulls for $\mathcal{T}$.)

Let E be the sum of the minimal projections in $\mathcal{A}_1$. The core, $\mathcal{C}$, of $\mathcal{T}$ contains $\mathcal{A}_1 \otimes I$ since it contains each $E_\alpha \otimes I$ and $\{E_\alpha\}$ generates $\mathcal{A}_1$. Moreover, if G is a minimal projection in $\mathcal{A}_1$, then

$$h(G) \otimes h(F_\gamma) + (h(G) - G) \otimes (I - h(F_\gamma))$$

and

$$h(G) \otimes (h(F_\gamma) - F_\gamma) + (h(G) - G) \otimes (I + F_\gamma - h(F_\gamma))$$

are hulls whose difference is $G \otimes F_\gamma$. Thus $\mathcal{C}$ contains $\mathcal{A}_1 E \otimes \mathcal{A}_2$, so that $\mathcal{C}$ contains $(\mathcal{A}_{1c} \otimes I) \oplus (\mathcal{A}_{1d} \otimes \mathcal{A}_2)$. On the other hand,

$$E_{\alpha'} \otimes I = (E_{\alpha'} E \otimes I) + (E_{\alpha'}(I - E) \otimes I)$$

and

$$E_\alpha \otimes I - G \otimes (I - F_\beta)$$

lie in $(\mathcal{A}_{1c} \otimes I) \oplus (\mathcal{A}_{1d} \otimes \mathcal{A}_2)$. Thus $\mathcal{C} = (\mathcal{A}_{1c} \otimes I) \oplus (\mathcal{A}_{1d} \otimes \mathcal{A}_2)$, and the proof is complete.

Several questions arise from the considerations of this section. We shall list them without comment as to their interrelations (although they are often manifest) and without comment as to our own conjecture concerning them.

Throughout, $\mathcal{T}$ is a maximal triangular, hyperreducible algebra with non-atomic diagonal, $\mathcal{A}$.

1. Is there a unique diagonal process on $\mathcal{T}$ relative to $\mathcal{A}$?
2. Is there a unique maximal diagonal-0 ideal in $\mathcal{T}$?
3. Is each maximal diagonal-0 ideal in $\mathcal{T}$ a maximal $\mathcal{A}$-disjoint ideal?
4. Do the set of elements of $\mathcal{T}$ with 0-diagonal form an ideal (an algebra)?
5. Is there a unique maximal $\mathcal{A}$-disjoint ideal?
6. If S is an integer-ordered basis and $\mathcal{I}$ is a maximal $\mathcal{A}$-disjoint ideal in $\mathcal{T}$, is there a unique maximal triangular extension of $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{S}$?
7. If so, what is its description?
8. How should a triangular tensor product for maximal triangular algebras which are not hyperreducible be defined?
9. Is $\mathcal{T} \otimes \mathcal{T}$ contained in a triangular algebra with core $\mathcal{A} \otimes I$ (something of the nature of $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{S}$, where $\mathcal{S}$ is an integer ordered basis)?
10. Is each strongly-closed maximal triangular algebra hyperreducible?

In connection with Question 10, if we specialize the construction of $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{T}_2$ to the case where $\mathcal{T}_2$ is a finite ordered basis, then $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{T}_2$ appears as the algebra of $n \times n$ matrices with entries from $\mathcal{I}$ below the diagonal and from $\mathcal{T}$ elsewhere. From Lemma 4.3.2, $\mathcal{I}$ has strong closure $\mathcal{T}$, whence the strong closure of $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{T}_2$ is $\mathcal{T} \otimes \mathcal{B}_2$. Note also that if $\mathcal{J}$ is a maximal $\mathcal{A}$-disjoint ideal distinct form $\mathcal{I}$ in $\mathcal{T}$, then $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{T}_2$ and $\mathcal{T}_{\mathcal{J}} \otimes \mathcal{T}_2$ have the same hulls and the same intersection with the commutant of the core. It is trivially the case that irreducible algebras having the same intersection with the commutant of their core coincide; and from Theorem 3.1.1, we have that maximal hyperreducible algebras with the same hulls and the same intersection with the commutant of the core are identical. The non-uniqueness which would follow from the existence of distinct maximal $\mathcal{A}$-disjoint ideals in $\mathcal{T}$ would introduce many pathological features from the point of view of a reduction theory for triangular algebras relative to the core.

With the notation just employed we may ask:

11. Are all maximal triangular extensions of $\mathcal{T} \otimes \mathcal{T}_2$ with core $\mathcal{A} \otimes I$ of the form $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{T}_2$?
12. If $\mathcal{S}$ is an infinite integer-ordered basis, are all triangular extensions of $\mathcal{T} \otimes \mathcal{S}$ with core $\mathcal{A} \otimes I$ extensions of $\mathcal{T}_{\mathcal{I}} \otimes \mathcal{S}$, for some $\mathcal{I}$?

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Added in proof (May 20, 1991): J. L. Orr announces that restriction to a core projection need not be maximal (see *Triangular algebras and nest algebras*, Bull.A.M.S.**23**(1990), 461-467). There are many interesting diagonal-disjoint-ideal results stated as well.