Reflections Relating a von Neumann Algebra and Its Commutant

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1. Introduction

The initial development of the theory of von Neumann algebras, proposed by von Neumann [12] and carried out by him in collaboration with F.J. Murray [9,10,11,13] can be viewed as consisting of two parts, an "algebraic theory" and a "spatial theory." In the algebraic theory, the results refer to the von Neumann algebra $\mathcal{R}$ and make no reference to the commutant; in the spatial theory, the results involve the commutant either explicitly or implicitly. Recognizing this mathematical dichotomy, Kaplansky [7,8] studied the algebraic structure of von Neumann algebras, without reference to their action on a space, isolating and putting in sharp focus many of the natural techniques that are basic to our subject. Of course, Murray and von Neumann had taken the algebraic theory to an advanced stage in their own way [9,10,11,13].

The spatial theory was developed by Murray and von Neumann, in splendid detail, for von Neumann algebras with no central summand of type III. Just two points were left undone for such algebras: the trace-scaling (non-spatial) automorphisms of a II_∞ factor (with a II_1 commutant) [4], and the structure of the II_1 commutant in that case, when the II_∞ factor is matricial [1,3,14]. The basic element in the Murray-von Neumann arguments is the trace. Their key result in this connection is:

Theorem 0. *If $\mathcal{R}$ and $\mathcal{R}'$ are von Neumann algebras of type II_1 acting on a Hilbert space $\mathcal{H}$ and x_0 is a separating and generating unit trace vector for $\mathcal{R}$, then there is a *anti-isomorphism φ of $\mathcal{R}$ onto $\mathcal{R}'$ such that $Ax_0 = \varphi(A)x_0$ for each A in $\mathcal{R}$.*

In effect, Murray and von Neumann construct their "reflection" about the trace vector x_0, for each A in $\mathcal{R}$, there is a unique $\varphi(A)$ in $\mathcal{R}'$ such that

$Ax_0 = \varphi(A)x_0$, and observe that φ is a *anti-isomorphism. The burden of the argument falls on finding $\varphi(A)$ given A. With that theorem and appropriate reductions to the II_1 case, Murray and von Neumann can prove that the *anti-isomorphisms are present in all the cases where there is no central summand of type III. Their experience with specifically constructed factors of type III led them to ask whether such a *anti-isomorphism might not be present for all von Neumann algebras. This question received a spectacularly positive answer by Tomita [16,17] who associates, with a separating and generating vector u for $\mathcal{R}$, a modular structure $\{J, \Delta\}$ (cf. [6; Section 9.2]), where J is a conjugate-linear, involutory isometry of $\mathcal{H}$ onto itself and Δ is a positive, self-adjoint operator (generally, unbounded). The mapping that associates JA^*J with A in $\mathcal{R}$ is the *anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$ associated with u. The mapping σ_t, whose value at A in $\mathcal{R}$ is $\Delta^{it}A\Delta^{-it}$, is a *automorphism of $\mathcal{R}$ for each real t; $t \to \sigma_t$ is a one-parameter group of *automorphisms of $\mathcal{R}$.

While the subalgebra of $\mathcal{R}$ consisting of those elements A such that $Au = A'u$ for some A' in $\mathcal{R}'$ plays an important role in the deep and complicated arguments that establish the results of Tomita, just noted, this subalgebra is by no means all of $\mathcal{R}$. Tomita's work broadens "Murray-von Neumann reflection," taking it away from simple reflection about a trace vector, and deepens it significantly. It replaces it by "Tomita reflection," the mapping $A \to JA^*J$.

There is, however, another direction in which one can take Murray-von Neumann reflection, which retains the elements of simple reflection and a trace. It, too, is a reflection extending Murray-von Neumann reflection. In this context, the centralizer of a state ω on $\mathcal{R}$ is used in an essential way. Let $\mathcal{R}_\omega$ be this centralizer, that is, the set of those A in $\mathcal{R}$ such that $\omega(AT) = \omega(TA)$ for all T in $\mathcal{R}$. With ω the restriction of a vector state ω_x to $\mathcal{R}$ and ω' the restriction of ω_x to $\mathcal{R}'$, we show (Theorem 5) that there is a *anti-isomorphism φ of $\mathcal{R}_\omega E$ onto $\mathcal{R}'_{\omega'}E'$ such that $Ax = \varphi(A)x$ for each A in $\mathcal{R}_\omega E$, where E and E' are the supports of ω and ω'. Again, the burden of the argument falls on finding $\varphi(A)$ in $\mathcal{R}'_{\omega'}E'$ given A in $\mathcal{R}_\omega E$, and the main element of that process is Sakai's ingenious Proposition 1 in his proof [15] of Dixmier's Radon-Nikodým conjecture [2; p. 63].

In the last part of this paper, we show that the reflection we construct in Theorem 5 (extending Murray-von Neumann reflection from the case of a trace vector to that of an arbitrary vector) and the restriction of Tomita reflection ($A \to JA^*J$) to the centralizer are identical by the techniques of modular theory.

In the next section, we establish some results about supports and centralizers of normal states that allow us to draw conclusions about general normal states rather than just those that are faithful.

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2. Centralizers and Supports

In this section, we prove three lemmas relating the support of a normal state to its centralizer. We shall use these lemmas to reduce to the case of a faithful state when proving our main results.

Lemma 1. *The support of a normal state of a von Neumann algebra lies in the center of the centralizer of that state.*

Proof. Let $\mathcal{R}$ be a von Neumann algebra, ω be a normal state of $\mathcal{R}$, E be the support of ω, and A be an element of the centralizer of ω. Since $\omega(I-E)=0$ and $0 \le I-E$, $I-E$ and E are in the centralizer of ω (for $0=\omega((I-E)B)=\omega(B(I-E))$, when $B \in \mathcal{R}$). Hence $EA(I-E)$ is in the centralizer of ω, and

$$0=\omega((I-E)A^*EA(I-E))=\omega(EA(I-E)A^*E).$$

Since E is the support of ω and $0 \le EA(I-E)A^*E$, we have, as a consequence, that $EA(I-E)A^*E=0$. Hence $EA(I-E)=0$. As A^* is also in the centralizer of ω, $EA^*(I-E)=0$ and $(I-E)AE=0$. It follows that

$$A=EAE+(I-E)A(I-E)$$

whence

$$EA=EAE=AE. \qquad \blacksquare$$

Lemma 2. *If ω is a normal state of a von Neumann algebra $\mathcal{R}$, E is the support of ω, and $\mathcal{R}_\omega$ is the centralizer of ω, then $\mathcal{R}_\omega$ is the direct sum of $(I-E)\mathcal{R}(I-E)$ and $\mathcal{R}_\omega E$.*

Proof. From Lemma 1, E is in the center of $\mathcal{R}_\omega$. Thus $\mathcal{R}_\omega$ is (isomorphic to) the direct sum of $\mathcal{R}_\omega(I-E)$ and $\mathcal{R}_\omega E$. We complete the proof by showing that $\mathcal{R}_\omega(I-E)=(I-E)\mathcal{R}(I-E)$. Since $I-E$ is in the center of $\mathcal{R}_\omega$,

$$\mathcal{R}_\omega(I-E)=(I-E)\mathcal{R}_\omega(I-E)\subseteq(I-E)\mathcal{R}(I-E).$$

Suppose S and T are in $\mathcal{R}$. Since $\omega(I-E)=0$, $I-E$ is in the left and right kernels of ω. Thus

$$0=\omega(S(I-E)T(I-E))=\omega((I-E)T(I-E)S).$$

In particular, $(I-E)T(I-E) \in \mathcal{R}_\omega$, whence $(I-E)T(I-E) \in \mathcal{R}_\omega(I-E)$. It follows that

$$(I-E)\mathcal{R}(I-E) \subseteq \mathcal{R}_\omega(I-E).$$

Combining this with the reverse inclusion, noted above, we conclude that $\mathcal{R}_\omega(I-E) = (I-E)\mathcal{R}(I-E)$. ∎

Lemma 3. *If $\mathcal{R}$ is a von Neumann algebra, ω is a normal state of $\mathcal{R}$, and E is the support of ω, then the centralizer of $\omega \mid E\mathcal{R}E$ is $\mathcal{R}_\omega E$.*

Proof. From Lemma 1, E is the center of the centralizer of ω so that

$$E\mathcal{R}_\omega E = \mathcal{R}_\omega E \subseteq \mathcal{R}_\omega, \quad \mathcal{R}_\omega E \subseteq E\mathcal{R}E.$$

Hence $\mathcal{R}_\omega E$ is contained in the centralizer of $\omega \mid E\mathcal{R}E$ $(= \omega_0)$.

Suppose T in $\mathcal{R}$ is such that ETE is in the centralizer of ω_0. With S in $\mathcal{R}$, we have that

$$\begin{aligned}\omega(SETE) &= \omega((I-E)SETE) + \omega(ESETE)\\ &= \omega(ESETE)\\ &= \omega(ETESE)\\ &= \omega(ETESE) + \omega(ETES(I-E))\\ &= \omega(ETES).\end{aligned}$$

Thus $ETE \in \mathcal{R}_\omega$ and $ETE \in \mathcal{R}_\omega E$. It follows that the centralizer of ω_0 is contained in $\mathcal{R}_\omega E$. From these inclusions, we have that the centralizer of ω_0 is $\mathcal{R}_\omega E$. ∎

3. Main Results

The theorem that follows details the construction of the reflection of an operator in the centralizer of a vector state. The argument makes crucial use of Sakai's proposition [15].

Theorem 4. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and x be a unit vector in $\mathcal{H}$. Let E be the support of $\omega_x \mid \mathcal{R}$ $(= \omega)$ and E' be the support of $\omega_x \mid \mathcal{R}'$. Then A is in the centralizer of ω if and only if $AE = EA$ and there is an A' in $\mathcal{R}'$ such that $E'A' = A'E'$ and $EAx = E'A'x$, $EA^*x = E'A'^*x$.*

Proof. Suppose, first, that for a given A in $\mathcal{R}$ commuting with E, there is an A' as described. Then,

$$\begin{aligned}\omega(AB) &= \langle ABx, x\rangle = \langle Bx, A^*x\rangle = \langle Bx, A^*Ex\rangle \\ &= \langle Bx, EA^*x\rangle = \langle Bx, E'A'^*x\rangle = \langle BA'x, x\rangle \\ &= \langle BE'A'x, x\rangle = \langle BEAx, x\rangle = \langle BAx, x\rangle \\ &= \omega(BA) \qquad (B \in \mathcal{R}).\end{aligned}$$

Thus A is in the centralizer of ω.

Suppose, now, that A is in the centralizer of ω. From Lemma 1, $AE = EA$. We begin by studying the case in which x is a separating and generating vector for $\mathcal{R}$ (and, hence, for $\mathcal{R}'$ as well). In this case, the ranges $[\mathcal{R}x]$ and $[\mathcal{R}'x]$ of E' and E are $\mathcal{H}$, so that $E = E' = I$. We define an operator R $(= R_{Ax})$ with domain $\mathcal{R}x$ by

$$RBx = BAx \qquad (B \in \mathcal{R}).$$

(See [6; p. 632].) Note that, with H self-adjoint in $\mathcal{R}$,

$$\omega_x(HAA^*) = \omega_x(AA^*H) = \overline{\omega_x(HAA^*)},$$

since AA^* is in the centralizer of ω. Thus $T \to \omega_x(TAA^*)$ is a hermitian functional on $\mathcal{R}$. By Sakai's proposition [15] (cf. [6; Lemma 7.3.4]),

$$\begin{aligned}\|RBx\|^2 &= \|BAx\|^2 = \langle A^*B^*BAx, x\rangle \\ &= \langle B^*BAA^*x, x\rangle = \omega_x(B^*BAA^*) \\ &= |\omega_x(AA^*B^*B)| \le \|AA^*\|\omega_x(B^*B) \\ &= \|A\|^2\|Bx\|^2 \qquad (B \in \mathcal{R}).\end{aligned}$$

Thus R extends uniquely to a bounded operator A' on $[\mathcal{R}x]$ (which is $\mathcal{H}$, under the present assumption). From [6; Lemma 9.2.28], $A' \in \mathcal{R}'$ (though, this is immediate in the bounded case). Moreover, $A'x = Rx = Ax$. At the same time,

$$\begin{aligned}\langle A'^*x, Bx\rangle &= \langle x, A'Bx\rangle = \langle x, RBx\rangle \\ &= \langle x, BAx\rangle = \langle A^*B^*x, x\rangle \\ &= \omega_x(A^*B^*) = \omega_x(B^*A^*) \\ &= \langle B^*A^*x, x\rangle = \langle A^*x, Bx\rangle \qquad (B \in \mathcal{R}).\end{aligned}$$

Thus $A'x = Ax$ and $A'^*x = A^*x$, under the present assumption.

We reduce the general situation to the case just studied. Note for this that $E\mathcal{R}E$ acting on $[\mathcal{R}'x]$ has $\mathcal{R}'E$ as commutant, and $E'E(\mathcal{R}'E)E'E$

$(= E'\mathcal{R}'E'E)$, acting on $[\mathcal{R}'x]\cap[\mathcal{R}x]$ $(= \mathcal{H}_0)$, has $(E\mathcal{R}E)E'E$ $(= E\mathcal{R}EE')$ as its commutant. Moreover, $E'\mathcal{R}'E'E$ and $E\mathcal{R}EE'$ acting on $\mathcal{H}_0$ have x as a joint generating and separating vector. Let ω_0 be $\omega_x \mid E\mathcal{R}EE'$. Then

$$\begin{aligned}\omega_0(EBEE'EAEE') &= \omega_0(EBEAEE') = \omega(EBEA)\\ &= \omega(AEBE) = \omega(AEB)\\ &= \omega_0(EAEE'EBEE') \qquad (B \in \mathcal{R}).\end{aligned}$$

Thus $EAEE'$ is in the centralizer of ω_0. From the case where x is a joint generating and separating vector, there is an element $E'A'E'E$ in $E'\mathcal{R}'E'E$ (with A' in $E'\mathcal{R}'E' \subseteq \mathcal{R}'$) such that

$$EAx = EAEE'x = E'A'E'Ex = E'A'x,$$

$$EA^*x = EA^*EE'x = E'A'^*E'Ex = E'A'^*x. \qquad \blacksquare$$

Theorem 5. *In the notation of Theorem 4, let ω' be $\omega_x \mid \mathcal{R}'$, $\mathcal{R}_\omega$ be the centralizer of ω, and $\mathcal{R}'_{\omega'}$, be the centralizer of ω'. With A in $\mathcal{R}_\omega E$, there is a unique A' in $\mathcal{R}'_{\omega'}E'$ such that $Ax = A'x$ and $A^*x = A'^*x$; the mapping $A \to A'$ is a *anti-isomorphism of $\mathcal{R}_\omega E$ onto $\mathcal{R}'_{\omega'}E'$.*

Proof. From Lemma 1, E and E' are in the centers of $\mathcal{R}_\omega$ and $\mathcal{R}'_{\omega'}$, respectively. Thus $\mathcal{R}_\omega E$ and $\mathcal{R}'_{\omega'}E'$ are von Neumann algebras. With A in $\mathcal{R}_\omega E$, A is in $\mathcal{R}_\omega$. From Theorem 4, there is an A' in $\mathcal{R}'$ such that

$$\begin{aligned}Ax &= EAx = E'A'x = E'A'E'x,\\ A^*x &= EA^*Ex = E'A'^*x = E'A'^*E'x.\end{aligned}$$

If we use $E'A'E'$ in place of A', we may assume that

$$Ax = A'x,\ A^*x = A'^*x,\ E'A' = A'E'.$$

Now, applying Theorem 4, again, we conclude that $A' \in \mathcal{R}'_{\omega'}$, from which $A' \in \mathcal{R}'_{\omega'}E'$.

If $B' \in \mathcal{R}'_{\omega'}E'$ and $Ax = B'x$, then $(B' - A')x = 0$. It follows that $(B' - A')\mathcal{R}x = 0$, and $B' - A' = (B' - A')E' = 0$. Thus A', as described, is unique. With A and B in $\mathcal{R}_\omega E$,

$$ABx = AB'x = B'Ax = B'A'x,$$

whence $(AB)' = B'A'$. The linearity of $A \to A'$ is evident. Moreover, $(A^*)' = (A')^*$ since $A^*x = A'^*x$. With B' in $\mathcal{R}'_{\omega'}E'$, there is a B in

$\mathcal{R}_\omega E$ such that $Bx = B'x$, $B^*x = B'^*x$, by symmetry. Thus the mapping $A \to A'$ is a *anti-isomorphism of $\mathcal{R}_\omega E$ onto $\mathcal{R}'_{\omega'}E'$. ∎

4. Relating Reflections

The relation of the reflection between centralizers, developed in Section 3, to the other reflections is established with the aid of the following proposition. Its proof requires the results and techniques of modular theory.

Proposition 6. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, u be a separating and generating unit vector for $\mathcal{R}$, and ω be $\omega_u\,|\,\mathcal{R}$. With $\{J, \Delta\}$ the modular structure for $\{\mathcal{R}, u\}$, the following are equivalent:*

(i) $Au = \Delta^{1/2}Au$;

(ii) $Au = JA^*Ju$;

(iii) *A is the closure of* $\Delta^{1/2}A\Delta^{-1/2}$;

(iv) $A\Delta \subseteq \Delta A$;

(v) *A is in the centralizer of ω.*

Proof. (i) $\leftrightarrow$ (ii) Suppose $Au = \Delta^{1/2}Au$. From [6; Theorem 9.2.9], $Ju = u$ and $J^2 = I$. Thus

$$JAu = J\Delta^{1/2}Au = SAu = A^*u,$$

and

$$Au = JJAu = JA^*u = JA^*Ju.$$

Assuming that $Au = JA^*Ju$, we have that

$$Au = JA^*u = JSAu = JJ\Delta^{1/2}Au = \Delta^{1/2}Au.$$

(i) $\to$ (iii) From [6; Theorem 9.2.9], $\Delta u = u$. Thus $u = \Delta^{1/2}u = \Delta^{-1/2}u$ from [6; Remark 5.6.32], and

$$Au = \Delta^{1/2}Au = \Delta^{1/2}A\Delta^{-1/2}u.$$

Recall that $J\Delta^{-1/2} = F$ (see the discussion following [6; Remark 9.2.2]) and $J\mathcal{R}'J = \mathcal{R}$ [6; Theorem 9.2.9]. Thus with B' in $\mathcal{R}'$, we have

$$\begin{aligned}\Delta^{1/2}A\Delta^{-1/2}B'u &= \Delta^{1/2}AJFB'u = \Delta^{1/2}AJB'^*u\\ &= JSAJB'^*Ju = J(JB'^*J)^*A^*u\\ &= J(JB'J)A^*u = B'JA^*u\\ &= B'JA^*Ju = B'Au = AB'u.\end{aligned}$$

Thus $\mathcal{R}'u \subseteq \mathcal{D}(\Delta^{1/2}A\Delta^{-1/2})$ and

$$\Delta^{1/2}A\Delta^{-1/2}\,|\,\mathcal{R}'u = A\,|\,\mathcal{R}'u.$$

Since $\mathcal{R}'u$ is a core for $\Delta^{-1/2}$, if $x \in \mathcal{D}(\Delta^{1/2}A\Delta^{-1/2})$, there is a sequence $\{A'_n\}$ in $\mathcal{R}'$ such that $A'_nu \to x$ and $\Delta^{-1/2}A'_nu \to \Delta^{-1/2}x$. But then $A\Delta^{-1/2}A'_nu \to A\Delta^{-1/2}x$ and $\Delta^{1/2}A\Delta^{-1/2}A'_nu = AA'_nu \to Ax$. Since $\Delta^{1/2}$ is closed,

$$\Delta^{1/2}A\Delta^{-1/2}A'_nu \to \Delta^{1/2}A\Delta^{-1/2}x.$$

Thus $\Delta^{1/2}A\Delta^{-1/2}x = Ax$ and $\Delta^{1/2}A\Delta^{-1/2} = A\,|\,\mathcal{D}(\Delta^{1/2}A\Delta^{-1/2})$. It follows that A is the closure of $\Delta^{1/2}A\Delta^{-1/2}$.

(iii) $\to$ (iv) By assumption, $\Delta^{1/2}A\Delta^{-1/2} \subseteq A$. From [6; 5.6.(13)], $\Delta^{-1/2}\Delta^{1/2}A\Delta^{-1/2} \subseteq \Delta^{-1/2}A$. Thus

$$A\Delta^{-1/2}\,|\,\mathcal{D}(\Delta^{1/2}A\Delta^{-1/2}) \subseteq \Delta^{-1/2}A.$$

We show, next, that $\mathcal{R}'u \subseteq \mathcal{D}(\Delta^{1/2}A\Delta^{-1/2})$. If $B' \in \mathcal{R}'$,

$$\Delta^{-1/2}B'u = JFB'u = JB'^*u = JB'^*Ju.$$

Now $JB'^*J \in \mathcal{R}$ from [6; Theorem 9.2.9], whence

$$A\Delta^{-1/2}B'u = AJB'^*Ju \in \mathcal{D}(\Delta^{1/2}).$$

Thus $B'u \in \mathcal{D}(\Delta^{1/2}A\Delta^{-1/2})$ and $\mathcal{R}'u \subseteq \mathcal{D}(\Delta^{1/2}A\Delta^{-1/2})$. It follows that $A\Delta^{-1/2}\,|\,\mathcal{R}'u \subseteq \Delta^{-1/2}A$. With x in $\mathcal{D}(\Delta^{-1/2})$, we can choose B'_n in $\mathcal{R}'$ such that $B'_nu \to x$ and $\Delta^{-1/2}B'_nu \to \Delta^{-1/2}x$. Since A is bounded, $A\Delta^{-1/2}B'_nu \to A\Delta^{-1/2}x$. But $A\Delta^{-1/2}B'_nu = \Delta^{-1/2}AB'_nu$ and $\Delta^{-1/2}A$ is closed. (With B bounded and T closed, BT need not be closed, but TB is closed [6; Example 5.6.33].) Thus $x \in \mathcal{D}(\Delta^{-1/2}A)$, $\Delta^{-1/2}Ax = A\Delta^{-1/2}x$, and $A\Delta^{-1/2} \subseteq \Delta^{-1/2}A$. From [6; Lemma 5.6.17], A commutes with the spectral resolution of $\Delta^{-1/2}$. Hence $A \in \mathcal{A}'$, where $\mathcal{A}$ is the abelian von Neumann algebra generated by $\Delta^{-1/2}$ (cf. [6; Theorem 5.6.18]). We have, from [6; Theorem 5.6.26], that $\Delta\ \eta\ \mathcal{A}$. Thus $A\Delta \subseteq \Delta A$.

(iv) $\to$ (v) If $A\Delta \subseteq \Delta A$, then A commutes with the abelian von Neumann algebra generated by Δ (as at the end of the preceding argument). That algebra contains Δ^{it} for each real t. Thus, $\Delta^{it}A\Delta^{-it} = A$. From [6; Theorem 9.2.13, Proposition 9.2.14], A is in the centralizer of ω.

(v) $\to$ (i) If A is in the centralizer of ω, then A commutes with Δ^{it} for each real t from [6; Proposition 9.2.14]. Using the formula from [6; Theorem 5.6.36],

$$\langle \hat{f}(H)x, y\rangle = \int_R f(t)\langle e^{itH}x, y\rangle dt,$$

with $\log\Delta$ in place of H and Ax in place of x, we have that

$$\begin{aligned}\langle \hat{f}(\log\Delta)Ax, y\rangle &= \int_R f(t)\langle \Delta^{it}Ax, y\rangle dt \\ &= \int_R f(t)\langle \Delta^{it}x, A^*y\rangle dt \\ &= \langle \hat{f}(\log\Delta)x, A^*y\rangle;\end{aligned}$$

whence $\hat{f}(\log\Delta)A = A\hat{f}(\log\Delta)$ for each f in $L_1(\mathbb{R})$. Pursuing this reasoning with appropriate choices for f, one can conclude that A commutes with the (abelian) von Neumann algebra generated by $\log\Delta$, and since Δ ($= \exp(\log\Delta)$) and $\Delta^{1/2}$ are affiliated with this algebra, that $A\Delta^{1/2} \subseteq \Delta^{1/2}A$. But careful use of Fourier transform arguments and formulae are needed for a complete proof that $A\Delta^{1/2} \subseteq \Delta^{1/2}A$. It is, perhaps, more convincing to employ a lemma from [5], which assures us that $\{\Delta^{it}: t \in \mathbb{R}\}$ and $(\Delta + I)^{-1}$ generate the same von Neumann algebra. Of course, Δ and $\Delta^{1/2}$ are affiliated with this algebra, so that $A\Delta^{1/2} \subseteq \Delta^{1/2}A$. Thus $Au = A\Delta^{1/2}u = \Delta^{1/2}Au$. ∎

The theorem that follows describes the relations among the various reflections.

Theorem 7. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, u be a generating and separating unit vector for $\mathcal{R}$, ω be $\omega_u \,|\, \mathcal{R}$, ω' be $\omega_u \,|\, \mathcal{R}'$, $\mathcal{R}_\omega$ be the centralizer of ω, and $\mathcal{R}'_{\omega'}$ be the centralizer of ω'. Let φ be the reflection of $\mathcal{R}_\omega$ onto $\mathcal{R}'_{\omega'}$ (about u) described in Theorem* 5.

(i) *If u is a trace vector for $\mathcal{R}$, then u is a trace vector for $\mathcal{R}'$, and $Au = \varphi(A)u$ for each A in $\mathcal{R}$. The mapping φ is Murray-von Neumann reflection in this case.*

(ii) *$A \in \mathcal{R}_\omega$ if and only if $Au = JA^*Ju$; and when $A \in \mathcal{R}_\omega$, $\varphi(A) = JA^*J$.*

Proof. (i) That u is a trace vector for $\mathcal{R}'$ is a consequence of [6; Lemma 7.2.14]. It follows that $\mathcal{R}_\omega = \mathcal{R}$ and $\mathcal{R}'_{\omega'} = \mathcal{R}'$. Since u is generating and separating, the supports of ω and ω' are both I. From Theorem 5, $Au = \varphi(A)u$ for each A in $\mathcal{R}$. Thus φ is Murray-von Neumann reflection about the trace vector u.

(ii) The first assertion of this part follows at once from the equivalence of (ii) and (v) of Proposition 6. With A in $\mathcal{R}_\omega$, $Au = \varphi(A)u$ from Theorem 5. But $Au = JA^*Ju$ so that $(\varphi(A) - JA^*J)u = 0$. Since $\varphi(A)$ and JA^*J are in $\mathcal{R}'$ and u is separating for $\mathcal{R}'$, $\varphi(A) = JA^*J$. ∎

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