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Operator Algebras – An Overview

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§**1. Introduction.** The subject popularly known as "operator algebras" (a shortened form of "algebras of Hilbert space operators") made its first appearance in a paper [**31**] published in the 1929–1930 Mathematische Annalen. In that article, a construct which von Neumann calls a "ring of operators" is defined, and a theorem, the Double Commutant Theorem, that is, arguably, the most basic theorem of the subject is proved. Somewhat later (the late 1940s), "rings of operators" was used interchangeably with "W^*-algebras," the 'W^*' referring to "weak-operator closed" and '*' to the central role of the adjoint operation, denoted by '*,' in the definition of this construct. In his classic text [**D**], the first in the subject, Dixmier referred to these algebras as 'von Neumann algebras' following a suggestion made to him by J. Dieudonné.

Let $\mathscr{H}$ be a Hilbert space over the complex numbers **C** and let $< x, y >$ denote the (positive-definite) inner product of two vectors x and y in $\mathscr{H}$. The "length" or "norm" of a vector x is $< x, x >^{\frac{1}{2}}$ (denoted by $\|x\|$). The mapping $x \rightarrow \|x\|$ provides $\mathscr{H}$ with a norm. By definition, $\mathscr{H}$ is *complete* relative to the metric that assigns $\|x - y\|$ as the distance between the vectors x and y.

If $\mathscr{K}$ is another such Hilbert space and T is a linear transformation of $\mathscr{H}$ into $\mathscr{K}$, then T is continuous relative to the metric topologies, just described, if and only if

$$\sup\{\|Tx\| : x \in \mathscr{H}, \|x\| \leq 1\} < \infty.$$

This supremum, denoted by $\|T\|$, is referred to as the "norm" or "bound" of T. We say that T is *bounded* in this case. (The terms 'bounded' and 'continuous' are used interchangeably for linear transformations.)

The family $\mathscr{B}(\mathscr{H}, \mathscr{K})$ of all bounded linear transformations of $\mathscr{H}$ into $\mathscr{K}$ is itself a linear space, where $(aT + S)(x)$ is defined as $a(Tx) + Sx$. When $\mathscr{K} = \mathscr{H}$, we write '$\mathscr{B}(\mathscr{H})$' in place of '$B(\mathscr{H}, \mathscr{H})$.' In this case, $\mathscr{B}(\mathscr{H})$ is

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an algebra, where $(TS)x$ is defined as $T(Sx)$. This algebra has the operator I (the identity transform, $Ix = x$ for all x in $\mathscr{H}$) as a unit element. The mapping $T \to \|T\|$ endows $\mathscr{B}(\mathscr{H},\mathscr{K})$ with a norm relative to which it is a Banach apace; $\mathscr{B}(\mathscr{H})$ is a Banach algebra ($\|TS\| \le \|T\|\,\|S\|$ and $\|I\| = 1$). The metric on $\mathscr{B}(\mathscr{H},\mathscr{K})$ that assigns $\|T - S\|$ as the distance between T and S gives rise to the *norm* or *uniform* topology on $\mathscr{B}(\mathscr{H},\mathscr{K})$.

The linear transformations of $\mathscr{H}$ into $\mathbf{C}$ (a one-dimensional Hilbert space) are called *(linear) functionals.*

The bounded linear functionals on $\mathscr{H}$ (elements of $\mathscr{B}(\mathscr{H},\mathbf{C})$) are described by the celebrated Riesz Representation Theorem. If φ is in $\mathscr{B}(\mathscr{H},\mathbf{C})$, there is a unique vector x in $\mathscr{H}$ such that $\varphi(y) = \langle y, x\rangle$ for each y in $\mathscr{H}$. The mapping $\varphi \to x$ is a conjugate-linear (that is, $a\varphi$ corresponds to $\bar{a}x$), isometric (that is, $\|\varphi\| = \|x\|$), isomorphism of $\mathscr{B}(\mathscr{H},\mathbf{C})$ onto $\mathscr{H}$. If T is in $\mathscr{B}(\mathscr{H},\mathscr{K})$ and z is in $\mathscr{K}$, then $x \to \langle Tx, z\rangle$ is in $\mathscr{B}(\mathscr{H},\mathbf{C})$. It follows that there is a unique vector T^*z in $\mathscr{H}$ such that, for all x in $\mathscr{H}$, $\langle Tx, z\rangle = \langle x, T^*z\rangle$. The mapping T^* is in $\mathscr{B}(\mathscr{K},\mathscr{H})$. The linear transformation T^* of $\mathscr{K}$ into $\mathscr{H}$ is called *the adjoint* of T. The equalities, $(a\,T + S)^* = \bar{a}\,T^* + S^*, (T^*)^* = T, \|T\| = \|T^*\|$, $\|T^*T\| = \|T^*\|\,\|T\|$, are easily proved. When $\mathscr{H} = \mathscr{K}$, we have that $(TS)^* = S^*T^*$. In this same case, we say that T is *selfadjoint* when $T = T^*$.

With $\mathscr{S}$ a subset of $\mathscr{B}(\mathscr{H})$, we define the *weak-operator closure* $\mathscr{S}^-$ of $\mathscr{S}$ to be the set of those T in $\mathscr{B}(\mathscr{H})$ such that, given a positive ε and vectors $x_1, \cdots, x_n; y_1, \cdots, y_n$ in the Hilbert space $\mathscr{H}$, there is an S in $\mathscr{S}$ for which $|\langle (T - S)x_j, y_j\rangle| < \varepsilon$, $j = 1, \cdots, n$. This closure operation defines a topology on $\mathscr{B}(\mathscr{H})$ called the *weak-operator topology.* A subalgebra $\mathscr{R}$ of $\mathscr{B}(\mathscr{H})$ that is weak-operator closed, contains I, and contains T^* when it contains T, is called a *von Neumann algebra.*

§**2. Motivation.** The reducibility and irreducibility results of I. Schur and the Peter-Weyl theory that figured so prominently during the period of von Neumann's university training led him to consider the analogous questions for families of operators (linear transformations) on a Hilbert space. This, in turn, led him to his Double Commutant Theorem, the heart of his 1929–1930 Mathematische Annalen paper, where "rings of operators" are first introduced.

In a sense, then, von Neumann algebras grew out of the early period of group representations. It is popular and natural, in view of von Neumann's involvement with the more rigorous mathematical formulation of the basics of quantum mechanics, to ascribe an important place to the "new physics" in motivating the introduction of "rings of operators," but that interpretation does not stand up to scrutiny. It seems equally clear that von Neumann's commitment to pursuing the study of his "rings of operators" was not, at first, large; he did not return to it for five years.

In 1935, F. J. Murray, a gifted young mathematician, who had just received his Ph.D. at Columbia University with a thesis on linear operators and spectral theory, arrived in Princeton as a National Research Fellow intent on studying under von Neumann. In the spirit of a postdoctoral research project, von Neumann proposed to Murray that he examine "rings of operators" from the point of view of the Wedderburn structure theory for matrix algebras. In particular, Murray was to concentrate on those von Neumann algebras whose centers consist of scalar multiples of I, the *factors*. Their most primitive guess, at the outset, was that a factor is isomorphic to $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$; that would be a worthy result and the end of the project. Before the first year of the investigation had passed, a rich and intriguing world of mathematical phenomena had opened before Murray and von Neumann.

At an early stage, von Neumann joined Murray in the investigation on a full-time basis in what became a seven-year project, one of this country's most successful mathematical collaborations. As their research proceeded, other layers of motivation developed. From the examples of factors they constructed, the very clear applicability of the subject to the study of transformations on measure spaces (ergodic theory) and to representations of groups (through a generalized group algebra) was assured. Factors of type II_1, a class of factors they discovered, have associated algebras of unbounded operators that permit all the formal algebraic manipulations being used by the founders of quantum mechanics in their mathematical formulation. Von Neumann felt that the factors of type II_1 might provide the natural framework for quantum mechanics. That hope was not to be realized, but the factors of type II_1 are basic to the analysis of the class, factors of type III, that are essential to a mathematical presentation of quantum statistical mechanics and quantum field theory.

§**3. Functions and operators.** At the same time as the Murray-von Neumann collaboration began, M. H. Stone recognized the possibility of extracting topological spaces from algebraic structures **[42]** and representing the algebraic structures by functions on the spaces. The earlier extension of the spectral theorem to include unbounded selfadjoint operators and the resulting function calculus of such operators developed by both Stone and von Neumann provided a major impetus to Stone's ideas and to his analysis of $C(X)$, the algebra, under pointwise operations, of continuous complex-valued functions on a locally compact Hausdorff space X. Stone proved **[43]** that an algebra with a partial ordering, satisfying some natural conditions, is isomorphic to the algebra of real-valued functions in $C(X)$ and applied this to yield such an isomorphism of a commutative algebra of bounded selfadjoint operators.

Combining Stone's ideas with his work on Banach algebras and his study of the first three articles in the series of Murray and von Neumann, I. M.

Gelfand undertook the characterization of those norm-closed subalgebras of $\mathcal{B}(\mathcal{H})$ that contain T^* when they contain T. He and M. Neumark **[15]** proved that the Banach algebras in question are those with an involution $(A \to A^*)$ having the properties of the adjoint operation on $\mathcal{B}(\mathcal{H})$. The Banach algebras of Gelfand and Neumark have a unit and the subalgebras of $\mathcal{B}(\mathcal{H})$ contain I. The norm-closed subalgebras of $\mathcal{B}(\mathcal{H})$ stable under the adjoint operation have come to be known as C^*-algebras. They occupy a key position in the subject. Each von Neumann algebra is, in particular, a C^*-algebra. The Gelfand-Neumark article **[15]** recast Stone's result **[43]** in Banach algebra form and used it as a key lemma.

THEOREM. *Let $\mathcal{A}$ be a commutative Banach algebra with a unit element I and an involution $A \to A^*$ satisfying $(aA+B)^* = \bar{a}A^* + B^*, (AB)^* = B^*A^*, A^{**} = A, \|A^*A\| = \|A^*\|\,\|A\|$. Then there is a compact Hausdorff space X and an algebraic isomorphism φ of $\mathcal{A}$ onto $C(X)$ such that*

$$\varphi(A^*) = \overline{\varphi(A)}, \quad \|A\| = \|\varphi(A)\| (= \sup\{|\varphi(A)(x)| : x \in X\}) \quad (A \in \mathcal{A}).$$

If 'commutative' is omitted from the hypothesis of the theorem, there is a C^*-algebra $\mathcal{A}_0$ and an isometric isomorphism ψ of $\mathcal{A}$ onto $\mathcal{A}_0$ such that $\psi(A^*) = \psi(A)^*$. We refer to ψ as 'a * isomorphism.' It follows, in particular, that each $C(X)$ is * isomorphic to a commutative (abelian) C^*-algebra (with complex conjugation of functions as the involution on $C(X)$).

The identification of the family of commutative C^*-algebras with the family of function algebras $C(X)$ underlies the interpretation of the general study of C^*-algebras as 'noncommutative (real) analysis,' the point of view that has dominated the subject since the late 1940s. Analysis is very largely a study of $C(X)$, its associated structures, and operations (differentiation and integration) on these. If $C(X)$ is replaced, in such considerations, by a "noncommutative" $C(X)$, that is, a noncommutative C^*-algebra, then the definitions of the related structures and operations are usually forthcoming. The results of classical analysis can then be reformulated as questions and conjectures involving a C^*-algebra. The answers to these questions and the proofs of the conjectures (or counterexamples) tend not to be so forthcoming.

The view of the study of C^*-algebras as noncommutative (real) analysis guides the research and provides a large template for the motivation of the subject. When noncommutative analysis is the appropriate analysis, as in quantum theory or classical (commutative) analysis situations with nonabelian groups acting, operator algebras provide the mathematical framework.

As noted, each von Neumann algebra is a C^*-algebra. An abelian von Neumann algebra is isomorphic to some $C(X)$. In this case, the X has very special properties; each open set in X has a closure that is open. The space X is said to be *extremely disconnected*. The converse is not true; there are extremely disconnected spaces X for which $C(X)$ is not isomorphic to a von Neumann algebra. There are subtle "measure-theoretic" requirements that X must fulfill for there to be such an isomorphism. The function representation of an abelian von Neumann algebra can be formulated cogently in terms of measure theory.

Let S be a set and μ a (σ-finite) measure on S. The linear space $L_2(S, \mu)$ of functions (absolutely) square integrable with respect to μ, provided with the inner product, $\langle f, g\rangle = \int f\bar{g}d\mu$, is a Hilbert space $\mathscr{H}$. With f an essentially bounded measurable function on S and g in $\mathscr{H}$, we define $M_f(g)$ to be the product $f \cdot g$. Then $fg \in \mathscr{H}$ and M_f is a bounded linear operator on $\mathscr{H}$. The family $\mathscr{A}$ of these *multiplication operators* is an abelian von Neumann algebra on $\mathscr{H}$, with the property that each T commuting with all the multiplication operators is a multiplication operator. We say that $\mathscr{A}$ is *maximal abelian.*

If we specialize the construction, taking for S and μ one of $[0, 1]$ with Lebesgue measure, a finite or countably infinite set of points each with positive measure, or the union of two such measure spaces, we arrive at what seem to be some special abelian von Neumann algebras. Surprisingly, each abelian von Neumann algebra on a separable Hilbert space is isomorphic to one of the few just constructed. If the von Neumann algebra is maximal abelian, then there is even an isomorphism (unitary transformation) of the separable Hilbert space onto $L_2(S, \mu)$ that transforms the algebra onto the multiplication algebra. The family of essentially bounded measurable functions on S is an algebra, the *multiplication algebra* of (S, μ), under the pointwise operations of addition and multiplication of functions. The mapping that assigns M_f to f is an isomorphism of this algebra onto the maximal abelian algebra of multiplication operators.

From this discussion, we see that the theory of abelian von Neumann algebras is a version of measure theory. As with general C^*-algebras and continuous function theory, we recognize that the theory of the general von Neumann algebra is *noncommutative measure theory.* This point of view leads us to identify the selfadjoint idempotent operators ($E = E^*$ and $E = E^2$), the projections, in a von Neumann algebra $\mathscr{R}$ with the characteristic functions of measurable sets in a measure space. In effect, the projections are the "measurable sets" in our "noncommutative measure space." It is a basic technical fact that there are many projections in a von Neumann algebra. In fact, the linear span of the set of projections in a von Neumann algebra is norm dense in that algebra.

§**4. Factors.** In studying the most noncommutative von Neumann algebras, the factors, an examination of the "measurable sets," the projections provide us with a first glimpse of the underlying structure. By analogy with the "atoms" of a measure space, sets of positive measure with no proper subsets of smaller positive measure, we define a *minimal projection* in a von Neumann algebra $\mathscr{R}$ to be a nonzero projection E in $\mathscr{R}$ such that if F is a nonzero projection in $\mathscr{R}$ whose range is contained in the range of E (equivalently, $FE = F$), we say that F is a subprojection of E in this case and write ($F \leq E$), then $F = E$. We study our factors for minimal projections.

THEOREM. *If the factor $\mathscr{M}$ contains a minimal projection, then $\mathscr{M}$ is isomorphic to $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.*

Projections E and F such that $EF = 0$ are called (*mutually*) *orthogonal*. The dimension of $\mathscr{H}$ in the preceding theorem is the cardinality of a maximal orthogonal family of minimal projections in $\mathscr{M}$. This theorem tells us most of what we can expect to learn about factors with a minimal projection concerning isomorphisms (about the factors themselves, not the von Neumann subalgebras, of course). The problem that Murray and von Neumann faced at this (early) point in their investigation was whether or not all factors had minimal projections. Certainly, $\mathscr{B}(\mathscr{H})$ has minimal projections, the projections with one-dimensional range, but does the fact of being central (that is, having center consisting of scalar's multiples of I) force the factor to have a minimal projection? They solved the problem in the negative during their first year of work by constructing an important class of examples through the use of ergodic theory techniques.

With hindsight, we can see that there should be factors without minimal projections, just as there are measure spaces without atoms. But, of course, had this not been the case, the concept of a noncommutative measure space would have little validity.

Murray and von Neumann conceived the idea of comparing the "sizes" of projections in a von Neumann algebra. Two projections E and F in a von Neumann algebra are said to be *equivalent* (*modulo* $\mathscr{R}$) when some A in $\mathscr{R}$ maps the range of E isometrically onto the range of F. In this case, $AE(= V \in \mathscr{R})$ has the same properties and $V^*V = E$, $VV^* = F$. Murray and von Neumann indicate this equivalence by writing $E \sim F$. When $E \sim F_0$ and $F_0 \leq F$, they write $E \precsim F$(and $E \prec F$ when E is, in addition, not equivalent to F).

Each T in $\mathscr{B}(\mathscr{H})$ has a unique decomposition as VH, the *polar decomposition* of T, where H is a positive selfadjoint operator (that is, $\langle Hx, x\rangle \geq 0$ for each x in $\mathscr{H}$) with square equal to T^*T and V maps the closure of the range of H (we write '$r(H)$' for this closure) isometrically onto $r(T)$ and maps the orthogonal complement of $r(H)$ to 0. We say that V is a *partial isometry* with *initial space* $r(H)$ and *final space* $r(T)$. If $R(H)$ is the projection with range $r(H)$ (we call $R(H)$ the *range projection* of H), then $V^*V = R(H)$ and $VV^* = R(T)$. We note that the components V and H of this polar decomposition lie in $\mathscr{R}$ when T does, from which we conclude that $R(H) \sim R(T)$, as a consequence of the uniqueness of those components and the main theorem of von Neumann's original paper **[31]**.

THEOREM (DOUBLE COMMUTANT). *If $\mathscr{R}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and $\mathscr{R}'$ is the family of bounded operators T that* *commute with all the operators in $\mathscr{R}$, then $\mathscr{R}'$ is a von Neumann algebra and $(\mathscr{R}')' = \mathscr{R}$.*

The notation $\mathscr{F}'$ for the set of bounded operators T commuting with all the operators in the family $\mathscr{F}$ of bounded operators will be used in the sequel; $\mathscr{F}'$ is called the *commutant* of $\mathscr{F}$. When $\mathscr{F}$ is selfadjoint (that is, $\mathscr{F} = \mathscr{F}^*$), $\mathscr{F}'$ is a von Neumann algebra.

With T in $\mathscr{R}$ and VH the polar decomposition of T, it is easy to see that $R(H) = R(T^*)$. It follows that $R(T) \sim R(T^*)$, a very useful fact for dealing with the relations $\sim$ and $\precsim$. Using this fact, the range projections of EAF and FA^*E are equivalent for each A in a factor $\mathscr{M}$, where E and F are projections in $\mathscr{M}$. Thus nonzero subprojections of E and F are equivalent unless EAF is 0 for each A in $\mathscr{M}$. The fact that $\mathscr{M}$ has center the scalars allows us to conclude that EAF is 0 for each A in $\mathscr{M}$ if and only if one of E or F is 0. It follows that each pair of nonzero projections in a factor $\mathscr{M}$ have equivalent nonzero subprojections. A measure-theoretic-like argument now shows that either $E \precsim F$ or $F \prec E$. An argument patterned on the Cantor-Schröeder-Bernstein argument from the theory of cardinals shows that $E \sim F$ when $E \precsim F$ and $F \precsim E$ for projections E and F in a general von Neumann algebra. It follows that $\precsim$ is a partial ordering of the equivalence classes of projections in a von Neumann algebra and a total ordering of these classes in a factor.

Murray and von Neumann define infinite and finite projections in this framework modeled on the set-theoretic approach. The projection E in $\mathscr{R}$ is *infinite* (relative to $\mathscr{R}$) when $E \sim E_0 < E$ (that is, $E_0 \leq E$ and $E_0 \neq E$), for some subprojection E_0 of E in $\mathscr{R}$, and finite otherwise. It is a difficult theorem (proved by Murray and von Neumann) that the union of two finite projections (this union is the projection with range spanned by the ranges of both projections) is a finite projection.

At this stage, the possibility of several types of factors presented itself to Murray and von Neumann. If the factor $\mathscr{M}$ has a minimal projection (and is isomorphic, therefore, to some $\mathscr{B}(\mathscr{H})$), they refer to it as a *factor of type I* (of *type I_n* if $\mathscr{H}$ is n-dimensional, of *type I_∞* when $\mathscr{H}$ is infinite dimensional). If $\mathscr{M}$ has no minimal projection but contains a (nonzero) finite projection, they refer to $\mathscr{M}$ as a *factor of type II* (of type II_1 if I is finite relative to the factor, of type II_∞ otherwise). The remaining possibility is the case where all nonzero projections in the factor are infinite. They refer to these as factors of type III. Of course, no two factors of different types could be isomorphic. But the paramount question was the existence of factors of the various types.

§**5. Examples.** Murray and von Neumann were able to construct factors without minimal projections and identify them as factors of types II_1 and II_∞ by means of examples built from groups acting "freely" by measure (and measurability) - preserving transformations on a measure space. These examples appear in the first paper [27] resulting from their collaboration.

Let G be a (discrete) group with unit e, and $\mathscr{A}$ be a maximal abelian (selfadjoint) subalgebra of $\mathscr{B}(\mathscr{H})$. (We use $\mathscr{A}$ to replace an explicit measure space, as we know we can from our earlier discussion.) Let $\mathscr{K}$ be the direct sum of copies of $\mathscr{H}$ indexed by elements of G. (In effect, $\mathscr{K}$ is the Hilbert space of functions x on G with values in $\mathscr{H}$, such that $\sum_{g\in G}\|x(g)\|^2 < \infty$, provided with the obvious linear structure and the inner product $\langle x,y\rangle = \sum_{g\in G}\langle x(g),y(g)\rangle$.) With S in $\mathscr{B}(\mathscr{H})$, there is a naturally associated operator $\Phi(S)$ in $\mathscr{B}(\mathscr{K})$ defined by $(\Phi(S)x)(g) = S(x(g))$. Let $U : g \to U(g)$ be a unitary representation of G on $\mathscr{H}$ (that is, a homomorphism of G into the group of unitary transformations of $\mathscr{H}$ into itself) and let $(V(g)x)(g')$ be $U(g)x(g^{-1}g')$. We assume that $U(g)AU(g)^*$ is in $\mathscr{A}$ for each A in $\mathscr{A}$ and g in G (that is, each $U(g)$ implements an automorphism of $\mathscr{A}$ and U gives rise to a representation of G by automorphisms of $\mathscr{A}$) and that $\mathscr{A} \cap (U(g)\mathscr{A}) = \{0\}$ for each $g(\neq e)$ in G (that is, G acts *freely*, by automorphisms, on $\mathscr{A}$). The mapping Φ "copies" $\mathscr{B}(\mathscr{H})$ isomorphically into $\mathscr{B}(\mathscr{K})$. It is easily checked that $V(gg') = V(g)V(g')$ and $V(g)\Phi(S)V(g)^* = \Phi(U(g)SU(g)^*)$ for each S in $\mathscr{B}(\mathscr{H})$. Thus the representation V gives rise to the "same" representation of G by automorphisms of the copy $\Phi(\mathscr{A})$ of $\mathscr{A}$ as U does. We say that G acts ergodically on $\mathscr{A}$ (through the representation U) when the only elements A in $\mathscr{A}$ such that $U(g)AU(g)^* = A$, for all g in G, are the scalars.

THEOREM. *The von Neumann algebra $\mathscr{R}$ generated by $\Phi(\mathscr{A})$ and the group $\{V(g)\}_{g\in G}$ is a factor if and only if G acts ergodically on $\mathscr{R}$. In this case, it is a factor of type I if and only if $\mathscr{A}$ has a minimal projection. In any event, $\mathscr{A}$ is a maximal abelian subalgebra of $\mathscr{R}$.*

Specific examples of the structures described above are obtained from a measure space $(S,\mathscr{S},m)$ that is countably separated ($\mathscr{S}$, the family of measurable sets, contains a countable family of $E_1, E_2, \ldots$ of non-null sets of finite measure such that if s and t are distinct points of S, then $t \in E_j$ and $s \notin E_j$ for some j) and a group G of one-to-one mappings of S onto S that preserves measurability and measure 0 subsets and acts freely on S (that is, $m(\{s \in S : g(s) = s\}) = 0$ when g is not the unit element e of G). In this case, $\mathscr{A}$ is the multiplication algebra of the measure space (acting on $L_2(S,m)$). The Radon-Nikodým theorem yields, for each g, a non-negative, real-valued, measurable function φ_g on S such that $\int x(g(s))\,dm(s) = \int x(s)\varphi_g(s)\,dm(s)$ for each x in $L_1(S,m)$. If U_g is defined by $(U_g x)(s) = [\varphi_g(s)]^{\frac{1}{2}}x(g^{-1}(s))$, for each x in $L_2(S,m)$, then $g \to U_g$ is a unitary representation of G that gives rise to automorphisms of $\mathscr{A}$ satisfying $\mathscr{A} \cap (U_g\mathscr{A}) = \{0\}$ for each $g(\neq e)$ in G. We say that G acts *ergodically* on S when $m(g(S_0)\backslash S_0) > 0$ for some g in G unless $m(S_0) = 0$ or $m(S\backslash S_0) = 0$. The representation $g \to U_g$ acts ergodically on $\mathscr{A}$ if and only if G acts ergodically on S.

With U_g in place of $U(g)$, $L_2(S,m)$ for $\mathscr{H}$, and the multiplication algebra of (S,m) for $\mathscr{A}$, the conditions for the construction of $\mathscr{R}$ described earlier are satisfied.

THEOREM. *If G acts ergodically on S, then $\mathscr{R}$ is a factor and*

(i) *$\mathscr{R}$ is of the type I if and only if some point in S has positive measure; in this case, $\mathscr{R}$ is of type I_n where n is the number of points in S.*

(ii) *$\mathscr{R}$ is of type II when S admits a G-invariant measure m_0 such that $m_0(S_0) = 0$ if $m(S_0) = 0$. In this case, $\mathscr{R}$ is of type II_1 when $m_0(S) < \infty$ and of type II_∞ when $m_0(S) = \infty$.*

(iii) *$\mathscr{R}$ is of type III when there is no m_0 as described in* (ii).

Explicit examples can be displayed by choosing explicit measure spaces and group actions. We describe some.

(a) With S the group of integers, each integer assigned measure 1, let G be the group of all translations of S (under addition). This example results in a factor of type I_∞. If S is replaced by a finite cyclic group of order n, the resulting von Neumann algebra is a factor of type I_n.

(b) Let S be the circle (in **C**) with Haar-Lebesgue measure and let G be the group of rotations of S through angles that are rational multiples of π. The action of G is ergodic and the resulting factor is of type II_1. The same is true if we replace G by the group generated by a single rotation through an angle that is an irrational multiple of π.

(c) Let S be **R**, the set of real numbers, with Lebesgue measure and G be the (countable) group of translations by rationals. Again, G acts ergodically, and the resulting factor is of type II_∞.

In each of the examples (a), (b) and (c), the measure is invariant under the group action. In the example that follows, we augment the group of (c) by transformations that do not preserve the measure. Since the group of rational translations acts ergodically, Lebesgue measure is the only invariant measure (up to positive scalar multiples). Thus the augmented group admits no invariant measure.

(d) Let S be **R** with Lebesgue measure and let G be the (countable) group of all "rational" affine transformations $s \to as + b$ with a and b rational. The resulting factor is of type III.

It is not easy to show that the factor in Example (d) is of type III (or to prove (iii) of the more general theorem on types). Some of the techniques and constructs needed will appear when we discuss traces and weights. Several years elapsed before the factors of type III were constructed. They appear in the third paper of the Murray-von Neumann series **[30]**, a paper authored by von Neumann alone.

Another, more simply described, class of examples was found later by Murray and von Neumann. These appear in **[29]**. They provide one of the possible extensions of the notion of group algebra (over **C**) from finite to infinite (discrete) groups. Let G be an infinite (discrete) group and $\mathscr{H}$ be the family of complex-valued functions φ on G such that $\sum_{g\in G}|\varphi(g)|^2 < \infty$. With the inner product, $\langle\varphi,\psi\rangle = \sum_{g\in G}\varphi(g)\overline{\psi(g)}$, $\mathscr{H}$ becomes a Hilbert space (referred to as $l_2(G)$). Let $(L_{g_0}\varphi)(g)$ be $\varphi(g_0^{-1}g)$ for each g in G. Then L_g is a unitary operator on the Hilbert space $\mathscr{H}$ (for $L_{g^{-1}}$ is its inverse and $< L_g\varphi, L_g\psi > = < \varphi,\psi >$ for all φ and ψ in $\mathscr{H}$). Moreover, $L_gL_{g'} = L_{gg'}$; the mapping $g \to L_g$ is a (group) isomorphism of G into the group of unitary operators on $\mathscr{H}$. In the same way, we can define the unitary operators R_{g_0} by $(R_{g_0}\varphi)(g) = \varphi(gg_0)$. Let $\mathscr{L}_G$ and $\mathscr{R}_G$ be the weak-operator closures of the algebras of finite, complex linear combinations of the operators $\{L_g : g \in G\}$ and $\{R_g : g \in G\}$, respectively. Then $\mathscr{L}_G$ and $\mathscr{R}_G$ are von Neumann algebras. In addition, each of $\mathscr{L}_G$ and $\mathscr{R}_G$ is the commutant of the other. (Thus $(\mathscr{L}_G')' = \mathscr{L}_G$, an illustration of the Double Commutant Theorem.)

THEOREM. *The von Neumann algebra $\mathscr{L}_G$ is a factor if and only if each conjugacy class (other than the class of the group identity) is infinite. In this case, $\mathscr{L}_G$ is a factor of type II_1.*

The groups G satisfying the infinite conjugacy class condition are called *i.c.c. groups*. Some examples of such groups are $\mathscr{F}_n$, the free (nonabelian) groups on $n(\geq 2)$ generators, and Π, the group of those permutations of the integers that move at most a finite number of integers. In **[29]**, with the aid of these examples, Murray and von Neumann answer the most important question raised in their earlier work: Has the type classification of factors settled the problem of algebraic isomorphism of factors? More specifically, are all factors of type II_1 isomorphic? They answer this last question in the negative.

THEOREM. *$\mathscr{L}_\Pi$ is not isomorphic to $\mathscr{L}_{\mathscr{F}_n}$.*

It was abundantly clear from this result of Murray and von Neumann that there are an infinite number of mutually nonisomorphic factors of type II_1 and that infinite groups, distinguished by subtle commutativity properties, would provide such examples. It was equally clear that the implementation of that program would not be easy. Twenty-five years after the Murray-von Neumann result, that program was completed [**25, 26, 39**], following basic advances in the program [**5, 14, 38, 40, 51**]. In another direction, a more innovative addition to the II_1 factor archives comes from an automorphism-group invariant invented by Connes [**9**] that produces an infinite number of nonisomorphic II_1 factors.

As this is written, we do not know the answer to the question of whether $\mathscr{L}_{\mathscr{F}_n}$ and $\mathscr{L}_{\mathscr{F}_m}$ are isomorphic when $n \neq m$.

§**6. States, weights, and traces.** A factor $\mathscr{M}$ of type I_n, with n finite, is isomorphic to the algebra $\mathscr{M}_n(\mathbf{C})$ of $n \times n$ complex matrices over $\mathbf{C}$. A key element of structure for $\mathscr{M}_n(\mathbf{C})$ (and $\mathscr{M}$) is the linear functional τ with the properties

$$\tau(AB) = \tau(BA) \quad (A, B \in \mathcal{M})$$
$$\tau(I) = 1.$$

We refer to τ as the *normalized trace* on $\mathcal{M}$. With the properties noted, τ is unique. In addition, τ takes non-negative real values at positive matrices. If we denote by $[a_{jk}]$ a matrix in $\mathcal{M}_n(\mathbf{C})$, where a_{jk} is the entry in row j and column k, then $\tau([a_{jk}])$ is $n^{-1}(\sum_{j=1}^{n} a_{jj})$.

A discovery that intrigued Murray and von Neumann greatly was the existence of a functional on a factor $\mathcal{M}$ of type II_1 with the main properties exhibited by the trace on $\mathcal{M}_n(\mathbf{C})$. They referred to this functional on $\mathcal{M}$ as the (normalized) trace. It was relatively easy to prove that τ is unique, and it was not difficult to determine the value that τ must assume at each element of $\mathcal{M}$. Proving that τ, so determined, is additive ($\tau(A + B) = \tau(A) + \tau(B)$) was quite another matter. The additivity of τ had Murray and von Neumann stopped throughout the development of their first paper **[27]**. They surmounted their difficulties and proved the additivity of τ in **[28]**, their second paper. It was a very hard argument. (In conversation with this author, von Neumann remarked that "that was Murray.")

To define their trace, Murray and von Neumann proceeded in a measure-theoretic manner. With $\mathcal{M}$ a factor of type II_1, it can be shown that for each positive integer n and each projection E in $\mathcal{M}$ there are n equivalent mutually orthogonal projections in $\mathcal{M}$ with sum E. If we assign to I the measure (or "normalized dimension") 1 and use I in place of E, then each of the n equivalent projections should be assigned measure n^{-1}. Each projection in $\mathcal{M}$ is a (possibly infinite) sum of such (rational) projections, which provides it with a measure. There are, of course, obvious technical problems, but these can be overcome without great trouble. Murray and von Neumann arrived at a "dimension function" d that assigns to each projection E in $\mathcal{M}$ a number in $[0, 1]$. They noted that the range of d is precisely $[0, 1]$, and recognized that they were dealing with "continuous dimensionality." These matrix-like algebras in which the associated subspaces (projections) could assume a continuous range of dimensions fascinated Murray and von Neumann, as well they might.

From another point of view, the function d is a (noncommutative) measure on the "measurable" sets of (that is, projections in) $\mathcal{M}$ – and the unique such measure compatible with the equivalence relation on projections. It enjoys a Haar-measure-like status relative to the noncommutative measure space underlying $\mathcal{M}$. Examining $M_n(\mathbf{C})$, we see that $\tau(E)$ is the dimension of the subspace on which the projection E projects, normalized so that the full space has dimension 1. Thus in $\mathcal{M}$, $d(E)$ must be the value assigned as $\tau(E)$. The spectral theorem, which tells us. in effect, that each selfadjoint operator is a (limit of) linear combinations of mutually orthogonal projections ($A = \int \lambda \, dE_\lambda$), determines that value of τ at each selfadjoint operator in $\mathscr{M}$: ($\tau(A) = \int \lambda \, d\tau(E_\lambda)$). Finally, each T in $\mathscr{M}$ is a sum $A + iB$ where $A(= (T + T^*)/2)$ and $B(= (T - T^*)/2i)$ are selfadjoint. If τ is to be linear, we must define $\tau(T)$ as $\tau(A) + i\tau(B)$. As remarked, proving that $\tau(R + S) = \tau(R) + \tau(S)$ for each pair of operators R and S in $\mathscr{M}$ was a considerable challenge.

Murray and von Neumann defined a dimension function d on the projections of all factors by devices similar to those described for a factor of type II_1. They note that the type of the factor is completely determined by the range of the dimension function. In the case of a factor $\mathscr{M}$ of type I, with a (necessarily finite) minimal projection E they normalize the dimension function d so that $d(E) = 1$. In this case, d has range $\{0, 1, \ldots, n\}$ when $\mathscr{M}$ is of type I_n, and range $\{0, 1, 2, 3, \ldots\}$ when $\mathscr{M}$ is of type I_∞. If $\mathscr{M}$ is of type II_∞, there is no preferred finite projection on which to normalize the dimension function, whichever (equivalence class of a) finite projection is chosen to have dimension 1, the range of the dimension function is $[0, \infty]$. With $\mathscr{M}$ of type III, the range of the dimension function consists of just 0 and ∞.

Inspired by the construction of the Gelfand-Neumark article **[15]**, Segal **[41]** singled out special linear functionals on a C^*-algebra from which one can construct Hilbert spaces and adjoint-preserving homomorphisms (*representations*) of the C^*-algebra into the algebras of all bounded operators on the Hilbert spaces. (It was through such representations that Gelfand and Neumark built the * isomorphism of their Banach algebra with a C^*-algebra.) The functionals on which Segal focused have the property that they assume non-negative real values on positive operators on the C^*-algebra and assign the value 1 to I. He called such functionals *states*. (They correspond to the expectation functionals associated with states of quantum mechanical systems.) The states turn out to be norm continuous without further assumptions. As noted, the Murray-von Neumann trace τ on a II_1 factor and the normalized trace on $\mathscr{M}_n(\mathbf{C})$ are states.

If ρ is a state of the C^*-algebra $\mathscr{A}$ and we let $\langle A, B\rangle_\rho$ be $\rho(B^*A)$, then $\langle , \rangle_\rho$ is a positive (semidefinite) inner product on $\mathscr{A}$. As such, $<, >_\rho$ satisfies the Cauchy-Schwarz inequality. It follows from this that

$$\{A \in \mathscr{A} : \rho(A^*A) = 0\} (= \mathscr{L}_\rho)$$

is a (norm-closed) left ideal in $\mathscr{A}$ (the *left kernel* of ρ), for $B^*B \leq \|B\|^2 I$ whence $A^*B^*BA \leq \|B\|^2 A^*A$. The quotient vector space $\mathscr{A}/\mathscr{L}_\rho$ inherits the positive-definite inner product

$$\langle T + \mathscr{L}_\rho, S + \mathscr{L}_\rho \rangle = \langle T, S\rangle_\rho = \rho(S^*T) \quad (T, S \in \mathscr{A})$$

relative to which its completion $\mathscr{H}_\rho$ is a Hilbert space. If $\varphi(A)(T + \mathscr{L}_\rho)$ is defined to be $AT + \mathscr{L}_\rho$, the resulting linear operator on $\mathscr{A}/\mathscr{L}_\rho$ has norm not exceeding $\|A\|$ (relative to the Hilbert space norm on $\mathscr{A}/\mathscr{L}_\rho$); this operator extends, uniquely, to a bounded linear operator $\varphi(A)$ on $\mathscr{H}_\rho$. It is not difficult to verify that $\varphi(aA+B) = a\varphi(A)+\varphi(B)$, $\varphi(AB) = \varphi(A)\varphi(B)$, $\varphi(A^*) = \varphi(A)^*$, and $\varphi(I) = I$ (φ is a *representation* of $\mathscr{A}$ on $\mathscr{H}_\rho$). In addition, $\|\varphi(A)\| \le \|A\|$ and the image $\varphi(\mathscr{A})$ of $\mathscr{A}$ is norm-closed in $\mathscr{B}(\mathscr{H}_\rho)$ (so that $\varphi(\mathscr{A})$ is a C^*-algebra).

Through this construction, the state ρ, itself, has been "represented" in a special form:

$$\rho(A) = \langle \varphi(A)(I+\mathscr{L}_\rho), I+\mathscr{L}_\rho\rangle.$$

In this way, ρ "has become" $\omega_{x_\rho}|\varphi(\mathscr{A})$, the *vector state* ω_{x_ρ} of $\mathscr{B}(\mathscr{H}_\rho)$ restricted to the C^*-algebra $\varphi(\mathscr{A})$, where x_ρ is the vector $I+\mathscr{L}_\rho$ in $\mathscr{H}_\rho$ and $\omega_{x_\rho}(T) = \langle Tx_\rho, x_\rho\rangle$ for each T in $\mathscr{B}(\mathscr{H}_\rho)$. Note, too, that $\varphi(\mathscr{A})x_\rho$ is $\mathscr{A}/\mathscr{L}_\rho$, a dense subspace of $\mathscr{H}_\rho$. We say that x_ρ is a *generating* (or *cyclic*) *vector* for $\varphi(\mathscr{A})$. The vector states have a key place in the study of von Neumann algebras, a place whose importance became clear throughout the 1950s.

The construction associating a representation of a C^*-algebra with a state of that algebra, is basic to the study of C^*-algebras. It is known as the GNS construction (for Gelfand, Neumark, and Segal). It was not available, as such, to Murray and von Neumann though they had found a vector representation for the normalized trace τ on a factor $\mathscr{M}$ of type II_1. (See **[28;** Theorem II].) In the case of τ, the left kernel is (0). We speak of τ as a *faithful* state of $\mathscr{M}$; the corresponding representation is an isomorphism (is *faithful*) and τ is "represented" by a vector x_τ. From the basic property of the trace, we see that $\langle ABx_\tau, x_\tau\rangle = \langle BAx_\tau, x_\tau\rangle$ for all A and B in $\mathscr{M}$; we call such a vector x_τ a *trace vector* for $\mathscr{M}$. In any case, with $\mathscr{M}$ acting on a Hilbert space $\mathscr{H}$, Murray and von Neumann show that τ is a convex combination $\sum_{j=1}^n a_j\omega_j$ of vector states ω_j.

A von Neumann algebra $\mathscr{R}$ is more closely tied to a representation on a Hilbert space than a C^*-algebra need be. As a consequence, the states most relevant for the analysis of the structure of $\mathscr{R}$ have a certain continuity property related to the weak-operator topology on $\mathscr{R}$, which is associated with the action of $\mathscr{R}$ on its underlying Hilbert space. These are the states of $\mathscr{R}$, termed *normal*, that are weak-operator continuous on the unit ball of $\mathscr{R}$. It is fairly difficult to prove the following characterization of normal states.

THEOREM. *With $\mathscr{R}$ a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, each normal state ω of $\mathscr{R}$ is a sum $\sum\omega_j$ of linear functionals $\omega_j (= \omega_{x_j}|\mathscr{R})$, where $\sum\|x_j\|^2 = 1$. (This sum may be (countably) infinite, in which case, $\sum_{j=1}^\infty \omega_j(A)$ converges to $\omega(A)$ for each A in $\mathscr{R}$.) If $\mathscr{R}$ has at least one faithful vector state, then each normal state of $\mathscr{R}$ is a vector state.*

If $\{E_n : n = 1, 2, \dots\}$ is an orthogonal family of projections in the von Neumann algebra $\mathscr{R}$, then $\sum E_n x$ converges to a vector Ex for each x in the underlying Hilbert space $\mathscr{H}$. In this case, E is a projection in $\mathscr{R}$; we write $E = \sum_{n=1}^\infty E_n$. If ω is a normal state of $\mathscr{R}$, then $\omega(\sum E_n) = \sum\omega(E_n)$. Conversely, if the state ω satisfies the preceding equality for each orthogonal family of projections in $\mathscr{R}$, then ω is a normal state of $\mathscr{R}$. Thinking, again, of $\mathscr{R}$ as a noncommutative measure algebra, we see that the normal states of $\mathscr{R}$ must be viewed as the noncommutative integration processes on $\mathscr{R}$. A normal state of $\mathscr{R}$ restricted to the projections in $\mathscr{R}$ serves as a noncommutative (probability) measure on the (noncommutative) measurable sets of $\mathscr{R}$.

The inverse situation (given a "measure" on the projections of a von Neumann algebra, is there a corresponding integral—that is, normal state) was discussed by Mackey, in the case where the von Neumann algebra is $\mathscr{B}(\mathscr{H})$, at the end of the 1940s in connection with his rigorous development of the foundations of quantum mechanics [**24**]. It was answered in the affirmative (for $\mathscr{B}(\mathscr{H})$) when $\mathscr{H}$ has dimension greater than 2 by Gleason [**16**]. Many years elapsed before this fundamental question received a positive answer (largely by Christensen [**6**] and Yeadon [**50**]) for general von Neumann algebras (with no two-dimensional representations). When we note that this answer includes, in particular, passing from the dimension function on the projections of a factor of type II_1 to the normalized trace on the factor, some of the difficulty involved can be appreciated.

Of course, the noncommutative measure-theoretic interpretation of the theory of von Neumann algebras cannot rest with an analogue of finite measures; the infinite measure spaces must be understood as well. The replacement for states has been developed in that context. A *weight* ρ of a C^*-algebra $\mathscr{A}$ is a function on the positive elements in $\mathscr{A}$ taking non-negative real values and the value $+\infty$ and satisfying $\rho(A+B) = \rho(A) + \rho(B)$ and $\rho(tA) = t\rho(A)$ when A and B are positive operators in $\mathscr{A}$ and $t > 0$. By analogy with infinite measures, the analogues of the sets of finite measure, the positive integrable functions, and the absolutely square integrable functions have noncommutative analogues of special importance. Various left ideals arise from these and a construction akin to the GNS construction can be made giving rise to a representation corresponding to the given weight.

As with states, we can speak of *faithful weights* (if $\rho(A^*A) = 0$, then $A = 0$). If $\mathscr{A}$ is a von Neumann algebra $\mathscr{R}$ and the linear space consisting of the positive operators in $\mathscr{R}$ on which ρ assumes finite values is weak-operator dense in $\mathscr{R}$, we say that ρ is *semifinite.* For a von Neumann algebra $\mathscr{R}$, we are interested, as with states, in weights related to the action of $\mathscr{R}$ on the underlying Hilbert space. These weights ρ, called *normal*, have the property that there is a family $\{\rho_a\}$ of normal states ρ_a of $\mathscr{R}$ such that $\rho(A) = \sum_a \rho_a(A)$ for each positive A in $\mathscr{R}$.

Again, a trace-like functional on $\mathscr{B}(\mathscr{H})$ provides us with our most prominent example of a weight. Let $\{x_a\}$ be an orthonormal basis for $\mathscr{H}$ and define $\tau(A)$ to be $\sum_a \langle Ax_a, x_a\rangle$ for a positive A in $\mathscr{B}(\mathscr{H})$. In this case, τ is a faithful, normal, semifinite weight on $\mathscr{B}(\mathscr{H})$. In addition, $\tau(AA^*) = \tau(A^*A)$ for each A in $\mathscr{B}(\mathscr{H})$. We say that τ, with this property, is a *tracial weight* (on $\mathscr{B}(\mathscr{H})$).

The existence of a semifinite, faithful, normal, tracial weight on a factor $\mathscr{M}$ entails the existence of a nonzero, finite projection in $\mathscr{M}$. Thus $\mathscr{M}$ admits no such tracial weight if it is of type III. If $\mathscr{M}$ is of type II_∞, there is an orthogonal family $\{E_a\}$ of (equivalent) finite projections in $\mathscr{M}$ with sum I. The family $E_a\mathscr{M}E_a$ of operators of the form E_aTE_a $(T \in \mathscr{M})$ is a von Neumann algebra. It is not difficult to show that $E_a\mathscr{M}E_a$ (acting on $E_a(\mathscr{H})$) is a factor $\mathscr{M}_a$ of type II_1. With τ_a the normalized trace on $\mathscr{M}_a$, the function τ on $\mathscr{M}$ that assigns $\sum_a \tau_a(A)$ to each positive A in $\mathscr{M}$ is a faithful, normal, semifinite, tracial weight on $\mathscr{M}$ (the unique such weight up to positive multiples).

The fact that a factor that admits no normal, semifinite tracial weight must be of type III is the key to von Neumann's argument that some of the factors described in Section 5 are of type III. In the notation of the general construction at the beginning of Section 5, we can view the elements of $\mathscr{R}$ as matrices, indexed by elements of G, with entries from $\mathscr{B}(\mathscr{H})$. The entry at the g, h position is $U(gh^{-1})A(gh^{-1})$, where $g' \to A(g')$ is a mapping from G into $\mathscr{A}$; the matrix should represent an element of $\mathscr{B}(\mathscr{K})$. Let η be the mapping that assigns to this element (matrix) of $\mathscr{R}$ the element with matrix all of whose entries are 0 except for the diagonal entries and these are all equal to $A(e)$. (In other words, η is the mapping that changes all off-diagonal entries to 0.) Several properties of η are noteworthy: $\eta(I) = I, \eta(H) \geq 0$ if $H \geq 0$, and $\eta(ATB) = A\eta(T)B$ when $A, B \in \mathscr{A}$. The mapping η is the noncommutative analogue of a conditional expectation (from $\mathscr{R}$ onto $\Phi(\mathscr{A})$) and is referred to as a *conditional expectation.* If ρ_0 is a normal, semifinite weight on $\mathscr{A}$ that satisfies

$$\rho_0(A) = \rho_0(U(g)AU(g)^*) \quad (A \in \mathscr{A}),$$

then $\rho(T)$ defined as $\rho_0(\Phi^{-1}(\eta(T)))$ yields the normal, semifinite, tracial weight ρ on $\mathscr{R}$. Moreover, each such tracial weight on $\mathscr{R}$ arises in this way. The more difficult part of the proof of this assertion is the argument establishing that $\rho_0(A)$, defined as $\rho(\Phi(A))$ for each A in $\mathscr{A}$, gives rise to a *semifinite* weight ρ_0 on $\mathscr{A}$ when the normal, semifinite, tracial weight ρ on $\mathscr{R}$ is given.

A normal, semifinite weight on $\mathscr{A}$ with the given group-invariance property corresponds to a group-invariant measure on the underlying measure space, absolutely continuous with respect to the underlying measure, in the situation of the specific examples of Section 5 constructed from a measure space and a group acting on it. In some of those examples, there is no such invariant measure, from which we must conclude that the factors constructed in those cases admit no normal, semifinite, tracial weight and are, accordingly, of type III.

§7. **Von Neumann algebras and their commutants.** In [27], Murray and von Neumann established that a factor of type I, II, or III has commutant a factor of the corresponding type, though a factor of type II_1 or II_∞ may have a factor of type II_∞ or II_1, respectively, as commutant. From the type I case, where it seems most natural to choose the representation of the factors as $\mathscr{B}(\mathscr{H})$, with $\mathscr{H}$ of the appropriate dimension, we might conclude that the "most natural" representation of a factor has as small a commutant as possible. With $\mathcal{M}$ of type II_1 or of type III, there is not much that can be sensibly effected in that direction. Some further reflection leads to the conclusion that the most "standard" representation to choose for a von Neumann algebra is one in which it and its commutant have the "same size." Just what that means is most simply illustrated in the case of a factor $\mathcal{M}$ of type I_n. In this case, $\mathcal{M}$ is isomorphic to $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ of dimension n. Let $\mathcal{K}$ be the n-fold direct sum of $\mathcal{H}$ with itself. We can view $\mathcal{B}(\mathcal{K})$ as $n \times n$ matrices with entries from $\mathcal{B}(\mathcal{H})$. The algebra of matrices with the same element of $\mathcal{B}(\mathcal{H})$ at each diagonal entry and 0 at all off-diagonal positions is a factor isomorphic to $B(\mathcal{H})$ and to $\mathcal{M}$. Its commutant consists of those matrices all of whose entries are scalar multiples of I (in $\mathcal{B}(\mathcal{H})$); it, too, is a factor of type I_n. For $\mathcal{M}$ of type I_∞, $\mathcal{H}$ is infinite dimensional; we restrict ourselves to the separable case with the dimension of $\mathcal{H}$ the cardinality of the natural numbers. Following the same procedure as in the type I_n case, but paying attention to convergence, we arrive at a representation of $\mathcal{M}$ with commutant $\mathcal{M}'$ of type I_∞.

For the "standardization" we are after, it no longer suffices, in the case of a factor of type II_1, to work toward a representation in which the commutant is of type II_1. We notice, however, that in the type I case, "standardization" is reached (recall, we are in the *separable* case) precisely when there is a vector x in the representing Hilbert space $\mathcal{H}$ such that $\mathcal{M}x$ and $\mathcal{M}'x$ are both dense, linear subspaces of $\mathcal{H}$. When $\mathcal{M}$ is of type III, it is automatically represented in standard form (separable case): when $\mathcal{M}$ is of type II a standard representation can be arranged by a process of forming direct sums and "copying" on the diagonal (as we did in the type I_n case) and restricting $\mathcal{M}$ to the range of a projection in $\mathcal{M}'$.

The importance of studying $\mathcal{M}$ in its standard representation for understanding its possible action on a space can be seen from the following result (known as the Unitary Implementation Theorem).

THEOREM. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, in standard form and φ is an isomorphism of $\mathcal{R}_1$ onto $\mathcal{R}_2$, then there is a unitary transformation U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $\varphi(A) = UAU^{-1}$ for each A in $\mathcal{R}_1$.*

We say that U of this theorem *implements* the isomorphism φ. the uniqueness of the standard form of a von Neumann algebra (up to unitary equivalence) is assured by the Unitary Implementation Theorem.

When we discussed the GNS construction applied to the normalized trace τ on a factor $\mathcal{M}$ of type II_1, we arrived at a generating trace vector x_τ for $\mathcal{M}$. It follows (easily) from the fact that τ is a faithful state of $\mathcal{M}$ that x_τ is generating for $\mathcal{M}'$ as well. Thus, as represented, $\mathcal{M}$ is in standard form. Murray and von Neumann noted that, if $\mathcal{M}$ is a factor and x_τ is a generating trace vector for $\mathcal{M}$, then x_τ is a generating trace vector for $\mathcal{M}'$. Moreover, for each A in $\mathscr{M}$, there is a unique A' in $\mathscr{M}'$ such that $Ax_\tau = A'x_\tau$. Observing that

$$(AB)x_\tau = AB'x_\tau = B'Ax_\tau = (B'A')x_\tau,$$

they show that $A \to A'$ is an (adjoint-preserving) anti-isomorphism of $\mathscr{M}$ onto $\mathscr{M}'$. They were able to extend this result, using the techniques they had developed, to all factors of types I or II; in standard form, each is * anti-isomorphic to its commutant. Factors of type III were beyond their reach. They raised this question for all factors (indeed, for all von Neumann algebras in standard form).

This problem remained open until 1967 when M. Tomita, completing years of arduous research announced his affirmative solution [**47**]. This occurred at a general conference on the theory of operator algebras and its applications held at Louisiana State University in Baton Rouge, Louisiana (in honor of J. Dixmier who was visiting Tulane University in New Orleans that year). This same conference was attended by several of the physicists who had made great strides in applying operator algebras to the study of quantum mechanical systems with infinitely many degrees of freedom (cf. [**1, 4, 13, 19, 20, 49**]). In particular, R. Haag, N. Hugenholtz, and M. Winnink, who had just completed a penetrating analysis of the conditions for a state to be an equilibrium state of a quantum statistical mechanical system in the framework of the theory of operator algebras [**18**], were present.

A large contingent of the powerful Japanese Functional Analysis school was also present, among them M. Takesaki, an important young contributor to the subject of operator algebras, who was to become one of the great leaders of the subject. Takesaki listened carefully to Haag, Hugenholtz, and Winnink, and, later, during a yearlong visit to the University of Pennsylvania (1968–1969), refined the Tomita work and fused it with the equilibrium-state analysis of [**18**] to create the Tomita-Takesaki Modular Theory [**44**], the dominant theme in research on von Neumann algebras throughout the 1970s.

The modular theory of Tomita-Takesaki has as its basic ingredients a von Neumann algebra $\mathscr{R}$ and a normal, faithful, semifinite weight ρ. As discussed, the analogue for weights of the GNS construction produces a representation of $\mathscr{R}$ on a Hilbert space $\mathscr{H}_\rho$. Although no vector represents the weight (by contrast with the case of a state), the assumption about the weight yields a faithful representation of $\mathscr{R}$ as a von Neumann algebra in "standard form," in the sense of having a commutant of the "same size" as the algebra, although there may be no generating vector for both the representing algebra and its commutant. While each von Neumann algebra has a faithful, normal, semifinite weight, the existence of a faithful, normal state imposes a countability restriction on the von Neumann algebra (each orthogonal family of projections in the algebra is countable) that is automatically fulfilled for algebras acting on a separable Hilbert space. Even when the von Neumann algebra satisfies this countability restriction, it is advantageous, for certain purposes, to have the theory available for weights as opposed to states. Nevertheless, the main features of the theory can be described in terms of a faithful, normal *state*, and this simplifies the description considerably.

Suppose ω is a faithful, normal state of $\mathscr{R}$. The GNS construction applied to ω gives rise to a faithful representation φ of $\mathscr{R}$ on a Hilbert space $\mathscr{H}$ and a (unit) vector u generating for $\varphi(\mathscr{R})$ and $\varphi(\mathscr{R})'$ such that $\omega(A) = \langle\varphi(A)u, u\rangle$ for each A in $\mathscr{R}$. To simplify our notation, we write $\mathscr{R}$ for $\varphi(\mathscr{R})$ and ω for the restriction of the vector state ω_u to $\varphi(\mathscr{R})$. Tomita's key idea is to analyze the adjoint operations on $\mathscr{R}$ and $\mathscr{R}'$, reducing them to the Hilbert space level through the use of the vector u. Define S_0Au to be A^*u and $F_0A'u$ to be A'^*u for each A in $\mathscr{R}$ and A' in $\mathscr{R}'$. The mappings S_0 and F_0 are conjugate-linear ($S_0(ax + y) = \bar{a}S_0x + S_0y$) and defined only on the dense subspaces $\mathscr{R}u$ and $\mathscr{R}'u$, respectively. In general, they are not even bounded—a very unpromising start! Some simple calculation reveals that S_0 and F_0 behave, to a great extent, as adjoints of one another. Combined with this, the fact that they are densely defined implies that they have "closures" S and F, respectively. While not necessarily bounded, "closed operators" (those whose graphs are closed) that are densely defined are susceptible to some Hilbert space analysis. In particular, S has a polar decomposition $J\Delta^{\frac{1}{2}}$, as described in Section 4 for T in $\mathscr{B}(\mathscr{H})$, where $\Delta(= S^*S)$ is a (generally, unbounded) selfadjoint operator. Since S is conjugate-linear, densely defined, and $S^2Au = Au$ for each A in $\mathscr{R}$, J is a conjugate-linear, isometric mapping of $\mathscr{H}$ onto $\mathscr{H}$ such that $J^2 = I$. The main result of Tomita, stated in the notation we have developed, follows.

THEOREM. *For each A in $\mathscr{R}$ and each real t, $\Delta^{it}A\Delta^{-it} \in \mathscr{R}$ and $J\mathscr{R}J = \mathscr{R}'$.*

The notation $\Delta^{it}(= e^{it\log\Delta})$ indicates spectral theory and the function calculus applied to the (possibly unbounded) selfadjoint operator Δ, so that Δ^{it} is a unitary operator U_t. The mapping $t \to U_t$ is a homomorphism of the additive groups of $\mathbf{R}$ into the group of unitary operators on $\mathscr{H}$. Moreover, this mapping is continuous from $\mathbf{R}$ in its standard topology to the unitary group in the weak-operator topology. We refer to $t \to U_t$ as a *one-parameter unitary group* (on $\mathscr{H}$). The condition, $\Delta^{it}A\Delta^{-it} \in \mathscr{R}$, implies that $A \to U_tAU_{-t}$ is an (adjoint-preserving) automorphism σ_t^ω of $\mathscr{R}$. The mapping $t \to \sigma_t^\omega$ is, again, a homomorphism of $\mathbf{R}$, this time into the group of automorphisms of $\mathscr{R}$; it is a special one-parameter group of automorphisms of $\mathscr{R}$. We refer to it as the *modular group* of $\mathscr{R}$ corresponding to the (faithful, normal) state ω.

The mapping $A \to JA^*J$ is an (adjoint-preserving) anti-isomorphism of $\mathscr{R}$ onto $\mathscr{R}'$ by virtue of the equality, $J\mathscr{R}J = \mathscr{R}'$. This answers the question of Murray and von Neumann in the affirmative: If $\mathscr{R}$ is a von Neumann algebra in standard form, then $\mathscr{R}$ and $\mathscr{R}'$ are * anti-isomorphic. When $\mathscr{R}$ is a factor of type II_1 and ω is the normalized trace τ on $\mathscr{R}$, $\Delta = I$ and the mapping $A \to JA^*J$ is the mapping, $A \to A'$, where $Ax_\tau = A'x_\tau$ with A in $\mathscr{R}$, A' in $\mathscr{R}'$, and x_τ a trace vector for $\mathscr{R}$ and $\mathscr{R}'$ used to define S and F.

In the process of answering the Murray-von Neumann question, Tomita discovered the modular automorphism group; it has come to be recognized as an important tool in studying the structure of factors of type III. Takesaki's discussions with Haag, Hugenholtz, and Winnink (HHW) at the 1967 Baton Rouge conference led him to the crucial discovery that the modular automorphism group σ_t^ω and ω are related by the condition HHW described for a quantum statistical mechanical system whose (bounded) observables correspond to the selfadjoint elements in $\mathscr{R}$ and whose dynamical evolution has ω as its equilibrium state. (This comment makes only a fleeting allusion to the dynamical evolution-equilibrium state relation; temperature and other parameters must also be discussed in a more careful formulation.)

DEFINITION. A one-parameter group of (adjoint-preserving) automorphisms, $t \to \alpha_t$, of a von Neumann algebra $\mathscr{R}$ satisfies the *modular condition* relative to ω when, for each A and B in $\mathscr{R}$, there is a function F continuous and bounded on the (closed) strip $\{z \in \mathbf{C} : 0 \le \operatorname{Im} z \le 1\}$ holomorphic on the interior ($\{z \in \mathbf{C} : 0 < \operatorname{Im} z < 1\}$), and satisfying

$$F(t) = \omega(\alpha_t(A)B), \qquad F(t+i) = \omega(B\alpha_t(A))$$

for all real t.

THEOREM. *The modular automorphism group corresponding to ω is the unique one-parameter group of automorphisms of $\mathscr{R}$ that satisfies the modular condition relative to ω.*

The modular condition is called "the KMS boundary condition" by HHW, for Kubo, Martin, and Schwinger, who describe the corresponding condition in another context (in the more physics-oriented literature); ω is said to satisfy the KMS boundary condition relative to σ_t^ω. It should be remarked that neither Tomita's theorem nor the uniqueness theorem just stated are easy to prove; the ingredients of the arguments are spectral theory, von Neumann algebra theory, Fourier transforms, and complex variables.

In a major contribution to modular theory, A. Connes [7] analyzes the constructs of the theory in greater detail providing many new and powerful devices for classifying factors of type III. He notes that if ω_1 and ω_2 are faithful, normal states of $\mathscr{R}$ and $t \to \sigma_t^{(1)}$, $t \to \sigma_t^{(2)}$ are the corresponding modular automorphism groups, then for each t, $\sigma_t^{(1)}$ and $\sigma_t^{(2)}$ differ by an "inner automorphism" of $\mathscr{R}$, that is, there is a unitary operator U_t in $\mathscr{R}$ such that $\sigma_t^{(1)}(A) = U_t\sigma_t^{(2)}(A)U_t^*$ for each A in $\mathscr{R}$. In this case, $t \to U_t$, need not be a homomorphism of $\mathbf{R}$ into the unitary group of $\mathscr{R}$. It does satisfy the "cocycle" condition, $U_{s+t} = U_s\sigma_s^{(2)}(U_t)$ for each pair of real numbers s and t. (The mapping $t \to U_t$ is known as the *Connes cocycle.*)

Connes denotes by ε the quotient homomorphism of the group of (adjoint-preserving) automorphisms of $\mathscr{R}$ onto its quotient $\operatorname{Out}(\mathscr{R})$ by the (normal) subgroup of inner automorphisms of $\mathscr{R}$. From the preceding discussion, $\varepsilon(\sigma_t^{(1)}) = \varepsilon(\sigma_t^{(2)})$ for each real t. Thus, $t \to \varepsilon(\sigma_t^{(1)})$ defines a homomorphism, that Connes denotes by δ of $\mathbf{R}$ into Out($\mathscr{R}$). The mapping δ is called the *modular homomorphism* of $\mathscr{R}$; it is "invariantly" associated with $\mathscr{R}$. The kernel $T(\mathscr{R})$ of δ, a subgroup of $\mathbf{R}$, is an algebraic invariant of $\mathscr{R}$. When $\mathscr{R}$ is a factor of type I or II (that is, when $\mathscr{R}$ is *semifinite*), $T(\mathscr{R}) = \mathbf{R}$; each modular automorphism is inner. In the separable case, if each modular automorphism is inner the factor is semifinite.

Connes [7] defines another invariant for a factor $\mathscr{M}$. He denotes by $S(\mathscr{M})$ the intersection, over the faithful, normal weights ρ on $\mathscr{M}$, of the spectra of the operators Δ_ρ. The condition that $\mathscr{M}$ is not of type III is equivalent to the condition that $0 \notin S(\mathscr{M})$, in which case, $S(\mathscr{M}) = \{1\}$. In any event, $S(\mathscr{M})\backslash\{0\}$ is a closed subgroup [**11**] of the multiplicative group $\mathbf{R}_+^*$ of positive reals. Aside from $\{1\}$, the possibilities for this subgroup are $\mathbf{R}_+^*$ itself and $\{\lambda^n : n \in \mathbf{Z}\}$, where $\mathbf{Z}$ is the group of integers and $\lambda \in (0, 1)$. The factor $\mathscr{M}$ is said to be of type III_λ when the subgroup is the cyclic group with generator λ in $(0, 1)$; it is of type III_1 when the group is $\mathbf{R}_+^*$; it is of type III_0 when $S(\mathscr{M})$ consists of 0 and 1. In the final section, we shall see examples of these factors.

We conclude this section with a relation, established by Connes [7], between $S(\mathscr{M})$ and $T(\mathscr{M})$ for a factor $\mathscr{M}$ not of type III_0. In this case, $T(\mathscr{M})$ is the kernel of the homomorphism $t \to \chi_t$ of $\mathbf{R}$ into the dual of $S(\mathscr{M})\backslash\{0\}$, where $\chi_t(\lambda) = \lambda^{it}$. Thus $T(\mathscr{M}) = (-2\pi/\log\lambda)\mathbf{Z}$, when $\mathscr{M}$ is of type III_λ.

§**8. Crossed products.** In Section 5, we described a construction of factors arising from a homomorphism α of a discrete group G into the automorphism group of an abelian von Neumann algebra $\mathscr{A}$. That construction can be extended to the case where $\mathscr{A}$ is replaced by an arbitrary von Neumann algebra $\mathscr{R}$; everything remains the same, with $\mathscr{R}$ in place of $\mathscr{A}$. The resulting von Neumann algebra $\mathscr{R} \otimes \alpha$ is called the *crossed product* of $\mathscr{R}$ by G. In terms of this extended construction, Connes [7] presents an explicit "decomposition" of the factors of type III_λ, $\lambda \neq 1$.

THEOREM. *If $\mathscr{M}$ is a factor of type III_λ, ($\lambda \in (0,1)$), there is a factor $\mathscr{N}$ of type II_∞ and an automorphism α of $\mathscr{N}$ for which $tr \circ \alpha = \lambda tr$, where tr is the (infinite) trace on $\mathscr{N}$, such that $\mathscr{M}$ is $\mathscr{N} \otimes_\tau \mathbf{Z}(= \mathscr{N} \otimes \alpha)$, where $\alpha(n) = \alpha^n$. Conversely, such a crossed product is a factor of type III_λ. Moreover, if $(\mathscr{N}_1, \alpha_1)$ and $(\mathscr{N}_2, \alpha_2)$ are two such pairs corresponding to λ in $(0, 1)$, then $\mathscr{N}_1 \otimes \alpha_1$ is isomorphic to $\mathscr{N}_2 \otimes \alpha_2$ if and only if there is an isomorphism of $\mathscr{N}_1$ onto $\mathscr{N}_2$ carrying α_1 onto $\alpha_2 \circ \beta$ where β is some inner automorphism of $\mathscr{N}_2$.*

The type III_0 factors are also crossed products—but this time of a II_∞ von Neumann algebra (not a factor) by α. The factors of type III_1 elude the (discrete) crossed product analysis of Connes. Takesaki [**45**] encompassed these factors in a crossed product description by extending the crossed product construction to apply to locally compact groups (especially **R**) and defining a "dual action" of these groups in the abelian case. This culminated in the celebrated Takesaki Duality Theorem. Suppose τ is a "continuous action" of the locally compact group G on a von Neumann algebra $\mathscr{R}$ acting on a Hilbert space $\mathscr{H}$; that is, τ is a homomorphism of G into the group of (adjoint-preserving) automorphisms of $\mathscr{R}$ such that $g \to \tau_g(A)x$ is a continuous mapping of G into $\mathscr{H}$ for each A in $\mathscr{R}$ and x in $\mathscr{H}$. An inessential change of representation for $\mathscr{R}$ allows us to assume that τ_g is implemented by a unitary operator U_g on $\mathscr{H}$. The crossed product $\mathscr{R} \otimes \tau$ acts on $\mathscr{H} \otimes L_2(G)$ (the space of square-integrable mappings of G, with Haar measure, into $\mathscr{H}$) and is generated by the operators $A \otimes I$ and $U_g \otimes L_g$, where L_g is the left-translation unitary operator on $L_2(G)((L_g\xi)(g') = \xi(g^{-1}g'))$.

Turning to the case of the action of **R** on a von Neumann algebra $\mathscr{R}$ by a modular automorphism group, we let ρ be a faithful normal weight on the von Neumann algebra $\mathscr{R}$ with $t \to \sigma_t$ its corresponding modular group. Then $t \to \sigma_t$ is a continuous action of **R** on $\mathscr{R}$. If $\mathscr{R}$ is of type III, $\mathscr{R} \otimes \sigma$ is of type II_∞. Define $(U_t\xi)(s)$ to be $e^{-its}\xi(s)$: $(\xi \in L_2(\mathbf{R}))$. Then $I \otimes U_t$ induces an automorphism $\hat{\sigma}_t$ of $\mathscr{R} \otimes \sigma$ and $t \to \hat{\sigma}_t$ is a continuous action (the action *dual* to σ) of $\hat{\mathbf{R}}$ on $\mathscr{R} \otimes \sigma$.

THEOREM (TAKESAKI DUALITY). *If $\mathscr{R}$ is of type III, $(\mathscr{R} \otimes \sigma) \otimes \hat{\sigma}$ is isomorphic to $\mathscr{R}$.*

Each type III_1 factor is a crossed product of a type II_∞ factor with **R**.

§**9. Tensor Products.** Murray and von Neumann [**27**] introduce a construction involving several Hilbert spaces that produces another Hilbert space, their *tensor product*. With the aid of this construction, it is possible to combine von Neumann algebras to form a *tensor product of von Neumann algebras*. The investigation of tensor products has had an important influence on the understanding we have of von Neumann algebras.

From one point of view, there is no difficulty in defining the algebraic tensor products (over **C**) of two von Neumann algebras $\mathscr{R}$ and $\mathscr{S}$; the problem arises in providing the algebra with an appropriate norm and completing. If $\mathscr{R}$ and $\mathscr{S}$ act on the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, and we form the tensor product $\mathscr{H} \otimes \mathscr{K}$ in the style of Murray and von Neumann, there is a natural candidate for $\mathscr{R} \bar{\otimes} \mathscr{S}$, the von Neumann algebra tensor product of $\mathscr{R}$ and $\mathscr{S}$. To form that tensor product, observe that $\mathscr{R}$ and $\mathscr{S}$ have (natural) isomorphic images as von Neumann algebras $\mathscr{R}_0$ and $\mathscr{S}_0$ acting on $\mathscr{H} \otimes \mathscr{K}$. The von Neumann algebra generated by $\mathscr{R}_0$ and $\mathscr{S}_0$ is the one we choose for $\mathscr{R} \bar{\otimes} \mathscr{S}$. Other possibilities that present themselves as a tensor product of $\mathscr{R}$ and $\mathscr{S}$ (though not von Neumann algebras) are $\mathscr{R} \otimes_a \mathscr{S}$, the algebra generated by $\mathscr{R}_0$ and $\mathscr{S}_0$, and $\mathscr{R} \otimes \mathscr{S}$, the C^*-algebra generated by $\mathscr{R}_0$ and $\mathscr{S}_0$.

The tensor product $\mathscr{H} \otimes \mathscr{K}$ is best described, for accurate mathematical purposes, in terms of functionals on the spaces $\mathscr{H}$ and $\mathscr{K}$. For our purposes, and most easily described, we can think of the basic elements of $\mathscr{H} \otimes \mathscr{K}$ as the "simple tensors" $x \otimes y$, with x in $\mathscr{H}$ and y in $\mathscr{K}$, and the algebraic tensor product $\mathscr{H} \otimes_a \mathscr{K}$ as being the linear span of these. Certain identifications must be made in accordance with such rules as $(ax+x')\otimes y = a(x\otimes y)+x'\otimes y$. Providing $\mathscr{H} \otimes_a \mathscr{K}$ with an inner product defined by

$$\langle x \otimes y, z \otimes w\rangle = \langle x, z\rangle\langle y, w\rangle$$

and the associated norm, we can complete $\mathscr{H} \otimes_a \mathscr{K}$ relative to the norm to produce $\mathscr{H} \otimes \mathscr{K}$, the desired Hilbert-space tensor product.

A major purpose to constructing the tensor product $\mathscr{H} \otimes \mathscr{K}$ is to develop a tool for analyzing multilinear mappings from $\mathscr{H} \times \mathscr{K}$, the Cartesian product of $\mathscr{H}$ and $\mathscr{K}$, into some Hilbert space $\mathscr{L}$. Suppose φ is such a mapping (so that $x \to \varphi(x, y_0)$ and $y \to \varphi(x_0, y)$ are linear mappings of $\mathscr{H}$ and $\mathscr{K}$ into $\mathscr{L}$, respectively, for each fixed y_0 in $\mathscr{K}$ and x_0 in $\mathscr{H}$). If no topological assumptions are made about φ, the appropriate tensor product is $\mathscr{H} \otimes_a \mathscr{K}$; there is a unique *linear* mapping η of $\mathscr{H} \otimes_a \mathscr{K}$ into $\mathscr{L}$ with the property that $\varphi(x, y) = \eta(x \otimes y)$ for all x in $\mathscr{H}$ and y in $\mathscr{K}$. Moreover, $\mathscr{H} \otimes_a \mathscr{K}$ is the unique linear space with this "factorization" property. If φ is required to satisfy certain continuity restrictions (a boundedness condition among them), there is a unique *bounded* linear mapping η of $\mathscr{H} \otimes_a \mathscr{K}$ into $\mathscr{L}$ with the property that $\varphi(x, y) = \eta(x \otimes y)$ for all x in $\mathscr{H}$ and y in $\mathscr{K}$.

If A is a bounded linear operator on $\mathscr{H}$, there is a unique linear operator $A \otimes I$ on $\mathscr{H} \otimes \mathscr{K}$, with the same bound as A, determined by $(A \otimes I)(x \otimes y) = (Ax) \otimes y$ for each x in $\mathscr{H}$ and y in $\mathscr{K}$. Similarly, with B a bounded linear operator on $\mathscr{K}$, we can define $I \otimes B$ on $\mathscr{H} \otimes \mathscr{K}$. The algebras $\mathscr{R}_0$ and $\mathscr{S}_0$, alluded to before, are the von Neumann algebras $\{A \otimes I : A \in \mathscr{R}\}$ and $\{I \otimes B : B \in \mathscr{S}\}$. Each element of $\mathscr{R}_0$ commutes with all elements of $\mathscr{S}_0$ (that is, $\mathscr{R}_0 \subseteq \mathscr{S}_0'$). Despite the apparent dependence of $\mathscr{R}\bar{\otimes}\mathscr{S}$ on the underlying Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, if $\mathscr{R}$ is isomorphic to $\mathscr{R}_1$ and $\mathscr{S}$ is isomorphic to $\mathscr{S}_1$, then $\mathscr{R}\bar{\otimes}\mathscr{S}$ is isomorphic to $\mathscr{R}_1\bar{\otimes}\mathscr{S}_1$. Of course, these constructions can be iterated to yield tensor products of any finite number of Hilbert spaces or von Neumann algebras; this iterated process is "associative."

In [27], Murray and von Neumann observe that a factor $\mathscr{M}$ acting on a Hilbert space $\mathscr{H}$ and its commutant $\mathscr{M}'$ generate an algebra $\mathscr{B}$ isomorphic to $\mathscr{M} \otimes_a \mathscr{M}'$. By virtue of the Double Commutant Theorem, the von Neumann algebra generated by $\mathscr{M}$ and $\mathscr{M}'$ (the weak-operator closure of $\mathscr{B}$) is $\mathscr{B}(\mathscr{H})$. Murray and von Neumann note that $\mathscr{M}$ and $\mathscr{M}'$ generate a von Neumann algebra isomorphic to $\mathscr{M}\bar{\otimes}\mathscr{M}'$ just in the case where $\mathscr{M}$ is of type I.

In [**D**], Dixmier poses the problem of determining the commutant $(\mathscr{R}\bar{\otimes}\mathscr{S})'$ of the tensor product $\mathscr{R}\bar{\otimes}\mathscr{S}$. In the finite-dimensional case, it is easy to establish the formula

$$(\mathscr{R}\bar{\otimes}\mathscr{S})' = \mathscr{R}'\bar{\otimes}\mathscr{S}'.$$

It is a simple matter to establish the inclusion

$$\mathscr{R}'\bar{\otimes}\mathscr{S}' \subseteq (\mathscr{R}\bar{\otimes}\mathscr{S})'.$$

The validity of the reverse inclusion was an open question for a number of years. The difficult case occurs when $\mathscr{R}$ and $\mathscr{S}$ are factors of type III (the other cases were settled in several articles). The existence of a * anti-isomorphism between a von Neumann algebra and its commutant in standard form is basic to the problem. The tensor-product, commutant formula was established by Tomita's theorem [**44**, **47**].

If $\mathscr{R}$ and $\mathscr{S}$ are factors, $\mathscr{R}\overline{\otimes}\mathscr{S}$ is a factor. The type of $\mathscr{R}\overline{\otimes}\mathscr{S}$ is certainly determined by the types of $\mathscr{R}$ and $\mathscr{S}$; if either is of infinite type, so is the product, and if both are of type I, so is the product. When $\mathscr{R}$ and $\mathscr{S}$ are of type II, the same is true of $\mathscr{R}\overline{\otimes}\mathscr{S}$. When both $\mathscr{R}$ and $\mathscr{S}$ are of type III, the type of $\mathscr{R}\overline{\otimes}\mathscr{S}$ remained unclarified until Sakai [37] proved that $\mathscr{R}\overline{\otimes}\mathscr{S}$ is a factor of type III.

It is also of interest to define a tensor product for C^*- algebras. The right choices are more involved in this case. Once again, we can consider the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ as acting on the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, form the C^*- algebras $\{A \otimes I : A \in \mathscr{A}\}$ and $\{I \otimes B : B \in \mathscr{B}\}$ on $\mathscr{H} \otimes \mathscr{K}$ and the C^*-algebra $\mathscr{A} \otimes \mathscr{B}$ generated by these C^*-algebras. If $\mathscr{A}_1$ and $\mathscr{B}_1$ are C^*-algebras acting on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{K}_1$, respectively, and we form $\mathscr{A}_1 \otimes \mathscr{B}_1$, then $\mathscr{A} \otimes \mathscr{B}$ is * isomorphic to $\mathscr{A}_1 \otimes \mathscr{B}_1$ when $\mathscr{A}$ and $\mathscr{B}$ are * isomorphic to $\mathscr{A}_1$ and $\mathscr{B}_1$, respectively. The algebraic tensor product $\mathscr{A} \otimes_a \mathscr{B}$ of the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is * isomorphic to the algebra generated by $\mathscr{A}_1$ and $\mathscr{B}_1$ (as a consequence of the results in [27]). Thus the norm on $\mathscr{A} \otimes \mathscr{B}$ induces on $\mathscr{A} \otimes_a \mathscr{B}$ a norm with the properties that $\mathscr{A} \otimes_a \mathscr{B}$ becomes a normed algebra and, for each T in $\mathscr{A} \otimes_a \mathscr{B}$, $\|T^*T\| = \|T\|^2$. A norm on $\mathscr{A} \otimes_a \mathscr{B}$ with these properties is called a *C^*-norm*. The completion of $\mathscr{A} \otimes_a \mathscr{B}$ relative to a C^*-norm is a reasonable choice as the C^*-algebra tensor product of $\mathscr{A}$ and $\mathscr{B}$. The question arises, of course, as to whether or not all these choices are the same; equivalently, is there just one C^*-norm on $\mathscr{A} \otimes_a \mathscr{B}$. In general, there are many C^*-norms on $\mathscr{A} \otimes_a \mathscr{B}$, among them the *spatial norm* σ, described before, obtained from faithful representations of $\mathscr{A}$ and $\mathscr{B}$ on Hilbert spaces. It is a nontrivial fact that σ is the smallest of all the C^*-norms on $\mathscr{A} \otimes_a \mathscr{B}$. Each C^*-norm α is a *cross norm*; that is, $\alpha(A \otimes B) = \alpha(A)\alpha(B)$ for all A in $\mathscr{A}$ and B in $\mathscr{B}$. These properties of tensor products of C^*-algebras and C^*-norms require some clever techniques. They were developed in an article by Takesaki [**46**]. The tensor product and its basic properties were introduced by Turumaru in [**48**].

It is the case, though not trivial, that for $\mathscr{A}$ finite-dimensional or abelian, $\mathscr{A} \otimes_a \mathscr{B}$ admits just on C^*-norm no matter which C^*-algebra $\mathscr{B}$ we choose. In case $\mathscr{A}$ is finite-dimensional, $\mathscr{A} \otimes_a \mathscr{B} = \mathscr{A} \otimes \mathscr{B}$; that is, $\mathscr{A} \otimes_a \mathscr{B}$ is complete relative to its (unique) C^*-norm, the spatial norm. When $\mathscr{A}$ has the property that $\mathscr{A} \otimes_a \mathscr{B}$ has a unique C^*-norm for all C^*-algebras $\mathscr{B}$, we say that $\mathscr{A}$ is a *nuclear* C^*-algebra. It is easy to show that if $\mathscr{A}$ is the norm closure of a family of nuclear C^*-subalgebras directed by inclusion, then $\mathscr{A}$ is nuclear. Thus, if $\mathscr{A}$ has as a norm-dense subalgebra the union of an ascending sequence of C^*-algebras $\mathscr{A}_k$, each containing I and each isomorphic to some $\mathscr{M}_{n(k)}(\mathbf{C})$, then $\mathscr{A}$ is nuclear.

This class of C^*-algebras, the *matricial* C^*-algebras (also called *uhf algebras*), developed and classified by Glimm **[17]**, is a basic class for the study of operator algebras. We shall return to it in the next section. In **[46]**, Takesaki shows that the C^*-algebra generated by $\{L_g : g \in \mathscr{F}_2\}$, in the notation of the group-algebra examples described at the end of **Examples**, is not nuclear.

In **[32]**, von Neumann introduces infinite tensor products of finite type I factors. The infinite tensor product of a family of C^*- algebras can be defined as the (C^*-algebra) inductive limit of the family of (spatial) tensor products of finite subfamilies. With $\{\mathscr{A}_a : a \in \mathbb{A}\}$ a family of C^*-algebras, we denote this tensor product as $\otimes_a \mathscr{A}_a$. If ρ_a is a state of $\mathscr{A}_a$ for each a in $\mathbb{A}$, there is a unique state $\otimes_{a \in \mathbb{A}} \rho_a (= \rho)$ of the tensor product such that

$$\rho(A_1 \otimes \cdots \otimes A_n) = \rho_{a(1)}(A_1) \cdots \rho_{a(n)}(A_n)$$

whenever $A_j \in \mathscr{A}_{a(j)}$, $\{a(1), \ldots, a(n)\} \subseteq \mathbb{A}$. Such states ρ are called *product states* of the tensor product. They occupy a central position in the analysis of infinite tensor products.

§**10. Matricial algebras.** In **[29]**, Murray and von Neumann introduce and study a family of factors of type II_1 very closely related to matrix algebras. In the terminology of Section 9, each of these factors is the weak-operator closure of a matricial C^*-algebra. Following an impressive sequence of technical lemmas, Murray and von Neumann prove that all matricial factors of type II_1 are * isomorphic. They also show that none of the factors $\mathscr{L}_{\mathscr{F}_n}$ is matricial and that $\mathscr{L}_\Pi$ is matricial (cf. Section 5).

If $\mathscr{A}$ is a matricial C^*-algebra and $\{\mathscr{A}_k\}$ is a family of C^*-subalgebras with $\mathscr{A}_k$ isomorphic to $\mathscr{M}_{n(k)}(\mathbf{C})$ such that $\mathscr{A}_k \subseteq \mathscr{A}_{k+1}$, then $n(k)$ divides $n(k+1)$ (recall that $I \in \mathscr{A}_k$). If $\mathscr{M}_{k+1}$ is the subalgebra of $\mathscr{A}_{k+1}$ consisting of those elements that commute with all elements of $\mathscr{A}_k$, then $\mathscr{M}_{k+1}$ is itself isomorphic to the full matrix algebra of order $n(k+1)/n(k)$ and $\mathscr{A}$ is isomorphic to the (infinite) tensor product of the family $\{\mathscr{M}_k\}$ (where $\mathscr{M}_1 = \mathscr{A}_1$). If $n(k')$ can be factored as $m \cdot n$, then $\mathscr{M}_{k'}$ has a subfactor $\mathscr{M}$ of type I_m and the set of elements in $\mathscr{M}_{k'}$ commuting with $\mathscr{M}$ is a subfactor $\mathscr{N}$ of type I_n. Replacing $\mathscr{M}_{k'}$ by $\mathscr{M}$ and $\mathscr{N}$ in the family $\{\mathscr{M}_k\}$ and forming the tensor product, we arrive, again, at the (matricial) C^*-algebra $\mathscr{A}$. Thus the orders $n(k+1)/n(k)$ of the algebras $\mathscr{M}_{k+1}$ are not, themselves, invariants for the algebra, though the "total" power to which a given prime may appear in these orders may be. The remaining problem in analyzing matricial C^*-algebras can be illustrated by the question of whether the tensor product of a countable family of $\mathscr{M}_2(\mathbf{C})$ algebras is isomorphic to the tensor-product of that algebra with $\mathscr{M}_3(\mathbf{C})$—or even whether that algebra contains a C^*-subalgebra (containing I) isomorphic to $\mathscr{M}_3(\mathbf{C})$. Using approximation techniques combined with the trace and projections in the algebra, it can be shown that both questions have a negative answer.

With $\mathscr{A}, \mathscr{A}_k$, and $n(k)$ as before, let m_p be $\sup\{r : (\exists k)(p^r | n(k))\}$, where p is a prime. Thus m_p may be 0 or ∞, as well as any positive integer. Let $n(\mathscr{A})$ be $2^{m_2}3^{m_3}5^{m_5}\cdots$. Glimm **[17]** proves the following result.

THEOREM. *Two matricial C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are * isomorphic if and only if $n(\mathscr{A}) = n(\mathscr{B})$.*

By forming the (infinite) tensor product of matrix algebras of suitable orders, a matricial C^*-algebra $\mathscr{A}$ can be constructed with any given invariant $n(\mathscr{A})$; there are many nonisomorphic matricial C^*-algebras. The result of Murray and von Neumann that all matricial II_1 factors are * isomorphic should be contrasted with this situation for matricial C^*-algebras; a substantial "coalescing" takes place under the "umbrella" of the trace.

If τ_k is the normalized trace on $\mathscr{M}_k$, then $\otimes_k \tau_k$ is the unique (normalized) trace τ on $\mathscr{A}$. The GNS construction applied to τ and $\mathscr{A}$ yields a representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$. It is not immediate, though not difficult, to show that each matricial C^*-algebra is simple. Thus each representation of $\mathscr{A}$ is faithful. For simpler notation, we may think of $\mathscr{A}$ as acting on $\mathscr{H}$ and τ as $\omega_u|\mathscr{A}$ for some unit generating vector u for $\mathscr{A}$. It follows that the weak-operator closure $\mathscr{M}$ of $\mathscr{A}$ is the matricial factor of type II_1 with u as a generating trace vector. Thus $\mathscr{M}'$ is of type II_1 with u as a trace vector.

Viewed in this way, it is natural to study the weak-operator closure of the image of a matricial algebra $\mathscr{A}$ under the representation obtained from the GNS construction applied to a product state of $\mathscr{A}$. As a byproduct of his thesis work on the (infinite) canonical anticommutation relations (CAR), Powers **[34]** did just this, at the same time, bringing into focus the pivotal role of product states. (See also **[35]**.) Powers works with the matricial C^*-algebra $\mathscr{A}$ for which $n(\mathscr{A}) = 2^\infty$. This algebra is the (infinite) tensor product of algebras $\mathscr{M}_k$ each of which is isomorphic to $\mathscr{M}_2(\mathbf{C})$. It is referred to as the *CAR algebra*; its representations are naturally associated with the representations of the CAR. With an isomorphism between $\mathscr{M}_k$ and $\mathscr{M}_2(\mathbf{C})$ specified, Powers considers the state ρ_k of $\mathscr{M}_k$ that assigns to A the value $ta + (1-t)b$, where $t \in [0, \frac{1}{2}]$ and a and b are the upper and lower diagonal entries of the matrix in $\mathscr{M}_k$ corresponding to A. The product state he studies is $\otimes_k \rho_k (= \rho_t)$. He shows that the (faithful) image of $\mathscr{A}$ under the GNS representation corresponding to ρ_t has weak-operator closure a (matricial) factor $\mathscr{M}_t$ of type III. He proves the following result.

THEOREM. *If $t, t' \in (0, \frac{1}{2})$ and $t \neq t'$, then $\mathscr{M}_t$ and $\mathscr{M}_{t'}$ are not isomorphic.*

In the notation of Section 7, $\mathscr{M}_t$ is a factor of type III_λ, where $\lambda = t/1 - t$. Although Pukanszky **[36]** had found two nonisomorphic factors of type III (along the lines of the Murray-von Neumann factors of type II_1 that are not isomorphic) at an early stage, the factors $\mathscr{M}_t$, the *Powers factors*, exhibited a continuum of nonisomorphic matricial factors of type III.

Inspired by the results of Powers, Araki and Woods [**2**, **3**] undertook a detailed analysis of the factors arising from the product states of matricial algebras. It is a fact, requiring an involved argument, that each product state of a matricial algebra $\mathscr{A}$ gives rise to a representation of $\mathscr{A}$ whose image has weak-operator closure a factor. From Araki and Woods, these factors have come to be known as *ITPFI factors* (for 'infinite tensor product of finite type I factors'). Araki and Woods develop the technique of Powers involving the states ρ_k on $\mathscr{M}_k$. As in the situation of quantum statistical mechanics, each state ρ on $\mathscr{M}_n(\mathbf{C})$ has a "density matrix" H (such that $\rho(A) = \tau(HA)$ for each A in $\mathscr{M}_n(\mathbf{C})$). Powers considers the eigenvalues of H. Araki and Woods note that the ratios of the eigenvalues of the density matrices for the 'tail' of the tensor product is what really counts in the arguments of Powers, and they define an "asymptotic ratio set" in terms of which they classify many of the ITPFI factors.

A vexing problem that emerged from these investigations concerned the relation between ITPFI factors of type III and matricial factors of type III. Certainly each ITPFI factor of type III is matricial; but is each matricial factor of type III an ITPFI factor? Krieger [**23**] constructed certain matricial factors and established results indicating that they might not be ITPFI factors. In [**7**], Connes completes the process showing that the Krieger factors are, indeed, not ITPFI.

In connection with the Powers article [**34**], another vexing problem arose that was to puzzle most of the research workers in the area for the next eight years. The von Neumann algebra tensor product of the matricial factor of type II_1 and $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is an infinite-dimensional, separable Hilbert space, is a factor of type II_∞. There is no difficulty in seeing that it is matricial. Are all matricial factors of type II_∞ isomorphic to that tensor product? Connes, along with a host of other experts, set their sights on that question. Connes had established that a number of other fascinating questions about matricial factors are equivalent to this question. Among them are the following problems: Is each subfactor of the matricial II_1 factor either matricial or of type I_n (such a subfactor must be of finite type)? Is the von Neumann group algebra of an amenable *i.c.c.* group matricial? Is a factor $\mathscr{M}$, acting on (separable) $\mathscr{H}$, with the property that there is a conditional expectation mapping $\mathscr{B}(\mathscr{H})$ onto $\mathscr{M}$ (such factors are said to be *injective*) matricial? In [**8**], Connes answers all these questions in the affirmative, and does much more. His argument is a brilliant *tour de force* relying heavily on his penetrating analysis of the automorphism group of certain factors. The intervening years have seen simpler proofs of these results [**21**, **33**] (none of which is very simple!), but the work of Connes [**8**] is so deep and rich in ideas and techniques that it remains a basic resource in the subject.

As noted, Connes had shown that some matricial factors of type III escape the ITPFI net and, consequently, escape the results on classification by Araki-Woods [**3**]. In [**8**], Connes shows that all matricial factors of type III_λ,

$\lambda \in (0, 1)$, are isomorphic. He had noted, in [7], that there are many matricial type III_0 factors depending on the action of an ergodic automorphism of an abelian von Neumann algebra. In [**8**], he completes the reduction to the (measure-theoretic) classification of such actions by showing that each type III_0 matricial factor is the crossed product of an abelian algebra and ($\mathbf{Z}$ through) an ergodic automorphism of the algebra.

The question of whether all matricial factors of type III_1 are isomorphic remained the most tantalizing open question of the subject until Haagerup [**22**] settled it in the affirmative.

§**11. Conclusion.** If we reckon the birth of the theory of operator algebras as occurring in [**31**], the subject is sixty years old. That is surely old enough to permit us to "look back" and assess the legacy of von Neumann in this area of mathematical thought. A substantial part of that legacy must be a collection of mathematical results of surpassing depth and beauty, many of them contributed by von Neumann himself and jointly with F. J. Murray, but a great many of them due to generations of mathematicians that followed von Neumann. Another significant part of the legacy is the host of applications and spin-offs of the theory of operator algebras; this theory touches virtually all areas of mathematics and many areas of physics. Legions of scientists were inspired by the ideas basic to the theory and fascinated by the power and elegance of the subject. A great many highly talented mathematicians and physicists have devoted much time and effort to bringing the subject to its present state. Several of them I do not (nor, I imagine, would von Neumann) hesitate to describe as brilliant. Perhaps that, the people and the enduring inspiration, is the most important part of the legacy; through them the ideas and work of von Neumann remain a living and vital force in present day science.

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