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Local Derivations

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1. INTRODUCTION

In this article, we initiate the study of a class of linear mappings of an (associative) algebra $\mathscr{R}$ into an $\mathscr{R}$-bimodule $\mathscr{M}$. We call these mappings *local derivations*. The defining property of a local derivation α is that for each A in $\mathscr{R}$, there is a derivation δ_A of $\mathscr{R}$ into $\mathscr{M}$ such that $\alpha(A) = \delta_A(A)$. One of our main results (Theorem A) states that each (norm-continuous) local derivation of a von Neumann algebra $\mathscr{R}$ into a dual $\mathscr{R}$-bimodule (see [2] or [4, Exercise 10.5.13] for definitions) is a derivation. In particular each local derivation of $\mathscr{R}$ into itself is a derivation. Employing the Derivation Theorem [5, 6], one concludes (Theorem B) that a norm-continuous, linear mapping of R into itself that maps each A in $\mathscr{R}$ into a commutator $(A, T_A]$ for some T_A in $\mathscr{R}$ has the form ad $T_0 | \mathscr{R}$ for some T_0 in $\mathscr{R}$. (This may be unexpected even when $\mathscr{R}$ is the algebra of all complex $n \times n$ matrices.) The module formulation is a worthwhile extension. For example, it permits us to draw a similar conclusion for mappings of $\mathscr{R}$ into $\mathscr{B}(\mathscr{H})$, the algebra of all bounded linear transformations on the complex Hilbert space $\mathscr{H}$ on which $\mathscr{R}$ acts.

The results just noted appear in Section 2; they rely on the analytic structure of von Neumann algebras. Sections 3 and 4 deal with purely algebraic situations involving local derivations. In Section 3, we present an example constructed by C. U. Jensen (in response to a question raised during a lecture in Copenhagen in 1986) of an algebra over the complex numbers $\mathbb{C}$ with many local derivations that are not derivations. If the conclusions of Theorem A or B were valid for all algebras, those results would have, at best, temporary status. In discussions with Irving Kaplansky (at MSRI, Berkeley, in January of 1985), Kaplansky produced first an infinite-dimensional algebra, then an algebra of dimension 24 over a field of

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characteristic 2 for which the conclusions fail. The computations in those examples are daunting.

C. U. Jensen's example uses the algebra of rational functions (over $\mathbb{C}$). This adds further interest to determining the local derivations of the polynomial rings in several variables over $\mathbb{C}$. In Section 4, we show, first, for $\mathbb{C}[x]$ and then for $\mathbb{C}[x_1, ..., x_n]$ that each local derivation is a derivation (Theorems C and D and Corollaries 17 and 18).

This investigation grew out of the cohomology program for operator algebras introduced and studied in [1–3]. That program requires the analysis of (norm-continuous) linear mappings of one operator algebra into another. In particular, such a mapping must be decomposed into other such mappings one term of which is a derivation (a 1-cocycle). These decompositions are constructed by the use of meaning processes. From a purely algebraic point of view, it is not difficult to decide which meaning process to use to produce the derivation component appropriate to a particular problem. For example, this is the case with the question raised by J. R. Ringrose and this author twenty years ago of whether or not the (norm-continuous) cohomology groups of a von Neumann algebra with coefficients in itself are 0. Our conjecture is that they are 0, but the question remains open (though some brilliant and decisive work on this question has been and is currently being done by Erik Christensen and Allan Sinclair). The difficulties lie precisely in the subtle interplay among the various topological algebraic structures that can be imposed on the (norm-continuous) 1-cochains and the resulting convergence questions for the meaning processes. A criterion that allows us to construct or recognize derivations in an element-by-element manner could prove useful for this and related problems. Theorem A provides such a criterion.

Numerous topics remain to be studied. Local higher cohomology (for example, local 2-cocycles) should be examined. Local derivations of other classes of algebras should be studied (a further publication will deal with some other algebras). Some finite-dimensional examples over $\mathbb{C}$ should be constructed with local derivations that are not derivations. (Note that if such an algebra is semi-simple, it is isomorphic to a von Neumann algebra and is included under Theorem A.) There are questions of a (possibly, extended) homological algebraic framework for such mappings and its application to "appropriate" dense subalgebras of operator algebras. This entails the study of "unbounded" local derivations and the related questions of "automatic continuity" for everywhere defined mappings.

We are indebted to J. R. Ringrose for the conversations during our collaboration on the cohomology of operator algebras that led us to the results appearing in the next section and to the NSF for partial support.

2. Local Derivations of von Neumann Algebras

Throughout this section, $\mathscr{R}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ with identity operator I and $\mathscr{M}$ is a dual $\mathscr{R}$-bimodule. We shall see (Lemma 3) that the conclusion of Theorem A is valid for a general $\mathscr{R}$-bimodule if it is valid for a *unital* dual $\mathscr{R}$-bimodule so that it suffices to deal with the case where $\mathscr{M}$ is unital.

We use the notation introduced in [2]. Thus $C_c^1(\mathscr{R}, \mathscr{M})$ is the linear space of norm-continuous linear mappings of $\mathscr{R}$ into $\mathscr{M}$, the 1-cochains on $\mathscr{R}$ with coefficients in $\mathscr{M}$, and $Z_c^1(\mathscr{R}, \mathscr{M})$ is the linear subspace of $C_c^1(\mathscr{R}, \mathscr{M})$ consisting of those mappings δ such that

$$A\delta(B) - \delta(AB) + \delta(A)B = 0 \qquad (A, B \in \mathscr{R}).$$

Such mappings δ are the 1-cocycles on $\mathscr{R}$ with coefficients in $\mathscr{M}$, the *derivations* of $\mathscr{R}$ into $\mathscr{M}$.

THEOREM A. *If $\delta \in C_c^1(\mathscr{R}, \mathscr{M})$ and for each A in $\mathscr{R}$ there is a δ_A in $Z_c^1(\mathscr{R}, \mathscr{M})$ such that $\delta(A) = \delta_A(A)$, then $\delta \in Z_c^1(\mathscr{R}, \mathscr{M})$.*

We refer to a mapping δ that agrees at each A with some derivation (varying with A) as a *local derivation.* Theorem A states that each local derivation is a derivation under the given conditions. Its proof is organized as five lemmas with the concluding argument following Lemma 16 and Lemma 4 requiring eleven sublemmas. Lemmas 1 and 2 apply without the assumption that $\mathscr{M}$ is unital. Lemma 3 uses Lemma 2 to reduce the general case to the case where $\mathscr{M}$ is unital. The notation δ will be used throughout for our local derivation.

LEMMA 1. *With E and F commuting idempotent elements in $\mathscr{R}$, we have that $\delta(EF) = \delta(E)F + E\delta(F)$.*

Proof. If $\delta_0 \in Z_c^1(\mathscr{R}, \mathscr{M})$, then

$$\delta_0(E) = \delta_0(E^2) = E\delta_0(E) + \delta_0(E)E.$$

Letting E act on the left and right sides of this equality, we have that $E\delta_0(E)E = E\delta_0(E)E + E\delta_0(E)E$, whence $E\delta_0(E)E = 0$. If δ_0 has been chosen such that $\delta_0(E) = \delta(E)$, then

$$E\delta(E)E = E\delta_0(E)E = 0 = (I-E)\,\delta_0(E)(I-E)$$
$$= (I-E)\,\delta(E)(I-E), \tag{1}$$
$$\delta(E) = \delta(E)E + E\delta(E). \tag{2}$$

Note that, if $EF=0$, then $\delta(E)F+E\delta(F)=E\delta(E)F+E\delta(F)F$ from (2); while, from (1), $E\delta(F)E$ and $E\delta(E)E$ are 0, whence

$$0=E(E+F)\,\delta(E+F)(E+F)=E\delta(E)F+E\delta(F)F.$$

Thus $0=\delta(EF)=\delta(E)F+E\delta(F)$ in this case.

If $EF=E$, then from (2) and what we have just proved, we have

$$\begin{aligned}\delta(E)F+E\delta(F)&=\delta(E)F+E\delta(E)+E\delta(F-E)\\&=\delta(E)F+E\delta(E)-\delta(E)(F-E)\\&=E\delta(E)+\delta(E)E\\&=\delta(E)\\&=\delta(EF).\end{aligned}$$

For an arbitrary commuting pair of idempotents E and F in $\mathscr{R}$, we have

$$\begin{aligned}\delta(E)F+E\delta(F)&=\delta(E)F+(E-EF)\,\delta(F)+EF\delta(F)\\&=\delta(E)F-\delta(E-EF)F+\delta(EF)-\delta(EF)F\\&=\delta(EF).\quad\blacksquare\end{aligned}$$

Lemma 2. *If $\mathscr{C}$ is an abelian von Neumann subalgebra of $\mathscr{R}$, then $\delta\,|\,\mathscr{C}\in Z_c^1(\mathscr{C},\mathscr{M})$.*

Proof. If $A=\sum_{j=1}^n a_jE_j$ and $B=\sum_{k=1}^m b_kF_k$, where E_j and F_k are projections in $\mathscr{C}$, then from Lemma 1,

$$\begin{aligned}\delta(A)B+A\delta(B)&=\sum_{j,k} a_jb_k[\delta(E_j)\,F_k+E_j\,\delta(F_k)]\\&=\delta\left(\sum_{j,k} a_jb_kE_jF_k\right)\\&=\delta(AB).\end{aligned}$$

Since the set of elements in $\mathscr{C}$ that are finite linear combinations of projections in $\mathscr{C}$ is norm dense in $\mathscr{C}$, and $A\to\delta(A)B+A\delta(B)-\delta(AB)$ is norm continuous and vanishes on this set when B lies in this set, it vanishes for all A in $\mathscr{C}$ when B lies in this set. Again, $B\to\delta(A)B+A\delta(B)-\delta(AB)$ is norm continuous for each A in $\mathscr{C}$; so that $\delta(AB)=\delta(A)B+A\delta(B)$ for all A and B in $\mathscr{C}$. Thus $\delta\,|\,\mathscr{C}\in Z_c^1(\mathscr{C},\mathscr{M})$. $\blacksquare$

Lemma 3. *If the statement of Theorem* A *is valid for each unital dual $\mathscr{R}$-bimodule, then it is valid for each (general) dual $\mathscr{R}$-bimodule.*

Proof. Suppose δ is a (norm-continuous) local derivation of $\mathscr{R}$ into a (general) dual $\mathscr{R}$-bimodule $\mathscr{M}$. Define two idempotents L and R on $\mathscr{M}$ by $L(m) = Im$ and $R(m) = mI$. Denote by ι the identity transformation on $\mathscr{M}$. Since $LR = RL$, we have that LR is an idempotent. By definition of "dual $\mathscr{R}$-bimodule," each of L and R is continuous mapping of $\mathscr{M}$ into itself provided with its w^*-topology. Thus LR is a w^*-continuous idempotent on $\mathscr{M}$, and its range $\mathscr{M}_0$, the null space of $\iota - LR$, is a w^*-closed subspace of $\mathscr{M}$. Let $\mathscr{M}_*$ be the predual of $\mathscr{M}$ and $\mathscr{M}_*^0$ be the subspace of $\mathscr{M}_*$ that $\mathscr{M}_0$ annihilates. Of course, $\mathscr{M}_*^0$ is norm closed in $\mathscr{M}_*$. From [4, Corollary 1.2.13, and Proposition 1.3.5], $\mathscr{M}_0$ is the annihilator of $\mathscr{M}_*^0$ in $\mathscr{M}$. Thus $\mathscr{M}_0$ is the (Banach) dual space of $\mathscr{M}_*/\mathscr{M}_*^0$ (cf. [4, Exercise 1.9.10(i)]). Since

$$AImIB = IAmBI \in \mathscr{M}_0 \qquad (m \in \mathscr{M})$$

and $IImI = ImII = ImI$, $\mathscr{M}_0$ is a unital dual $\mathscr{R}$-bimodule.

It follows from (1) of Lemma 1 that $I\delta(I)I = 0$. From Lemma 2, we have that $\delta(AI) = \delta(A)I + A\delta(I)$ (first for self-adjoint A, then, by linearity of δ, for all A in $\mathscr{R}$). We show that $LR\delta$ is a local derivation of $\mathscr{R}$ into the unital dual $\mathscr{R}$-bimodule $\mathscr{M}_0$. Since δ is a local derivation, for a given A in $\mathscr{R}$, there is a derivation δ_A of $\mathscr{R}$ into $\mathscr{M}$ such that $\delta(A) = \delta_A(A)$. We have that

$$\begin{aligned} LR\delta_A(A_1 B) &= I\delta_A(A_1)B + A_1\,\delta_A(B)I \\ &= I\delta_A(A_1)\,IB + A_1 I\delta_A(B)I \\ &= LR\delta_A(A_1)B + A_1 LR\delta_A(B). \end{aligned}$$

Thus $LR\delta_A$ is a derivation of $\mathscr{R}$ into $\mathscr{M}_0$ for which $LR\delta_A(A) = I\delta_A(A)I = I\delta(A)I = LR\delta(A)$. It follows that $LR\delta$ is a local derivation of $\mathscr{R}$ into $\mathscr{M}_0$. From our hypothesis, $LR\delta$ is a derivation of $\mathscr{R}$ into $\mathscr{M}_0$.

Let η be $(\iota - L)\,R\delta$, and note that

$$\begin{aligned} \eta(A) &= \delta(A)I - I\delta(A)I \\ &= \delta(IA)I - I\delta(A)I \\ &= \delta(I)AI + I\delta(A)I - I\delta(A)I \\ &= \delta(I)A. \end{aligned}$$

Now $A\delta(I)B = A(I\delta(I)I)B = 0$, whence

$$\eta(AB) = \delta(I)AB = \delta(I)\,AB + A\delta(I)B = \eta(A)B + A\eta(B).$$

Thus η is a derivation of $\mathscr{R}$ into $\mathscr{M}$. Similarly, $L(\iota - R)\,\delta(A) = A\delta(I)$, from which we have that $L(\iota - R)\delta$ is a derivation of $\mathscr{R}$ into $\mathscr{M}$. In addition, we have

$$\begin{aligned}
&(\iota - L)(\iota - R)\,\delta(A) \\
&\quad = \delta(IA) - \delta(IA)I - I\delta(A) + I\delta(A)I \\
&\quad = \delta(I)A + I\delta(A) - [\delta(I)A + I\delta(A)]I - I\delta(A) + I\delta(A)I \\
&\quad = 0.
\end{aligned}$$

Since $\delta = LR\delta + (\iota - L)\,R\delta + L(\iota - R)\delta + (\iota - L)(\iota - R)\delta$, we have that δ is a sum of derivations of $\mathscr{R}$ into $\mathscr{M}$. ∎

We assume, henceforth, that our dual $\mathscr{R}$-bimodule $\mathscr{M}$ is unital.

Lemma 4. *If E and F are projections such that $EF = 0$, and V is an operator in $\mathscr{R}$ such that $V^*V = E$ and $VV^* = F$, then $\delta \mid \mathscr{R}_0 \in Z_c^1(\mathscr{R}_0, \mathscr{M})$, where $\mathscr{R}_0$ is the algebra generated by E, F, V, and V^*.*

We establish Lemma 4 with the aid of the following sublemmas.

Sublemma 5. *If A, B, C, in $\mathscr{R}$, are such that $0 = AB = BC$, then $A\delta(B)C = 0$.*

Proof. With δ_0 a derivation of $\mathscr{R}$ into $\mathscr{M}$ such that $\delta_0(B) = \delta(B)$, we have

$$\begin{aligned}
0 = \delta_0(ABC) &= \delta_0(A)\,BC + A\delta_0(B)C + AB\delta_0(C) \\
&= A\delta_0(B)C = A\delta(B)C. \quad ∎
\end{aligned}$$

Applying sublemma 5, we have

$$\begin{aligned}
F\delta(V^*)(I - F) = F\delta(V^*)E = 0, \qquad & F\delta(V^*) = F\delta(V^*)F \\
(I - E)\,\delta(V^*)E = 0, \qquad & \delta(V^*)E = E\delta(V^*)E,
\end{aligned} \tag{3}$$

where E, F, and V are as in the statement of Lemma 4.

Sublemma 6. $F\delta(V^*)F = F\delta(E)\,V^*$.

Proof. Note that $(E + V^*)^2 = E + V^*$. From Lemma 1,

$$\begin{aligned}
\delta(E + V^*) &= \delta((E + V^*)^2) \\
&= \delta(E + V^*)[E + V^*] + [E + V^*]\,\delta(E + V^*).
\end{aligned} \tag{4}$$

Thus

$$F\delta(E+V^*)F = F[\delta(E)+\delta(V^*)]\, V^* = F\delta(E)\, V^* + F\delta(V^*)\, EV^*$$
$$= F\delta(E)\, V^*,$$

since $F\delta(V^*)E = 0$ from (3). From Sublemma 5, $F\delta(E)F = 0$. Hence $F\delta(V^*)F = F\delta(E)\, V^*$. ∎

SUBLEMMA 7. $E\delta(V^*)E = -V^*\delta(E)E$.

Proof. Multiplying (4) on the right by E, we have that

$$[E+V^*]\, \delta(E+V^*)E = 0.$$

From (1), $E\delta(E)E = 0$. Thus

$$E\delta(V^*)E = E[\delta(E)+\delta(V^*)]\, E + (E+V^*)][\delta(E)+\delta(V^*)]\, E$$
$$= E\delta(E)\, E + E\delta(V^*)E + E\delta(E)\, E + E\delta(V^*)\, E$$
$$+ V^*\delta(E)\, E + V^*\delta(V^*)\, E.$$

Thus, from (3),

$$E\delta(V^*)E = -V^*\delta(E)E - V^*F\delta(V^*)E = -V^*\delta(E)E. \quad ∎$$

SUBLEMMA 8. $\delta(EV^*) = \delta(V^*) = \delta(E)\, V^* + E\delta(V^*)$.

Proof. From (3) and Sublemma 6, we have that

$$F\delta(V^*) = F\delta(V^*)F = F\delta(E)\, V^* = F[\delta(E)\, V^* + E\delta(V^*)]. \tag{5}$$

From (1), we have

$$E\delta(V^*) = E\delta(E)\, EV^* + E\delta(V^*) = E[\delta(E)\, V^* + E\delta(V^*)]. \tag{6}$$

From Sublemma 5,

$$0 = (I-E-F)\, \delta(V^*)E = (I-E-F)\, \delta(V^*)\, V^*$$
$$= (I-E-F)\, \delta(E)(I-E). \tag{7}$$

Thus (recall that $\mathscr{M}$ is a *unital* $\mathscr{R}$-bimodule),

$$(I-E-F)\, \delta(E) = (I-E-F)\, \delta(E)E. \tag{8}$$

From (4), we have that

$$(I-E-F)[\delta(E)+\delta(V^*)] = (I-E-F)[\delta(E)+\delta(V^*)](E+V^*).$$

Combining this equality with (7) and (8), we have

$$(I-E-F)[\delta(E)+\delta(V^*)]=(I-E-F)\,\delta(E)+(I-E-F)\,\delta(E)\,V^*.$$

Hence

$$(I-E-F)\,\delta(V^*)=(I-E-F)[\delta(E)\,V^*+E\delta(V^*)]. \tag{9}$$

Our assertion follows by adding (5), (6), and (9). ∎

SUBLEMMA 9. $\delta(V^*F)=\delta(V^*)=V^*\delta(F)+\delta(V^*)F.$

Proof. From (1), we have that

$$\delta(V^*)F=\delta(V^*)F+V^*F\delta(F)F=[\delta(V^*)F+V^*\delta(F)]F. \tag{10}$$

From Sublemma 5, $V^*\delta(E)(I-E)=0$, whence $V^*\delta(E)=V^*\delta(E)E$. Using this and Lemma 1, we have

$$\begin{aligned}[\delta(V^*)F+V^*\delta(F)]E=V^*\delta(F)E=-V^*F\delta(E)=-V^*\delta(E)\\ =-V^*\delta(E)E.\end{aligned}$$

Thus, from (3) and Sublemma 7,

$$\delta(V^*)E=E\delta(V^*)E=-V^*\delta(E)E=[\delta(V^*)F+V^*\delta(F)]E. \tag{11}$$

Since $(F+V^*)^2=F+V^*$, we have, as with (4), that

$$\delta(F+V^*)=\delta(F+V^*)(F+V^*)+(F+V^*)\,\delta(F+V^*).$$

Using this with Sublemma 5, we have that

$$\begin{aligned}[\delta(F)+\delta(V^*)](I-E-F)&=(F+V^*)\,\delta(F+V^*)(I-E-F)\\ &=\delta(F)(I-E-F)+V^*\delta(F)(I-E-F).\end{aligned}$$

Hence

$$\delta(V^*)(I-E-F)=[\delta(V^*)F+V^*\delta(F)](I-E-F). \tag{12}$$

Our sublemma follows by adding (10), (11), and (12). ∎

SUBLEMMA 10. $0=\delta(FV^*)=\delta(F)\,V^*+F\delta(V^*)$ *and* $0=\delta(V^*E)=\delta(V^*)E+V^*\delta(E)$.

Proof. From Lemma 1, (3), and Sublemma 6, we have

$$\begin{aligned}\delta(F)\, V^* + F\delta(V^*) &= \delta(F)\, EV^* + F\delta(V^*)F \\ &= -F\delta(E)\, V^* + F\delta(E)\, V^* \\ &= 0.\end{aligned}$$

From (3) and Sublemmas 5 and 7 , we have that

$$\delta(V^*)E + V^*\delta(E) = E\delta(V^*)E + V^*\delta(E)E = 0. \quad ▮$$

SUBLEMMA 11.

$$\delta(FV) = \delta(V) = \delta(F)\, V + F\delta(V),$$

$$\delta(VE) = \delta(V) = \delta(V)E + V\delta(E), \quad 0 = \delta(EV) = \delta(E)\, V + E\delta(V),$$

and

$$0 = \delta(VF) = \delta(V)F + V\delta(F).$$

Proof. Replace V^* by V, E by F, and F by E, and apply Sublemmas 8, 9, and 10, respectively. ▮

SUBLEMMA 12. $F\delta(V)E = -V\delta(V^*)\, V.$

Proof. Let δ_0 be a derivation of $\mathscr{R}$ into $\mathscr{M}$ such that $\delta_0(V + V^*) = \delta(V + V^*)$. Since $(V^* + V)^2 = E + F$, we have

$$\delta_0(E+F) = \delta_0(V+V^*)[V+V^*] + [V+V^*]\,\delta_0(V+V^*),$$

whence

$$\begin{aligned}0 &= [E+F]\,\delta_0(E+F)[E+F] \\ &= [E+F]\,\delta_0(V+V^*)[V+V^*] + [V+V^*]\,\delta_0(V+V^*)[E+F].\end{aligned}$$

Thus

$$\begin{aligned}0 &= F[E+F]\,\delta(V+V^*)[V+V^*]\,V + F[V+V^*]\,\delta(V+V^*)[E+F]\,V \\ &= F\delta(V+V^*)E + V\delta(V+V^*)\,V.\end{aligned}$$

From Sublemma 5, $0 = F\delta(V^*)E = V\delta(V)\,V$, whence

$$F\delta(V)E = -V\delta(V^*)\,V. \quad ▮$$

SUBLEMMA 13. $\delta(V^*V) = \delta(E) = \delta(V^*)\,V + V^*\delta(V).$

Proof. From Sublemmas 8, 11, and 12, we have

$$\begin{aligned}\delta(V^*)V+V^*\delta(V)&=\delta(EV^*)V+V^*\delta(VE)\\&=[\delta(E)V^*+E\delta(V^*)]V+V^*[\delta(V)E+V\delta(E)]\\&=\delta(E)E+E\delta(V^*)V+V^*\delta(V)E+E\delta(E)\\&=\delta(E)+E\delta(V^*)V+V^*F\delta(V)E\\&=\delta(E)+E\delta(V^*)V-V^*V\delta(V^*)V\\&=\delta(E).\ \blacksquare\end{aligned}$$

SUBLEMMA 14. $\delta(VV^*)=\delta(F)=\delta(V)V^*+V\delta(V^*)$.

Proof. Replace E by F, V by V^*, and V^* by V, in the statement of Sublemma 13. ▮

SUBLEMMA 15. $\delta(V^2)=0=\delta(V)V+V\delta(V)$ *and* $\delta(V^{*2})=0=\delta(V^*)V^*+V^*\delta(V^*)$.

Proof. Let δ_0 be a derivation of $\mathscr{R}$ into $\mathscr{M}$ such that $\delta(V)=\delta_0(V)$. Then $V^2=0$ and

$$0=\delta(V^2)=\delta_0(V^2)=\delta_0(V)V+V\delta_0(V)=\delta(V)V+V\delta(V).$$

Replacing V by V^* in the foregoing, we have the second assertion of this sublemma. ▮

Proof of Lemma 4. From Lemma 2 and Sublemmas 8, 9, 10, 11, 13, 14, and 15, δ has the multiplicative derivation property for the linear generators E, F, V, and V^* of $\mathscr{R}_0$. Thus $\delta\,|\,\mathscr{R}_0\in Z_c^1(\mathscr{R}_0,\mathscr{M})$. ▮

LEMMA 16. *With* $\mathscr{R}_0$ *as in Lemma* 4 *and* $\mathscr{C}$ *an abelian von Neumann subalgebra of* $\mathscr{R}\cap\mathscr{R}_0'$, *we have that* $\delta\,|\,\mathscr{R}_1\in Z_c^1(\mathscr{R}_1,\mathscr{M})$, *where* $\mathscr{R}_1$ *is the von Neumann algebra generated by* $\mathscr{R}_0$ *and* $\mathscr{C}$.

Proof. Let E and F be projections, one in $\mathscr{C}$ and the other in $\mathscr{R}_0$. Since $EF=FE$, we have that $\delta(EF)=\delta(E)F+E\delta(F)$ from Lemma 1. Now each element of $\mathscr{R}_0$ and each element of a norm-dense subset of $\mathscr{C}$ are finite linear combinations of projections in $\mathscr{R}_0$ and $\mathscr{C}$, respectively, and δ is norm continuous on $\mathscr{R}$. Thus $\delta(CA)=\delta(C)A+C\delta(A)=\delta(AC)=\delta(A)C+A\delta(C)$, where $C\in\mathscr{C}$ and $A\in\mathscr{R}_0$. As $\mathscr{R}_1$ consists of finite linear combinations of elements of the form CA, with C in $\mathscr{C}$ and $A=I$ or A in $\mathscr{R}_0$, it remains to show that

$$\delta(CAC'A')=\delta(CA)C'A'+CA\delta(C'A'). \tag{13}$$

Now,

$$\begin{aligned}\delta(CAC'A') &= \delta(CC'AA')\\ &= \delta(C)\,C'AA' + C\delta(C')\,AA'\\ &\quad + CC'\delta(A)A' + CC'A\delta(A'),\end{aligned}$$

while

$$\begin{aligned}&\delta(CA)\,C'A' + CA\delta(C'A')\\ &\quad = \delta(C)\,AC'A' + C\delta(A)\,C'A' + CA\delta(C')\,A' + CAC'\delta(A').\end{aligned}$$

Since

$$\begin{aligned}C\delta(C')\,AA' + CC'\delta(A)\,A' &= C[\delta(C')A + C'\delta(A)]A'\\ &= C[\delta(A)\,C' + A\delta(C')]\,A',\end{aligned}$$

(13) follows. ∎

Proof of Theorem A. Since δ is norm continuous and the set of finite linear combinations of projections in $\mathscr{R}$ is norm dense in $\mathscr{R}$ (cf. [4, Theorem 5.2.2(v)]), it will suffice to show that $\delta(EF) = \delta(E)F + E\delta(F)$ for each pair of projections E and F in $\mathscr{R}$. The von Neumann algebra $\mathscr{R}_1$ generated by E and F is either abelian, of type I_2, or the direct sum of an abelian von Neumann algebra and one of type I_2 (cf. [4, Exercise 12.4.11]). If $\mathscr{R}_1$ is not abelian, it is generated by its center and a subalgebra $\mathscr{R}_0$ isomorphic to a factor of type I_2. Lemma 16 applies and $\delta \mid \mathscr{R}_1 \in Z_c^1(\mathscr{R}_1, \mathscr{M})$. Thus $\delta(EF) = \delta(E)F + E\delta(F)$. ∎

THEOREM B. *If δ is a norm-continuous linear mapping of a von Neumann algebra $\mathscr{R}$ into itself such that, for each A in $\mathscr{R}$, there is a T_A in $\mathscr{R}$ for which $\delta(A) = AT_A - T_A A$, then there is a T in $\mathscr{R}$ such that $\delta(A) = AT - TA$ for all A in $\mathscr{R}$.*

Proof. From the Derivation Theorem [5, 6], each derivation of $\mathscr{R}$ into itself has the form $A \to AT - TA$ for some T in $\mathscr{R}$. Now apply Theorem A. ∎

3. AN EXAMPLE

We present (a slightly modified version of) C. U. Jensen's example. He exhibits an infinite-dimensional, commutative algebra over $\mathbb{C}$, the complex numbers, admitting a mapping that is a local derivation but not a derivation. The algebra is $\mathbb{C}(x)$, the rational functions in the variable x over $\mathbb{C}$. Let $\mathbb{C}[x]$ be the subalgebra of polynomials. We note certain facts.

a. The derivations of $\mathbb{C}(x)$ into itself are mappings of the form $f \to gf'$ for some g in $\mathbb{C}(x)$, where f' is the usual derivative of f. Such a mapping is a derivation of $\mathbb{C}(x)$. Let δ be a derivation of $\mathbb{C}(x)$ into $\mathbb{C}(x)$ and let g be $\delta(x)$. If $p \in \mathbb{C}[x]$, then $\delta(p) = gp'$ (applying the multiplicative property of the derivation). At the same time, if $p \neq 0$, then

$$0 = \delta(1) = \delta(pp^{-1}) = \delta(p)\, p^{-1} + p\delta(p^{-1}),$$

whence $\delta(p^{-1}) = -\delta(p)\, p^{-2} = -gp'p^{-2}$. Thus, with p and q in $\mathbb{C}[x]$,

$$\begin{aligned}\delta(pq^{-1}) &= \delta(p)\, q^{-1} + p\delta(q^{-1}) = gp'q^{-1} - gpq'q^{-2}\\ &= g[p'q - pq']\, q^{-2} = g[pq^{-1}]'.\end{aligned}$$

b. The local derivations of $\mathbb{C}(x)$ are the linear mappings that annihilate the constants. If α is a local derivation, then for each c in $\mathbb{C}$, there is a derivation δ of $\mathbb{C}(x)$ such that $\alpha(c) = \delta(c) = 0$. Suppose, now, that α is a linear mapping of $\mathbb{C}(x)$ into $\mathbb{C}(x)$ that annihilates the constants. Of course α agrees with every derivation on all constants. If f, in $\mathbb{C}(x)$, is not a constant, then $f' \neq 0$. Let $\delta(h)$ be $(\alpha(f)/f')\, h'$. Then δ is a derivation of $\mathbb{C}(x)$ into $\mathbb{C}(x)$, and $\delta(f) = \alpha(f)$. Thus α is a local derivation of $\mathbb{C}(x)$.

c. We display a local derivation of $\mathbb{C}(x)$ into itself that is not a derivation. With $\mathbb{C}(x)$ considered as a vector space over $\mathbb{C}$, the 2-dimensional subspace X generated by 1 and x has a complement Y. Let α be the projection of $\mathbb{C}(x)$ on Y along X. Then α annihilates the constants, whence α is a local derivation from *b*. If α were a derivation, then from *a*, $\alpha(f)$ would be $\alpha(x) f'$, which is a 0 since $\alpha(x) = 0$. As $\alpha \neq 0$, α is not a derivation.

4. Polynomial Algebras

With C. U. Jensen's example in mind, it is of interest to study the local derivations of $\mathbb{C}[x]$. In the theorem that follows, we consider local derivations of $\mathbb{C}[x]$ into $\mathbb{C}[x, y, ..., w]$, the polynomial ring over $\mathbb{C}$ in an arbitrary set of variables $\{x, ..., w\}$.

THEOREM C. *Each local derivation of* $\mathbb{C}[x]$ *into* $\mathbb{C}[x, y, ..., w]$ *is a derivation.*

Proof. Let α be a local derivation of $\mathbb{C}[x]$ into $\mathbb{C}[x, y, ..., w]$. For each positive integer j, there is a derivation δ_j of $\mathbb{C}[x]$ into $\mathbb{C}[x, y, ..., w]$ such that

$$\alpha(x^j) = \delta_j(x^j) = jx^{j-1}\,\delta_j(x) = jx^{j-1}g_j,$$

where $g_j(=\delta_j(x))$ is some element of $\mathbb{C}[x, y, ..., w]$. Similarly, for a given p in $\mathbb{C}[x]$, there is a g_p in $\mathbb{C}[x, y, ..., w]$ such that $\alpha(p) = p'g_p$, where p' is the usual derivative of p. Thus for some h in $\mathbb{C}[x, y, ..., w]$ and each non-zero a in $\mathbb{C}$,

$$\begin{aligned} 2(x^j + ax^k)(jx^{j-1} + kax^{k-1})h &= \alpha([x^j + ax^k]^2) \\ &= \alpha(x^{2j} + 2ax^{j+k} + a^2x^{2k}) \\ &= 2jx^{2j-1}g_{2j} + 2(j+k)\,ax^{j+k-1}g_{j+k} \\ &\quad + 2ka^2x^{2k-1}g_{2k}, \end{aligned} \tag{14}$$

when $j, k \geqslant 1$. Suppose b in $\mathbb{C}$ is such that $b^{k-j} = -a^{-1}$. Then b is a root of the left side of (14). The right side of (14) can be rewritten as

$$2x^{2j-1}(jg_{2j} + (j+k)\,ax^{k-j}g_{j+k} + ka^2x^{2(k-j)}g_{2k}),$$

which must vanish when x is replaced by b. Thus

$$jg_{2j}(b, y, ..., w) + kg_{2k}(b, y, ..., w) - (j+k)\,g_{j+k}(b, y, ..., w) = 0. \tag{15}$$

Since a does not appear in (15) and b satisfies only $b^{k-j} = -a^{-1}$, if we choose a to be $-b^{j-k}$ for an arbitrary non-zero choice of b in $\mathbb{C}$, (15) is satisfied. Thus

$$jg_{2j} + kg_{2k} - (j+k)\,g_{j+k} = 0 \tag{16}$$

holds identically, when $j, k \geqslant 1$.

Considering $\alpha([1 + ax^k]^2)$ and proceeding as in the computation of (14), we conclude that $ax^{k-1}g_k + a^2x^{2k-1}g_{2k}$ vanishes when x is replaced by b in $\mathbb{C}$ such that $b^k = -a^{-1}$. The same is true of $x(ax^{k-1}g_k + a^2x^{2k-1}g_{2k})$. Arguing as at the end of the preceding paragraph, we have that $g_k = g_{2k}$ for each positive integer k. Thus $g_1 = g_2$.

With $k+1$ in place of j in (16), we have

$$(k+1)\,g_{2k+2} + kg_{2k} = (2k+1)\,g_{2k+1}. \tag{17}$$

With 1 for k in (17), we have that $2g_4 + g_2 = 3g_3$. But $g_2 = g_4$. Thus $g_2 = g_3$. Suppose we have established that $g_1 = g_2 = \cdots = g_{2k+1}$. Then $g_{2k} = g_{2k+1}$ (if $k \geqslant 1$) and from (17), it follows that $(k+1)\,g_{2k+2} =$

$(k+1)\, g_{2k+1}$. Thus $g_1 = g_2 = \cdots = g_{2k+2}$. Now $g_{2k+4} = g_{k+2}$, from the preceding paragraph, and we have just shown that $g_{k+2} = g_{2k+2}$. Thus from (17),

$$(2k+3)\, g_{2k+3} = (k+1)\, g_{2k+2} + (k+2)\, g_{2k+4} = (2k+3)\, g_{2k+2}.$$

Hence $g_1 = g_2 = \cdots = g_{2k+1} = g_{2k+2} = g_{2k+3}$. It follows by induction that $g_j = g_k$ for all non-zero j and k. If p in $\mathbb{C}[x]$ is $a_n x^n + \cdots + a_1 x + a_0$, then

$$\alpha(p) = n a_n x^{n-1} g_n + \cdots + a_1 g_1 = p' g_1.$$

Thus α is the derivation $p \to p' g_1$ of $\mathbb{C}[x]$ into $\mathbb{C}[x, y, ..., w]$. ∎

COROLLARY 17. *Each local derivation of* $\mathbb{C}[x]$ *into itself is a derivation.*

In the theorem that follows, we extend Theorem C to local derivations of $\mathbb{C}[x_1, ..., x_n]$ into $\mathbb{C}[x_1, ..., x_m]$ when $n \leqslant m$, where $\mathbb{C}[x_1, ..., x_n]$ is viewed as a subalgebra of $\mathbb{C}[x_1, ..., x_m]$. Theorem C plays a key role in its proof.

THEOREM D. *Each local derivation of* $\mathbb{C}[x_1, ..., x_n]$ *into* $\mathbb{C}[x_1, ..., x_m]$, *where* $n \leqslant m$, *is a derivation.*

Proof. If $n = 1$, Theorem C applies to show that our local derivation α restricted to $\mathbb{C}[x_j]$ is a derivation of $\mathbb{C}[x_j]$ into $\mathbb{C}[x_1, ..., x_m]$. Suppose we have proved that the restriction of α to $\mathbb{C}[x_{j(1)}, ..., x_{j(r-1)}]$ is a derivation for each $(r-1)$-element subset $\{x_{j(1)}, ..., x_{j(r-1)}\}$ of $\{x_1, ..., x_n\}$ for some r not greater than n. We show that the same is true for each r-element subset of $\{x_1, ..., x_n\}$. It will suffice to prove that the restriction of α to $\mathbb{C}[x_1, ..., x_r]$ is a derivation.

Let $\alpha(x_j)$ be g_j(in $\mathbb{C}[x_1, ..., x_m]$) for j in $\{1, ..., n\}$ and define δ_0 by

$$\delta_0(x_1^{k(1)} \cdots x_n^{k(n)}) = \sum_{j=1}^{n} k(j)\, x_1^{k(1)} \cdots x_{j-1}^{k(j-1)} x_j^{k(j)-1} x_{j+1}^{k(j+1)} \cdots x_n^{k(n)} g_j,$$

where the $k(j)$ are non-negative integers. (If some $k(j)$ is 0, the corresponding term of the sum is interpreted as 0.) With this definition, δ_0 has a (unique) linear extension to a derivation δ of $\mathbb{C}[x_1, ..., x_n]$ into $\mathbb{C}[x_1, ..., x_m]$ and $\delta(x_j^k) = k x_j^{k-1} g_j = \alpha(x_j^k)$ for all j in $\{1, ..., n\}$ and all positive integers k. Since δ is a derivation, $\alpha - \delta$ is a local derivation of $\mathbb{C}[x_1, ..., x_n]$ into $\mathbb{C}[x_1, ..., x_m]$ that maps $x_1, ..., x_n$ to 0. We shall show that $\alpha - \delta$ is 0, whence $\alpha = \delta$, and α is a derivation.

Changing notation, we may assume that α is a local derivation of $\mathbb{C}[x_1, ..., x_n]$ into $\mathbb{C}[x_1, ..., x_m]$ such that $\alpha(x_j) = 0$ for j in $\{1, ..., n\}$. Since α is, by assumption, a derivation of $\mathbb{C}[x_{j(1)}, ..., x_{j(r-1)}]$ into $\mathbb{C}[x_1, ..., x_m]$ for each $(r-1)$-element subset $\{x_{j(1)}, ..., x_{j(r-1)}\}$ of $\{x_1, ..., x_m\}$, we have that $\alpha(x_{j(1)}^{k(1)} \cdots x_{j(r-1)}^{k(r-1)})=0$. We shall show that the restriction of α to $\mathbb{C}[x_1, ..., x_r]$ is 0. Choose a non-zero a in $\mathbb{C}$ and note that, for some $h_1, ..., h_r, p$ in $\mathbb{C}[x_1, ..., x_m]$, we have that

$$\begin{aligned} 2&[x_1^{k(1)}+ax_2^{k(2)}\cdots x_r^{k(r)}]p \\ &=\alpha([x_1^{k(1)}+ax_2^{k(2)}\cdots x_r^{k(r)}]^2) \\ &=\alpha(x_1^{2k(1)}+2ax_1^{k(1)}\cdots x_r^{k(r)}+a^2x_2^{2k(2)}\cdots x_r^{2k(r)}) \\ &=2a\alpha(x_1^{k(1)}\cdots x_r^{k(r)}) \\ &=2a\sum_{j=1}^{r} k(j)\, x_1^{k(1)}\cdots x_j^{k(j)-1}\cdots x_r^{k(r)}h_j. \end{aligned} \tag{18}$$

Choose non-zero complex numbers $b_1, ..., b_r$ such that

$$b_1^{-k(1)}b_2^{k(2)}\cdots b_r^{k(r)}=-a^{-1}.$$

With $x_1, ..., x_r$ replaced by $b_1, ..., b_r$, respectively, the left side of (18) is 0. The right side is

$$2x_1^{2k(1)-1}\Bigg[k(1)\,ax_1^{-k(1)}x_2^{k(2)}\cdots x_r^{k(r)}h_1 + \sum_{j=2}^{r} k(j)\,ax_1^{-k(1)}x_2^{k(2)}\cdots x_r^{k(r)}x_1x_j^{-1}h_j\Bigg],$$

which is 0 when $x_1, ..., x_r$ are replaced by $b_1, ..., b_r$, respectively. Thus

$$\begin{aligned} 0&=k(1)\,h_1(b_1, ..., b_r, x_{r+1}, ..., x_m) \\ &\quad+\sum_{j=2}^{r} k(j)\,b_1b_j^{-1}h_j(b_1, ..., b_r, x_{r+1}, ..., x_m) \end{aligned}$$

and

$$0=\sum_{j=1}^{r} k(j)\,b_j^{-1}h_j(b_1, ..., b_r, x_{r+1}, ..., x_m). \tag{19}$$

Since a does not appear in (19) and a is an arbitrary non-zero complex number, (19) is valid for all non-zero complex $b_1, ..., b_r$. It follows that

$$0=\sum_{j=1}^{r} k(j)\,x_j^{-1}h_j. \tag{20}$$

That is, $\sum_{j=1}^{r} k(j)\, x_1 \cdots x_{j-1} x_{j+1} \cdots x_r h_j$ vanishes identically. But then

$$\begin{aligned}\alpha(x_1^{k(1)} \cdots x_r^{k(r)}) &= \sum_{j=1}^{r} k(j)\, x_1^{k(1)} \cdots x_j^{k(j)-1} \cdots x_r^{k(r)} h_j \\ &= x_1^{k(1)-1} \cdots x_r^{k(r)-1} \sum_{j=1}^{r} k(j)\, x_1 x_2 \cdots x_{j-1} x_{j+1} \cdots x_r h_j \\ &= 0\end{aligned}$$

and $\alpha = 0$. ∎

COROLLARY 18. *Each local derivation of* $\mathbb{C}[x_1, ..., x_n]$ *into itself is a derivation.*

Note added in proof. (January 29, 1990). I. Kaplansky has found (letter dated December 1, 1990) local derivations of $\mathbb{C}[x]/[x^3]$, a 3-dimensional algebra over $\mathbb{C}$, that are not derivations.

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