Some Notes on Non-commutative Analysis

Richard V. Kadison

Dedicated to the memory of Michel Sirugue

Abstract.
An unbounded non-commutative monotone convergence result (Theorem 9) is proved. A new derivation of the Friedrichs extension is given. The basics of the Takesaki cones are studied.

1. Introduction.

We study certain aspects of the theory of von Neumann algebras that emphasize its interpretation as non-commutative measure theory. In this interpretation, which is direct and unmistakable, the projections in the algebra are the characteristic functions of the (non-commuting) measurable sets (which do not appear!), the elements of $\mathcal{R}$ are the bounded measurable functions, and the (normal) states of $\mathcal{R}$ are the probability measures on the underlying (non-commutative) measure space (which, again, does not appear). An important result, in the early stages of the theory, states that if $\{A_n\}$ is a monotone increasing sequence of self-adjoint operators, bounded above (by some multiple of the identity operator I), then there is a bounded self-adjoint operator A such that $A_n x \to Ax$ for each x in the underlying Hilbert space. If each $A_n \in \mathcal{R}$, then, of course, $A \in \mathcal{R}$ (cf. [6; Lemma 5.1.4]). This is a primitive non-commutative monotone convergence theorem. In Section 3, the restriction that the sequence $\{A_n\}$ be bounded above is removed; the limit A is now an appropriate unbounded self-adjoint operator affiliated with $\mathcal{R}$. (We write $A \;\eta\; \mathcal{R}$ to indicate this affiliation.) The extension of the classical bounded non-commutative monotone convergence result to this unbounded version (Theorem 9) seems not to be routine. A simpler unbounded monotone convergence assertion (Proposition 7),that assumes the presence of a "well-placed" separating vector for $\mathcal{R}$, is also proved in Section 3.

The Friedrichs extension is a self-adjoint extension of a positive closed (densely defined) symmetric operator [4]. Subject to a certain domain condition, this extension is unique. The Friedrichs extension of a symmetric operator affiliated with a von Neumann algebra is also affiliated with that algebra, as follows from the uniqueness. This extension plays a somewhat hidden, but important, role in the theory of von Neumann algebras. In the first of the monumental series of papers by Murray and von Neumann, it supplies the basic ingredient of a crucial comparison result (cf. [7; Lemma 9.3.3]). the Friedrichs extension is vital in establishing the basic properties of the Takesaki cones [8; pp. 101–106].

This research was carried out with partial support of NSF (USA).

A new proof for the existence of the Friedrichs extension is given in Section 2. In Section 4, the Friedrichs extension is used to give a readily accessible account of the fundamental properties of the Takesaki cones. The view of these cone properties as a broad non-commutative Radon-Nikodým theorem is described.

2. The Friedrichs Extension.

Throughout this section, A_0 is a closed linear operator with domain $\mathcal{D}(A_0)$ dense in a Hilbert space $\mathcal{H}$ and $0 \le \langle A_0 x, x\rangle$ for each x in $\mathcal{D}(A_0)$. The lemma that follows is needed primarily for establishing the uniqueness of the Freidrichs extension. The conditions and relations that the extension must fulfill are explored in this lemma, and that points the way to defining the extension.

Lemma 1. *If A is a positive self-adjoint extension of A_0, then $A+I$ is one-to-one with range $\mathcal{H}$. The inverse B to $A+I$ is a positive operator, everywhere defined,* $\|B\| \le 1$, *and*

$$(*) \qquad \langle x, y\rangle = \langle (A_0+I)x, By\rangle \qquad (x \in \mathcal{D}(A_0), y \in \mathcal{H}).$$

The range of B is contained in $\mathcal{D}(A_0^*)$.

Proof. Since $\langle A_0 x, x\rangle$ is real for each x in $\mathcal{D}(A_0)$, polarization (cf. [6; Prop. 2.1.7 and 2.4 (3)]) yields that $\langle A_0 x, y\rangle = \langle x, A_0 y\rangle$, when x and y are in $\mathcal{D}(A_0)$. Hence $A_0 \subseteq A_0^*$, that is, A_0 is symmetric. As $\langle (A_0+I)x, x\rangle \ge 0$ for each x in $\mathcal{D}(A_0)$ $(= \mathcal{D}(A_0+I))$, A_0+I is symmetric. In addition, A_0+I is closed, since A_0 is closed. By the same token, $A+I$ is closed and thus, self-adjoint. Moreover,

$$\|(A+I)x\|\,\|x\| \ge \langle (A+I)x, x\rangle = \langle Ax, x\rangle + \|x\|^2 \ge \|x\|^2 \ge 0$$

for each x in $\mathcal{D}(A)$ $(= \mathcal{D}(A+I))$. Thus $A+I$ is a positive self-adjoint operator with null space (0), and $\|x\| \le \|(A+I)x\|$ for each x in $\mathcal{D}(A)$. As the closure of the range of $A+I$ $(= (A+I)^*)$ is the orthogonal complement of the null space of $A+I$ (cf. [6; Exercise 2.8.45]), $A+I$ has range dense in $\mathcal{H}$.

If $\{x_n\}$ is a sequence in $\mathcal{D}(A)$ such that $\{(A+I)x_n\}$ tends to y, then $\|x_n - x_m\| \le \|(A+I)(x_n - x_m)\|$, and $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$. It follows that $\{x_n\}$ converges to x. Since $A+I$ is closed, $x \in \mathcal{D}(A+I)$ and $(A+I)x = y$. Hence A+I has a closed range. From our earlier conclusion, this range is dense. Thus $A+I$ has range $\mathcal{H}$.

If B is the mapping inverse to $A+I$ and $y = (A+I)x$, then

$$0 \le \|By\|^2 = \|x\|^2 \le \langle x, (A+I)x\rangle = \langle By, y\rangle \le \|By\|\,\|y\|.$$

Thus $\|B\| \le 1$ and $B \ge 0$. For each x in $\mathcal{D}(A_0+I)$, $B(A_0+I)x = B(A+I)x = x$. Hence, with y in $\mathcal{H}$,

$$\langle x, y\rangle = \langle B(A_0+I)x, y\rangle = \langle (A_0+I)x, By\rangle.$$

It follows that $By \in \mathcal{D}((A_0+I)^*)$ and $(A_0+I)^*By = y$. Since $(A_0+I)^* = A_0^* + I$ (more generally, $(T+S)^* = T^* + S^*$ when S is bounded), $By \in \mathcal{D}(A_0^*)$. ∎

For the lemmas that follow, we define a positive definite inner product on $\mathcal{D}(A_0)$ by

$$\langle u, v\rangle' = \langle (A_0 + I)u, v\rangle$$

and denote by $\mathcal{D}'$ the completion of $\mathcal{D}(A_0)$ relative to this inner product.

Lemma 2. *The identity mapping of $\mathcal{D}(A_0)$ onto itself has a (unique) bounded extension ι mapping $\mathcal{D}'$ into $\mathcal{H}$, ι is one-to-one, and $\|\iota\| \le 1$. For each y in $\mathcal{H}$, the functional $x \to \langle x, y\rangle$ on $\mathcal{D}(A_0)$ extends to a bounded linear functional on $\mathcal{D}'$ whose norm does not exceed $\|y\|$. There is a (unique) vector By in $\mathcal{D}(A_0^*)$ and in $\iota(\mathcal{D}')$ satisfying*

$$\langle x, y\rangle = \langle (A_0 + I)x, By\rangle \qquad (x \in \mathcal{D}(A_0))$$

$$\langle x, y\rangle = \langle x, \iota^{-1}(By)\rangle' \qquad (x \in \mathcal{D}(A_0)).$$

Proof. With x in $\mathcal{D}(A_0)$,

$$\|x\|^2 = \langle x, x\rangle \le \langle x, x\rangle + \langle A_0 x, x\rangle = \langle x, x\rangle' = \|x\|'^2.$$

Thus the identity mapping of $\mathcal{D}(A_0)$ onto itself has a (unique) bounded extension ι mapping $\mathcal{D}'$ into $\mathcal{H}$ and $\|\iota\| \le 1$. To see that ι is one-to-one, choose x_n in $\mathcal{D}(A_0)$ tending to z' in $\mathcal{D}'$ and note that

$$\|x_n - \iota(z')\| = \|\iota(x_n) - \iota(z')\| \le \|x_n - z'\|' \to 0,$$

whence $\|x_n\| \to 0$ when $\iota(z') = 0$. Thus, for each m,

$$\begin{aligned}\langle z', x_m\rangle' &= \lim_n \langle x_n, x_m\rangle' \\ &= \lim_n \langle (A_0 + I)x_n, x_m\rangle = \lim_n \langle x_n, (A_0 + I)^* x_m\rangle = 0,\end{aligned}$$

since $x_m \in \mathcal{D}(A_0 + I) \subseteq \mathcal{D}((A_0 + I)^*)$. But,

$$\langle z', z'\rangle' = \lim_m \langle z', x_m\rangle' = 0,$$

whence $z' = 0$ and ι is one-to-one.

Since $|\langle x, y\rangle| \le \|x\|\,\|y\| \le \|x\|'\,\|y\|$ $(x \in \mathcal{D}(A_0), y \in \mathcal{H})$ (as we have just shown), the functional $x \to \langle x, y\rangle$ on $\mathcal{D}(A_0)$ has bound not exceeding $\|y\|$ relative to the norm $x \to \|x\|'$. This functional extends (uniquely) to a linear functional of norm not exceeding $\|y\|$ on $\mathcal{D}'$. From this and Riesz's representation of functionals on Hilbert space, there is a (unique) vector z' in $\mathcal{D}'$ such that $\langle x, y\rangle = \langle x, z'\rangle'$ for each x in $\mathcal{D}(A_0)$. Let By be $\iota(z')$. We can choose x_n in $\mathcal{D}(A_0)$ tending to z' in $\mathcal{D}'$. Then, as before,

$$\|x_n - \iota(z')\| = \|\iota(x_n) - \iota(z')\| \le \|x_n - z'\|' \to 0.$$

It follows that, for each x in $\mathcal{D}(A_0)$,

$$\begin{aligned}\langle x, y\rangle = \langle x, z'\rangle' = \lim_n \langle x, x_n\rangle' &= \lim_n \langle (A_0 + I)x, x_n\rangle \\ &= \langle (A_0 + I)x, \iota(z')\rangle = \langle (A_0 + I)x, By\rangle,\end{aligned}$$

whence $By \in \mathcal{D}((A_0 + I)^*) = \mathcal{D}(A_0^*)$ (cf. the proof of Lemma 1 where it is noted that $(A_0 + I)^* = A_0^* + I$). At the same time, $\iota^{-1}(By) = z'$, so that $\langle x, y\rangle = \langle x, \iota^{-1}(By)\rangle'$. ∎

Lemma 3. *With B as in Lemma 2, $B \in \mathcal{B}(\mathcal{H})$, $B \geq 0$, $\|B\| \leq 1$, and B is one-to-one.*

Proof. Choose y in $\mathcal{H}$ and let z' be the (unique) vector in $\mathcal{D}'$ found in the proof of Lemma 2, such that $\langle x, y\rangle = \langle x, z'\rangle'$ for each x in $\mathcal{D}(A_0)$ and in terms of which By was defined to be $\iota(z')$. Then

$$\|By\| = \|\iota(z')\| \leq \|z'\| \leq \|y\|.$$

Hence $\|B\| \leq 1$. Choose x_n in $\mathcal{D}(A_0)$ tending to z' in $\mathcal{D}'$. From Lemma 2,

$$\|x_n - By\| = \|\iota(x_n - z')\| \leq \|x_n - z'\|' \to 0$$

and

$$\begin{aligned}\langle By, y\rangle &= \lim_n \langle x_n, y\rangle = \lim_n \langle x_n, z'\rangle' \\ &= \lim_n \langle x_n, \iota^{-1}(By)\rangle' = \langle z', \iota^{-1}(By)\rangle' = \|\iota^{-1}(By)\|'^2 \geq 0.\end{aligned}$$

Thus $B \geq 0$.

If y is a non-zero element of $\mathcal{H}$, then for some x in (the dense linear manifold) $\mathcal{D}(A_0)$,

$$0 \neq \langle x, y\rangle = \langle x, \iota^{-1}(By)\rangle'.$$

Hence $\iota^{-1}(By) \neq 0$. From Lemma 2, ι is one-to-one, whence $By \neq 0$, and B is one-to-one. ∎

We use the notation ι and B, in the theorem that follows, to describe the Friedrichs extension.

Theorem 4. *If A_1 is the mapping inverse to B, then $A_1 - I$ ($= A$) is a positive self-adjoint extension (the Friedrichs extension) of A_0, and $\mathcal{D}(A) \subseteq \iota(\mathcal{D}')$. Moreover, A is the unique positive self-adjoint extension of A_0 satisfying $\mathcal{D}(A) \subseteq \iota(\mathcal{D}')$.*

Proof. Suppose $x = Bu$ and $y = Bv$. Then $x \in \mathcal{D}(A_1) = \mathcal{D}(A)$, and

$$\begin{aligned}\langle Ax, y\rangle &= \langle (A_1 - I)x, y\rangle = \langle u - x, Bv\rangle \\ &= \langle B(u - x), v\rangle = \langle x - Bx, v\rangle \\ &= \langle x, (I - B)v\rangle = \langle x, v - y\rangle \\ &= \langle x, (A_1 - I)y\rangle = \langle x, Ay\rangle.\end{aligned}$$

Thus $y \in \mathcal{D}(A^*)$ and $A^*y = Ay$. It follows that $A \subseteq A^*$.

With v in $\mathcal{H}$, let $x = Bv$. If $z \in \mathcal{D}(A^*)$, then

$$\begin{aligned}\langle v, BA^*z\rangle &= \langle Bv, A^*z\rangle = \langle x, A^*z\rangle \\ &= \langle Ax, z\rangle = \langle (A_1 - I)Bv, z\rangle \\ &= \langle v - Bv, z\rangle = \langle v, (I - B)z\rangle.\end{aligned}$$

Since this equality holds for all v in $\mathcal{H}$,

$$(I - B)z = BA^*z.$$

Thus $z = B(I + A^*)z$, z is in the range of B, and $z \in \mathcal{D}(A)$. Thus $A = A^*$.

If x and u are in $\mathcal{D}(A_0)$, then from Lemmas 2 and 3,

$$\langle u, x\rangle' = \langle (A_0 + I)u, x\rangle = \langle u, (A_0 + I)x\rangle = \langle u, \iota^{-1}(B(A_0 + I)x)\rangle'.$$

Since $\mathcal{D}(A_0)$ is dense in $\mathcal{D}'$, $x = \iota^{-1}(B(A_0 + I)x)$, whence

$$x = \iota(x) = B(A_0 + I)x.$$

Hence $x \in \mathcal{D}(A_1)$ and $A_1 x = (A_0 + I)x$. Thus $Ax = (A_1 - I)x = A_0 x$, and A is a self-adjoint extension of A_0.

The range of B is $\mathcal{D}(A_1)$ $(= \mathcal{D}(A))$. From Lemma 2, the range of B is contained in $\iota(\mathcal{D}')$. Thus $\mathcal{D}(A) \subseteq \iota(\mathcal{D}')$. For each y in $\mathcal{H}$,

$$\langle ABy, By\rangle = \langle (A_1 - I)By, By\rangle = \langle (I - B)y, By\rangle \geq 0,$$

since $I - B$ and B are positive, commuting, bounded operators on $\mathcal{H}$. Thus A is positive.

The restrictions on (that is "properties of") a positive self-adjoint extension of A_0 are noted in Lemma 1. Suppose now that A' is such an extension and that $\mathcal{D}(A') \subseteq \iota(\mathcal{D}')$. Let B' be the operator arising from A' with the properties corresponding to those of B in Lemma 1. Then

$$\langle (A_0 + I)x, (B - B')y\rangle = 0 \qquad (x \in \mathcal{D}(A_0), y \in \mathcal{H}).$$

Since $\mathcal{D}(A') \subseteq \iota(\mathcal{D}')$, there is a vector u' in $\mathcal{D}'$ such that $\iota(u') = (B - B')y$. Let $\{x_n\}$ be a sequence of vectors in $\mathcal{D}(A_0)$ tending to u' (in $\mathcal{D}'$). Then

$$\|x_m - (B - B')y\| = \|x_m - \iota(u')\| \leq \|x_m - u'\|' \to 0$$

and

$$\begin{aligned}\langle x_n, u'\rangle' &= \lim_m \langle x_n, x_m\rangle' \\ &= \lim_m \langle (A_0 + I)x_n, x_m\rangle \\ &= \langle (A_0 + I)x_n, (B - B')y\rangle \\ &= 0.\end{aligned}$$

Hence $\langle u', u'\rangle' = \lim_n \langle x_n, u'\rangle' = 0$, and $u' = 0$. It follows that $\iota(u') = (B - B')y = 0$ and $B = B'$. Since B and B' are the mappings inverse to A and A', respectively, $A = A'$. ∎

Corollary 5. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and A_0 be a closed, densely defined, symmetric operator affiliated with $\mathcal{R}$. Suppose $\langle A_0 x, x\rangle \geq 0$ for each x in $\mathcal{D}(A_0)$ and A is the Friedrichs extension of A_0. Then $A \,\eta\, \mathcal{R}$.*

Proof. Let V' be a unitary operator in $\mathcal{R}'$. Then $V'AV'^*$ is a positive self-adjoint extension of $V'A_0V'^*$ and $\mathcal{D}(V'AV'^*) \subseteq V'(\iota(\mathcal{D}'))$. Since $A_0 \,\eta\, \mathcal{R}$, $V'A_0V'^* = A_0$. From uniqueness of the Friedrichs extension, it remains to show that $V'(\iota(\mathcal{D}')) \subseteq \iota(\mathcal{D}')$.

Suppose $z \in \iota(\mathcal{D}')$ and $\iota(z') = z$ (with z' in $\mathcal{D}'$). Then $\{x_n\}$ tends to z' for some sequence $\{x_n\}$ in $\mathcal{D}(A_0)$. Since $A_0 \,\eta\, \mathcal{R}$, $V'(\mathcal{D}(A_0)) = \mathcal{D}(A_0)$ and $V'x_n \in \mathcal{D}(A_0)$. Now

$$\begin{aligned}\|V'x_n - V'x_m\|'^2 &= \langle (A_0 + I)V'(x_n - x_m), V'(x_n - x_m)\rangle \\ &= \langle (A_0 + I)(x_n - x_m), (x_n - x_m)\rangle \\ &= \|x_n - x_m\|'^2 \to 0\end{aligned}$$

as $m, n \to \infty$ since $\{x_n\}$ converges in $\mathcal{D}'$. Thus $\{V'x_n\}$ converges in $\mathcal{D}'$ to some u' and $\{V'x_n\}$ converges in $\mathcal{H}$ to $\iota(u')$. Since $\{x_n\}$ tends to z in $\mathcal{H}$, $\{V'x_n\}$ tends to $V'z$ in $\mathcal{H}$. Thus $V'z = \iota(u') \in \iota(\mathcal{D}')$ and $V'(\iota(\mathcal{D}')) \subseteq \iota(\mathcal{D}')$. ∎

3. Monotone Convergence.

We prove an unbounded non-commutative monotone convergence result (Theorem 9) and use it to give a proof of the Murray-von Neumann "BT-Lemma" [7; Lemma 9.2.1] (cf. [6; Theorem 7.2.1′]). A more easily proven unbounded monotone convergence result, with the assumption of a separating vector, is found in Proposition 7 and Corollary 8. The lemma that follows, blending weak and norm convergence of nets of vectors in a Hilbert space, will be useful throughout this section.

Lemma 6. *Suppose $\{y_a\}_{a\in\mathbb{A}}$ is a bounded net of vectors in a Hilbert space $\mathcal{H}$.*

(i) Suppose $\{x_b\}_{b\in\mathbb{B}}$ converges in norm to a vector x and y is a vector in $\mathcal{H}$ such that $\lim_a\langle y_a, x_b\rangle = \langle y, x_b\rangle$ for each b in $\mathbb{B}$. Then $\lim_a\langle y_a, x\rangle = \langle y, x\rangle$ and $\lim_{a,b}\langle y_a, x_b\rangle = \langle y, x\rangle$.

(ii) Suppose $\mathcal{D}$ is a dense linear submanifold of $\mathcal{H}$ such that $\lim_a\langle y_a, x\rangle$ converges for each x in $\mathcal{D}$. Then $\{y_a\}$ converges weakly to some y in $\mathcal{H}$.

Proof. (i) Choose k such that $\|y\| \le k$ and $\|y_a\| < k$ for each a in $\mathbb{A}$. Given a positive ϵ, choose b' in $\mathbb{B}$ such that $\|x - x_b\| < \epsilon/6k$ when $b \ge b'$. Now choose a' in $\mathbb{A}$ such that $|\langle y_a - y, x_{b'}\rangle| < \epsilon/6$ when $a \ge a'$. Then

$$\begin{aligned}|\langle y_a, x\rangle - \langle y, x\rangle| &\le |\langle y_a, x - x_{b'}\rangle| + |\langle y_a - y, x_{b'}\rangle| + |\langle y, x_{b'} - x\rangle| \\ &\le \|y_a\|\epsilon/6k + \epsilon/6 + \|y\|\epsilon/6k < \epsilon/2\end{aligned}$$

when $a \ge a'$. Thus $\lim_a\langle y_a, x\rangle = \langle y, x\rangle$. At the same time, when $a \ge a'$ and $b \ge b'$,

$$\begin{aligned}|\langle y_a, x_b\rangle - \langle y, x\rangle| &\le |\langle y_a, x_b\rangle - \langle y_a, x\rangle| + |\langle y_a, x\rangle - \langle y, x\rangle| \\ &< \|y_a\|\epsilon/6k + \epsilon/2 < \epsilon.\end{aligned}$$

Thus $\lim_{a,b}\langle y_a, x_b\rangle = \langle y, x\rangle$.

(ii) The mapping $x \to \lim_a\langle y_a, x\rangle$ is a conjugate-linear functional on $\mathcal{D}$. Since $|\langle y_a, x\rangle| \le \|y_a\|\,\|x\|$ and $\{\|y_a\|\}_{a\in\mathbb{A}}$ is bounded, this functional is bounded and extends, without change of norm, to a bounded conjugate-linear functional on $\mathcal{H}$. From Riesz's representation of such functionals, there is a vector y in $\mathcal{H}$ such that $\lim_a\langle y_a, x\rangle = \langle y, x\rangle$ for each x in $\mathcal{D}$. With z in $\mathcal{H}$, there is a sequence $\{x_n\}$ in $\mathcal{D}$ that converges in norm to z. Since $\lim_a\langle y_a, x_n\rangle = \langle y, x_n\rangle$, the condition (i) is satisfied and $\lim_a\langle y_a, z\rangle = \langle y, z\rangle$. Thus $\{y_a\}_{a\in\mathcal{A}}$ converges weakly to y. ∎

For the purposes of the following proposition, we define $H \leq K$ for positive symmetric operators H and K to mean that $\mathcal{D}(K) \subseteq \mathcal{D}(H)$ and $\langle Hx, x\rangle \leq \langle Kx, x\rangle$ for each x in $\mathcal{D}(K)$.

Proposition 7. *Let $\{H_a\}_{a\in\mathbb{A}}$ be a monotone increasing net of positive symmetric operators affiliated with a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ and x_0 be a separating unit vector for $\mathcal{R}$ in the domain of each H_a. If either of the following two conditions is satisfied,*

(i) $\{H_a x_0\}_{a\in\mathbb{A}}$ converges weakly in $\mathcal{H}$ to some vector ($\{H_a\}_{a\in\mathbb{A}}$ need not be required to be increasing in this case),

(ii) the net $\{\|H_a x_0\|\}_{a\in\mathbb{A}}$ is bounded,

then there is a positive self-adjoint operator H affiliated with $\mathcal{R}$ such that $\{H_a T' x_0\}_{a\in\mathbb{A}}$ converges weakly to $HT'x_0$ for each T' in $\mathcal{R}'$.

Proof. (i) By assumption $\{H_a x_0\}_{a\in\mathbb{A}}$ converges weakly to some vector $H_0 x_0$ in $\mathcal{H}$. Since bounded operators on $\mathcal{H}$ are continuous on $\mathcal{H}$ in its weak topology, $\{T' H_a x_0\}_{a\in\mathbb{A}}$ ($= \{H_a T' x_0\}_{a\in\mathbb{A}}$) converges to the vector $T' H_0 x_0$ in $\mathcal{H}$. We define $H_0 T' x_0$ to be $T' H_0 x_0$. Then H_0 is linear with (dense) domain $\mathcal{R}' x_0$. Moreover,

$$\langle H_0 T' x_0, T' x_0\rangle = \lim_a \langle H_a T' x_0, T' x_0\rangle \geq 0$$

since each H_a is positive. It follows that H_0 is symmetric and, therefore, has a closure H_1 that is positive. With V' a unitary operator in $\mathcal{R}'$, $H_1 V' T' x_0 = V' T' H_0 x_0 = V' H_1 T' x_0$. From [6; Remark 5.6.3], $H_1 \,\eta\, \mathcal{R}$ since $\mathcal{R}' x_0$ is a core for H_1. The Friedrichs extension H of H_1 is affiliated with $\mathcal{R}$ (Corollary 5) and has the properties required.

(ii) Suppose k is a bound for the net $\{\|H_a x_0\|\}_{a\in\mathbb{A}}$. Since the ball of radius k with centre 0 is weakly compact in $\mathcal{H}$, some cofinal subnet of $\{H_a x_0\}_{a\in\mathbb{A}}$ converges weakly to a vector in that ball. From (i), there is a positive self-adjoint H affiliated with $\mathcal{R}$ such that that cofinal subnet of $\{H_a T' x_0\}$ converges weakly to $HT' x_0$ for each T' in $\mathcal{R}'$. Since $\{H_a\}_{a\in\mathbb{A}}$ is monotone and that subnet is cofinal, $\{\langle H_a T' x_0, T' x_0\rangle\}_{a\in\mathbb{A}}$ converges to $\langle HT' x_0, T' x_0\rangle$ (over $\mathbb{A}$) for each T' in $\mathcal{R}'$. Polarizing, we have that $\{\langle H_a T' x_0, S' x_0\rangle\}_{a\in\mathbb{A}}$ converges to $\langle HT' x_0, S' x_0\rangle$ for all T' and S' in $\mathcal{R}'$. From Lemma 6 (ii), $\{H_a T' x_0\}_{a\in\mathbb{A}}$ converges weakly to some vector y in $\mathcal{H}$. Thus $\langle y - HT' x_0, S' x_0\rangle = 0$ for all S' in $\mathcal{R}'$. Since $\mathcal{R}' x_0$ is dense in $\mathcal{H}$, $y = HT' x_0$. Thus $\{H_a T' x_0\}_{a\in\mathbb{A}}$ converges weakly to $HT' x_0$ for each T' in $\mathcal{R}'$. ∎

Corollary 8. *Let $\{H_n\}$ be a sequence of positive symmetric operators affiliated with a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ and x_0 be a separating unit vector for $\mathcal{R}$ in the domain of each H_n^2. Suppose $\{H_n^2\}$ is monotone increasing. If some subsequence of $\{H_n x_0\}$ converges weakly, then there is a positive self-adjoint operator H affiliated with $\mathcal{R}$ such that $\{H_n T' x_0\}$ converges weakly to $HT' x_0$ for each T' in $\mathcal{R}$.*

Proof. The weakly convergent subsequence of $\{H_n x_0\}$ is bounded from [6; Theorem 1.8.10]. If $n \leq m$ then $\|H_n x_0\|^2 = \langle H_n^2 x_0, x_0\rangle \leq \langle H_m^2 x_0, x_0\rangle = \|H_m x_0\|^2$. Thus $\{H_n x_0\}$ is bounded, (ii) of Proposition 7 applies and completes the argument. ∎

Theorem 9. *Suppose $\{H_a\}_{a\in\mathbb{A}}$ is a monotone increasing net of operators in a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{F}$ be the family of vectors x in $\mathcal{H}$ such that $\{\langle H_a x, x\rangle : a \geq a_0\}$ is bounded for some a_0 in $\mathbb{A}$ and E be the projection with range $[\mathcal{F}]$. Then $E \in \mathcal{R}$, and there is a self-adjoint operator H affiliated with $\mathcal{R}$ such that $EH \subseteq HE$, $H(I-E) = 0$, $\mathcal{D}(H) \ominus (I-E)(\mathcal{H}) \subseteq \mathcal{F}$, and $\{\langle H_a x, x\rangle\}_{a\in\mathbb{A}}$ converges to $\langle Hx, x\rangle$ for each x in $\mathcal{D}(H)\cap\mathcal{F}$.*

Proof. We may replace $\{H_a\}_{a\in A}$ by $\{H_a\}_{a\geq a'}$ for some a' in $\mathbb{A}$. Since $-\|H_{a'}\|I \leq H_{a'}$, we may assume that $rI \leq H_a$ for each a in $\mathbb{A}$ and some (fixed) real number r. Let K_a be $H_a + (1-r)I$. Then $\{K_a\}$ is monotone increasing, $I \leq K_a$, and $x \in \mathcal{F}$ if and only if $\{\langle K_a x, x\rangle\}_{a\in\mathbb{A}}$ is bounded. If we find a self-adjoint operator K affiliated with $\mathcal{R}$ such that $EK \subseteq KE$, $K(I-E) = 0$, $\mathcal{D}(K)\ominus(I-E)(\mathcal{H}) \subseteq \mathcal{F}$, and $\{\langle K_a x, x\rangle\}$ converges to $\langle Kx, x\rangle$ for each x in $\mathcal{D}(K)\cap\mathcal{F}$, then $K-(1-r)E$ will serve as the required H.

Since $\langle H_a U'x, U'x\rangle = \langle U'^* U' H_a x, x\rangle$ for each unitary operator U' in $\mathcal{R}'$, $U'x \in \mathcal{F}$ when $x \in \mathcal{F}$. Thus $E \in \mathcal{R}'' = \mathcal{R}$. From [6; Proposition 4.2.8], $\{K_a^{-1}\}_{a\in\mathbb{A}}$ is monotone decreasing. Now $0 \leq K_a^{-1}$, whence $\{K_a^{-1}\}_{a\in\mathbb{A}}$ converges to its greatest lower bound S in the strong-operator topology. We have that $S \in \mathcal{R}$ and $0 \leq S \leq I$. Since $I \leq K_b \leq K_a$ when $b \leq a$, we have that $0 \leq K_a^{-1/2} K_b K_a^{-1/2} \leq I$. As $\{K_a^{-1/2} K_b K_a^{-1/2}\}_{a\in\mathbb{A}}$ tends to $S^{1/2} K_b S^{1/2}$ in the strong-operator topology, we have that $0 \leq S^{1/2} K_b S^{1/2} \leq I$ and $\|K_b^{1/2} S^{1/2}\| \leq 1$ for each b in $\mathbb{A}$. (See [6; Remark 2.5.10, Proposition 5.3.2].) It follows that $\|K_a^{1/2} S^{1/2} z\| \leq \|z\|$ for each z in $\mathcal{H}$. Thus $S^{1/2}z$ and Sz $(= S^{1/2}S^{1/2}z)$ are in $\mathcal{F}$, so that $F \leq E$, where F is the range projection of S.

Suppose $Fy = 0$. Then $Sy = 0$, since $S = S^*$, and $S^{1/2}y = 0$. From [6; Proposition 4.2.8], $\{K_a^{-1/2}\}_{a\in\mathbb{A}}$ is monotone decreasing with strong-operator limit $S^{1/2}$. Hence $\{\|K_a^{-1/2}y\|\}_{a\in\mathbb{A}}$ tends to 0. Let z_a be $K_a^{-1/2}y$ so that $y = K_a^{1/2} z_a$. With x in $\mathcal{F}$, we have that

$$|\langle y, x\rangle| = |\langle K_a^{\frac{1}{2}} z_a, x\rangle| = |\langle z_a, K_a^{\frac{1}{2}} x\rangle| \leq \|z_a\|\,\|K_a^{\frac{1}{2}} x\| \underset{a}{\to} 0$$

since $\{\|K_a^{1/2}x\|\}_{a\in\mathbb{A}}$ $(= \{\langle K_a x, x\rangle^{1/2}\}_{a\in\mathbb{A}})$ is bounded and $\|z_a\| \to 0$. Thus y is orthogonal to $\mathcal{F}$, and $I - F \leq I - E$. Combining this with the inequality $F \leq E$, established in the preceding paragraph, we conclude that $F = E$.

Define $K(Sx+y)$ to be x when $x \in E(\mathcal{H})$ and $y \in (I-E)(\mathcal{H})$. Since S is one-to-one on $E(\mathcal{H})$, K is well defined. The range of S is dense in $E(\mathcal{H})$, whence K is densely defined. Moreover,

$$\langle K(Sx+y), Sx+y\rangle = \langle x, Sx+y\rangle = \langle x, Sx\rangle \geq 0.$$

As in the proof of Lemma 1, K is symmetric and positive. Moreover, $EK(Sx+y) = Ex = x = KE(Sx+y)$ so that $EK \subseteq KE$, $K(I-E) = 0$, and $\mathcal{D}(K)\ominus(I-E)(\mathcal{H}) \subseteq \mathcal{F}$.

At this point, several possibilities present themselves for proceeding with the argument. Theorem 4 and Corollary 5 apply to yield a positive self-adjoint extension of K affiliated with $\mathcal{R}$. By using the Borel function f, defined at 0 as 0 and at a positive real t as $1/t$, and the Borel function calculus of [6; Section 5.6] (see, especially [6; Theorem 5.6.26]) to form $f(S)$, we arrive at a self-adjoint operator extending K (inverse to S on $E(\mathcal{H})$). If we restrict S to $E(\mathcal{H})$, then S is invertible in the sense of the discussion of p. 595 of [6], and the argument of the last paragraph of p. 596 applies to show that $f(S)$ restricted to $E(\mathcal{H})$ is the mapping inverse to the restriction of S. Thus $f(S) = K$, and K is a positive self-adjoint operator affiliated with $\mathcal{R}$. This last conclusion can also be argued directly, without appeal to the Borel function calculus, as presented at the top of p. 467 of [6] (with K in place of T_0).

By construction of K, $\mathcal{D}(K) \ominus (I - E)(\mathcal{H})$ is the range of S, which is contained in $\mathcal{F}$, as we have shown. Thus $\mathcal{D}(K) \cap \mathcal{F}$ is the range of S. With x in $E(\mathcal{H})$, $\{K_a^{1/2} S^{1/2} x\}_{a\in\mathbb{A}}$ lies in the closed ball in $\mathcal{H}$ of radius $\|x\|$ with center 0. Since this ball is weakly compact, some cofinal subnet $\{K_{a'}^{1/2} S^{1/2} x\}_{a'\in\mathbb{A}'}$ of $\{K_a^{1/2} S^{1/2} x\}_{a\in\mathbb{A}}$ converges weakly to a vector u. We shall show that $Eu = x$. Given z in $\mathcal{H}$, from Lemma 6 (i),

$$\begin{aligned}\langle u, S^{\frac{1}{2}} z\rangle &= \lim_{a',a} \langle K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x, K_a^{-\frac{1}{2}} z\rangle \\ &= \lim_{a'} \langle K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x, K_{a'}^{-\frac{1}{2}} z\rangle \\ &= \lim_{a'} \langle S^{\frac{1}{2}} x, z\rangle \\ &= \langle S^{\frac{1}{2}} x, z\rangle.\end{aligned}$$

Thus $S^{1/2} u = S^{1/2} x$, and

$$Eu = KSEu = KS^{\frac{1}{2}} S^{\frac{1}{2}} u = KS^{\frac{1}{2}} S^{\frac{1}{2}} x = x.$$

Suppose, now, that $\|x\| = 1$. Given a positive ϵ, we can choose a' in $\mathbb{A}'$ such that

$$|\langle K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x, x\rangle - \langle u, x\rangle| < \epsilon.$$

Then

$$\begin{aligned}1 - \epsilon = \langle x, x\rangle - \epsilon &= \langle Eu, x\rangle - \epsilon = \langle u, x\rangle - \epsilon \\ &< |\langle K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x, x\rangle| \le \|K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x\| \, \|x\| \\ &= \|K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x\| \le \|x\| = 1.\end{aligned}$$

Thus, if $a \ge a'$,

$$\begin{aligned}(1 - \epsilon)^2 &\le \|K_{a'}^{\frac{1}{2}} S^{\frac{1}{2}} x\|^2 = \langle S^{\frac{1}{2}} K_{a'} S^{\frac{1}{2}} x, x\rangle \\ &\le \langle S^{\frac{1}{2}} K_a S^{\frac{1}{2}} x, x\rangle \le \|x\|^2 = 1.\end{aligned}$$

It follows that $\{\langle S^{1/2} K_a S^{1/2} x, x\rangle\}_{a\in\mathbb{A}}$ tends to 1. As $\{S^{1/2} K_a S^{1/2}\}_{a\in\mathbb{A}}$ is monotone increasing and bounded above by E, this net converges to some positive A ($\subseteq E$) in the strong-operator topology. From what we have proved $\langle Ax, x\rangle = 1$ for each x of norm 1 in $E(\mathcal{H})$. Thus $A = E$ and

$$\begin{aligned}\langle K_a Sx, Sx\rangle &= \langle S^{\frac{1}{2}} K_a S^{\frac{1}{2}} S^{\frac{1}{2}} x, S^{\frac{1}{2}} x\rangle \\ &\xrightarrow[a]{} \langle ES^{\frac{1}{2}} x, S^{\frac{1}{2}} x\rangle = \langle x, Sx\rangle = \langle KSx, Sx\rangle.\end{aligned}$$

∎

Theorem 10. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, x_0 be a vector in $\mathcal{H}$ and z_0 be a vector in $[\mathcal{R}x_0]$. Then there is a B in $\mathcal{R}$ and a positive self-adjoint T affiliated with $\mathcal{R}$ such that $BTx_0 = z_0$.*

Proof. We may assume the $\|z_0\| = 1$. We first choose T_0 in $\mathcal{R}$ such that $\|z_0 - T_0x_0\| \le 4^{-1}$ and $\|T_0x_0\| \le \|z_0\| = 1$. We then choose T_1 in $\mathcal{R}$ such that $\|z_0 - T_0x_0 - T_1x_0\| \le 4^{-2}$ and $\|T_1x_0\| \le \|z_0 - T_0x_0\|$ ($\le 4^{-1}$). Continuing in this way, we choose T_n in $\mathcal{R}$ such that

$$\|z_0 - T_0x_0 - T_1x_0 - \cdots - T_{n-1}x_0 - T_nx_0\| \le 4^{-(n+1)}, \quad \|T_nx_0\| \le 4^{-n}.$$

Then $\sum_{n=0}^{\infty} T_nx_0$ converges to z_0.

Let V_nH_n be the polar decomposition of T_n. Then $\|H_nx_0\| = \|T_nx_0\| \le 4^{-n}$. Let K_m be $(I + \sum_{k=0}^{m} 4^kH_k^2)^{1/2}$. Then

$$\|K_mx_0\|^2 = \langle K_m^2x_0, x_0\rangle = \|x_0\|^2 + \sum_{k=0}^{m} 4^k\|H_kx_0\|^2 \le \|x_0\|^2 + \sum_{k=0}^{m} 4^{-k}.$$

Thus $\{K_mx_0\}$ is bounded. Let u be a weak limiting point of $\{K_mx_0\}$ in $\mathcal{H}$. We apply Theorem 9 with $\{K_m\}$ in place of $\{K_a\}_{a\in\mathbb{A}}$. Let T be the positive self-adjoint operator affiliated with $\mathcal{R}$ such that $\{\langle K_mx, x\rangle\}$ converges to $\langle Tx, x\rangle$ for each x in $\mathcal{D}(T) \cap \mathcal{F}$, S be the strong-operator limit of the monotone decreasing sequence $\{K_m^{-1}\}$, and z be a vector in $\mathcal{H}$. From Lemma 6, we have

$$\begin{aligned}
\langle u, Sz\rangle &= \lim_{m'}\langle K_{m'}x_0, Sz\rangle = \lim_{m'}\lim_{n}\langle K_{m'}x_0, K_n^{-1}z\rangle \\
&= \lim_{m',n}\langle K_{m'}x_0, K_n^{-1}z\rangle = \lim_{m'}\langle K_{m'}x_0, K_{m'}^{-1}z\rangle \\
&= \langle x_0, z\rangle. \quad (\{K_{m'}x_0\} \text{ a subnet of } \{K_mx_0\})
\end{aligned}$$

Thus $Su = x_0$. It follows that $x_0 \in \mathcal{D}(T) \cap \mathcal{F}$ and that $Eu = TSu = Tx_0$. Thus $x_0 = Su = SEu = STx_0$.

Since $4^nH_n^2 \le I + \sum_{k=0}^{m} 4^kH_k^2 = K_m^2$ when $n \le m$, we have that $K_m^{-1}H_n^2K_m^{-1} \le 4^{-n}I$. As the product of operators is jointly strong-operator continuous on bounded subsets of $\mathcal{B}(\mathcal{H})$, $\{K_m^{-1}H_n^2K_m^{-1}\}$ tends to SH_n^2S in the strong-operator topology as m tends to ∞. Thus $0 \le SH_n^2S \le 4^{-n}I$ and

$$\|T_nS\|^2 = \|ST_n^*T_nS\| = \|SH_n^2S\| \le 4^{-n}.$$

It follows that $\sum_{n=0}^{\infty} T_nS$ converges in norm to an operator B in $\mathcal{R}$. Moreover,

$$BTx_0 = \left(\sum_{n=0}^{\infty} T_nS\right)Tx_0 = \sum_{n=0}^{\infty} T_nSTx_0 = \sum_{n=0}^{\infty} T_nx_0 = z_0.$$

∎

4. Cones and States.

In this section, we study the Takesaki cones $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ associated with a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ and a separating and generating vector u. (See [8], where these cones were first introduced and investigated.) The notation and results of Tomita's modular theory [8,9], as described in [6; section 9.2], will be used. We recall that conjugate-linear operators S_0 and F_0, with dense domains $\mathcal{R}u$ and $\mathcal{R}'u$, respectively, are defined by $S_0Au = A^*u$ and $F_0A'u = A'^*u$ for A in $\mathcal{R}$ and A' in $\mathcal{R}'$. It is easy to show that $F_0 \subseteq S_0^*$ and $S_0 \subseteq F_0^*$, so that S_0 and F_0 have closures S and F, respectively. The polar decomposition $J\Delta^{1/2}$ of S, where $\Delta = S^*S$, supplies the main elements, J and Δ, of Tomita's theory.

With x a vector in $\mathcal{H}$, define functionals ϕ_x on $\mathcal{R}$ and ϕ'_x on $\mathcal{R}'$ by $\phi_x(A) = \langle Au, x\rangle$ and $\phi'_x(A') = \langle A'u, x\rangle$. The vectors x that give rise to positive functionals ϕ'_x form a cone $\mathcal{V}_u^0$ in $\mathcal{H}$. Symmetrically, those vectors x that give rise to positive functionals ϕ_x form a cone $\mathcal{V}_u^{1/2}$ in $\mathcal{H}$. The Friedrichs extension plays a key role in establishing that $x \in \mathcal{V}_u^0$ if and only if $x = Hu$ for some positive self-adjoint operator H affiliated with $\mathcal{R}$. This result is a crucial step in the exposition of Takesaki's results that follows.

Theorem 11. *The functional ϕ'_x on $\mathcal{R}'$ is positive if and only if $x = Hu$ for some positive self-adjoint H affiliated with $\mathcal{R}$.*

Proof. Suppose $x \in \mathcal{D}(F_0^*)$ and $F_0^*x = x$. With A' self-adjoint in $\mathcal{R}'$, we have that

$$\langle A'u, x\rangle = \langle F_0A'u, x\rangle = \langle F_0^*x, A'u\rangle = \langle x, A'u\rangle;$$

whence $\phi'_x(A')$ is real and ϕ'_x is hermitian.

Assume, now, that ϕ'_x is hermitian and $T' \in \mathcal{R}'$. Then

$$\langle F_0T'u, x\rangle = \langle T'^*u, x\rangle = \phi'_x(T'^*) = \overline{\phi'_x(T')} = \overline{\langle T'u, x\rangle} = \langle x, T'u\rangle,$$

whence $x \in \mathcal{D}(F_0^*)$ and $x = F_0^*x$. (That $F_0^* = S$ is proved in [6; Corollary 9.2.30].)

Suppose $x = Hu$, where H is a positive self-adjoint operator affiliated with $\mathcal{R}$. Let $\{E_\lambda\}$ be the resolution of the identity for H (cf. [6; pp. 310, 311]), and let H_n be HE_n for each positive integer n. Then $H_n \in \mathcal{R}$ and $E_nH \subseteq H_n$. Hence $H_nu = E_nHu \to Hu = x$ as n tends to ∞ (since E_n is strong-operator convergent to I). Now $\langle A'u, H_nu\rangle = \langle H_nA'u, u\rangle \geq 0$, when A' is a positive operator in $\mathcal{R}'$. Thus

$$0 \leq \lim_n \langle A'u, H_nu\rangle = \langle A'u, Hu\rangle = \langle A'u, x\rangle,$$

and $\phi'_x \geq 0$.

Suppose that $\phi'_x \geq 0$ for an x in $\mathcal{H}$. Then, in particular, ϕ'_x is hermitian. From what we have proved, $x \in \mathcal{D}(F_0^*)$ and $F_0^*x = x$. Let $L_x^0A'u$ be $A'x$ for each A' in $\mathcal{R}'$. Then L_x^0 is a linear operator with (dense) domain $\mathcal{R}'u$, and

$$\begin{aligned}\langle L_x^0A'u, B'u\rangle &= \langle x, A'^*B'u\rangle = \langle x, F_0B'^*A'u\rangle \\ &= \langle A'u, B'F_0^*x\rangle = \langle A'u, B'x\rangle,\end{aligned}$$

for all A' and B' in $\mathcal{R}'$. Thus $B'u \in \mathcal{D}(L_x^{0*})$ and $L_x^{0*}B'u = B'x$. It follows that the domain of L_x^{0*} is dense, whence L_x^0 has a closure L_x. (See [6; Theorem 2.7.8 (ii)].) If $T' \in \mathcal{R}'$,

$$\langle L_xT'u, T'u\rangle = \langle T'x, T'u\rangle = \overline{\phi'_x(T'^*T')} \geq 0,$$

since T'^*T' is a positive operator in $\mathcal{R}'$. But $\mathcal{R}'u$ is a core for L_x, whence L_x is a positive symmetric operator. Moreover, with U' a unitary operator in $\mathcal{R}'$,

$$U'L_xT'u = U'T'x = L_xU'T'u.$$

Since $\mathcal{R}'u$ is a core for L_x, $L_x \,\eta\, \mathcal{R}$ from [6; Remark 5.6.3]. Theorem 4 and Corollary 5 apply; L_x has a positive self-adjoint extension H, its Friedrichs extension, affiliated with $\mathcal{R}$. Finally, $x = L_xu = Hu$. ∎

With $\mathbf{X}$ a linear space and $\mathbf{X}^\#$ its dual space, cones $\mathcal{V}$ in $\mathbf{X}$ and $\mathcal{V}^\#$ in $\mathbf{X}^\#$ are said to be dual cones when $x \in \mathcal{V}$ if and only if $\phi(x) \geq 0$ for each ϕ in $\mathcal{V}^\#$, and $\eta \in \mathcal{V}^\#$ if and only if $\eta(y) \geq 0$ for each y in $\mathcal{V}$. In a sense, $\mathcal{V}$ and $\mathcal{V}^\#$ are dual when each "determines" the ordering induced by the other. This concept applies in various contexts of linear space and dual space. If X is a compact Hausdorff space and $C(X)$ is the algebra of continuous complex-valued functions on X, the cone of positive functions and the cone of positive linear functionals on X are dual. As interpreted in a Hilbert space $\mathcal{H}$, and using the fact that $\mathcal{H}$ can be identified with its dual, cones $\mathcal{V}$ and $\mathcal{V}'$ in $\mathcal{H}$ are said to be dual cones when $x \in \mathcal{V}$ if and only if $\langle x, x'\rangle \geq 0$ for all x' in $\mathcal{V}'$, and $y' \in \mathcal{V}'$ if and only if $\langle y, y'\rangle \geq 0$ for all y in $\mathcal{V}$.

Proposition 12. *The Takesaki cones $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ are norm-closed dual cones in $\mathcal{H}$.*

Proof. If A' is a positive operator in $\mathcal{R}'$, then $0 \leq \langle A'u, ax + y\rangle$ when $a \geq 0$ and $x, y \in \mathcal{V}_u^0$. Thus $ax+y \in \mathcal{V}_u^0$. If v and $-v$ are in $\mathcal{V}_u^0$, then $\langle A'u, v\rangle = 0$ for each positive A' in $\mathcal{R}'$. Since each operator T' in $\mathcal{R}'$ is a linear combination of (four) positive operators in $\mathcal{R}'$, $\langle T'u, v\rangle = 0$. Since $[\mathcal{R}'u] = \mathcal{H}$, $v = 0$. Thus $\mathcal{V}_u^0$ and, symmetrically, $\mathcal{V}_u^{1/2}$ are cones in $\mathcal{H}$.

If $\{x_n\}$ is a sequence of vectors in $\mathcal{V}_u^0$ tending to x in norm and A' is a positive operator in $\mathcal{R}'$, then $0 \leq \langle A'u, x_n\rangle \to \langle A'u, x\rangle$. Hence $x \in \mathcal{V}_u^0$, and $\mathcal{V}_u^0$ is norm closed. Symmetrically, $\mathcal{V}_u^{1/2}$ is norm closed.

If $v \in \mathcal{V}_u^{1/2}$, then $\langle Au, v\rangle \geq 0$ for each positive A in $\mathcal{R}$. If $w \in \mathcal{V}_u^0$, then from Theorem 11, $w = Hu$ for some positive self-adjoint H affiliated $\mathcal{R}$. With $\{E_\lambda\}$ the resolution of the identity for H,

$$0 \leq \langle H_nu, v\rangle = \langle E_nHu, v\rangle \to \langle Hu, v\rangle = \langle w, v\rangle,$$

where $H_n = HE_n \in \mathcal{R}$.

If $\langle w, v\rangle \geq 0$ for each v in $\mathcal{V}_u^{1/2}$, then

$$0 \leq \langle w, A'u\rangle = \langle A'u, w\rangle$$

for each positive A' in $\mathcal{R}'$, since $A'u \in \mathcal{V}_u^{1/2}$ (from Theorem 11 applied with $\mathcal{R}'$ in place of $\mathcal{R}$). Hence $\phi'_w \geq 0$ and $w \in \mathcal{V}_u^0$. Thus $w \in \mathcal{V}_u^0$ if and only if $\langle w, v\rangle \geq 0$ for each v in $\mathcal{V}_u^{1/2}$. Symmetrically, $v \in \mathcal{V}_u^{1/2}$ if and only if $\langle w, v\rangle \geq 0$ for each w in $\mathcal{V}_u^0$. ∎

We recall the notation $\mathcal{R}^+$ and $\mathcal{R}'^+$ for the sets of positive operators in $\mathcal{R}$ and $\mathcal{R}'$, respectively.

Proposition 13. *The Takesaki cones $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ are the respective norm closures of $\mathcal{R}^+u$ and $\mathcal{R}'^+u$. Moreover, $\Delta^{1/2}\mathcal{R}^+u = \mathcal{R}'^+u$ and $\Delta^{-1/2}\mathcal{R}'^+u = \mathcal{R}^+u$, whence $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ are the respective norm closures of $\Delta^{-1/2}\mathcal{R}'^+u$ and $\Delta^{1/2}\mathcal{R}^+u$.*

Proof. From Theorem 11, $\mathcal{R}^+u \subseteq \mathcal{V}_u^0$. If $x \in \mathcal{V}_u^0$, then $x = Hu$ for some positive self-adjoint H affiliated with $\mathcal{R}$, again from Theorem 11. With $\{E_\lambda\}$ the resolution of the identity for H, $H_n = HE_n \in \mathcal{R}^+$ and $H_nu = E_nHu \to Hu = x$. Thus x is in the norm closure of $\mathcal{R}^+u$. It follows that $\mathcal{V}_u^0$ is the norm closure of $\mathcal{R}^+u$ and, symmetrically, $\mathcal{V}_u^{1/2}$ is the norm closure of $\mathcal{R}'^+u$.

Let $\Phi(A)$ be JA^*J for A in $\mathcal{R}$. The mapping Φ is a *anti-isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$ [6; p. 591]. Thus $\Phi(\mathcal{R}^+) = \mathcal{R}'^+$. With A in $\mathcal{R}^+$, $Au \in \mathcal{D}(S) = \mathcal{D}(\Delta^{1/2})$, and

$$\Delta^{1/2}Au = JSAu = JA^*u = JA^*Ju = \Phi(A)u.$$

Hence $\Delta^{1/2}\mathcal{R}^+u = \mathcal{R}'^+u$. Symmetrically, with A' in $\mathcal{R}'^+$, $A'u \in \mathcal{D}(F) = \mathcal{D}(\Delta^{-1/2})$, and

$$\Delta^{-1/2}A'u = JFA'u = JA'^*u = JA'^*Ju = \Phi^{-1}(A')u.$$

Thus $\Delta^{-1/2}\mathcal{R}'^+u = \mathcal{R}^+u$. ∎

The notation we are using for the Takesaki cones is motivated by Proposition 13, which identifies $\mathcal{V}_u^0$ and $\mathcal{V}_u^{1/2}$ with the norm closures of $\mathcal{R}^+u$ $(= \Delta^0\mathcal{R}^+u)$ and $\Delta^{1/2}\mathcal{R}^+u$, respectively. This notation is Araki's who introduces [1,2] and subjects to a deep and penetrating analysis, the one-parameter family of cones $\mathcal{V}_u^a$ defined as the norm closures of $\Delta^a\mathcal{R}^+u$, where $a \in [0, 1/2]$. Araki shows that $\mathcal{V}_u^a$ and $\mathcal{V}_u^{a'}$ are dual cones, where $a' = 1/2 - a$. The cone $\mathcal{V}_u^{1/4}$, which is "self-dual," exhibits surprising and useful properties. Independently, and at the same time, Connes [3] introduces and studies the self-dual cone, proving an important order characterization result for von Neumann algebras. Haagerup, in an unpublished note, studies the self-dual cone at about this same time. (His clever techniques are incorporated in the solution to [6; Exercises 9.6.62-4].) In [5], Haagerup extends the scope of his self-dual-cone techniques and results to the non-countably decomposable case.

Theorem 14. *With ω a normal state of $\mathcal{R}$, there is a unique vector v in $\mathcal{V}_u^0$ such that $\omega_v \mid \mathcal{R} = \omega$. Moreover,*

$$\|v - u\| = \inf\{\|z - u\| : \omega_z \mid \mathcal{R} = \omega\}.$$

Proof. From [6; Theorem 7.2.3], there is a unit vector z in $\mathcal{H}$ such that $\omega = \omega_z \mid \mathcal{R}$ (as $\mathcal{R}$ admits the separating vector u). From [6; Theorem 7.3.2], there is a partial isometry V' in $\mathcal{R}'$ such that ω' is a positive normal linear functional on $\mathcal{R}'$, where $\omega'(A') = \phi'_z(V'A')$ for each A' in $\mathcal{R}'$, and such that $\phi'_z(A') = \omega'(V'^*A')$. Now

$$\omega'(A') = \phi'_z(V'A') = \langle V'A'u, z\rangle = \langle A'u, V'^*z\rangle,$$

whence $(v =)\, V'^*z \in \mathcal{V}_u^0$. In addition,

$$\langle A'u, z\rangle = \phi'_z(A') = \omega'(V'^*A') = \phi'_z(V'V'^*A') = \langle A'u, V'V'^*z\rangle.$$

Since u is generating for $\mathcal{R}'$, $z = V'V'^*z$. Thus

$$\omega_v \mid \mathcal{R} = \omega_z \mid \mathcal{R} = \omega.$$

If H is a positive self-adjoint operator affiliated with $\mathcal{R}$ and $u \in \mathcal{D}(H)$, then $u \in \mathcal{D}(H^{1/2})$, $H^{1/2}u \in \mathcal{D}(H^{1/2})$, $H^{1/2}H^{1/2}u = Hu$, and $A'H^{1/2}u = H^{1/2}A'u$ for each A' in $\mathcal{R}'$. Thus, if V' is a partial isometry in $\mathcal{R}'$.

$$\begin{aligned} |\langle V'Hu, u\rangle| &= |\langle V'H^{\frac{1}{2}}u, H^{\frac{1}{2}}u\rangle \\ &\le \|V'H^{\frac{1}{2}}u\|\,\|H^{\frac{1}{2}}u\| \\ &\le \|H^{\frac{1}{2}}u\|^2 \\ &= \langle Hu, u\rangle \end{aligned} \tag{1}$$

and

$$\operatorname{Re}\langle V'Hu, u\rangle \le \langle Hu, u\rangle. \tag{2}$$

Suppose z is a unit vector in $\mathcal{H}$ such that $\omega_z \mid \mathcal{R} = \omega_{Hu} \mid \mathcal{R}$. The mapping $AHu \to Az$ $(A \in \mathcal{R})$ extends to a partial isometry W' in $\mathcal{R}'$, with initial space $[\mathcal{R}Hu]$, such that $W'Hu = z$. From (2),

$$\operatorname{Re}\langle z, u\rangle = \operatorname{Re}\langle W'Hu, u\rangle \le \langle Hu, u\rangle,$$

so that

$$\|Hu - u\|^2 = 2 - 2\operatorname{Re}\langle Hu, u\rangle \le 2 - 2\operatorname{Re}\langle z, u\rangle = \|z - u\|^2. \tag{3}$$

From Theorem 11, there is a positive self-adjoint operator H, affiliated with $\mathcal{R}$, such that $v = Hu$. From (3),

$$\|v - u\| = \inf\{\|z - u\| : \omega_z \mid \mathcal{R} = \omega\}.$$

If v' is another vector in $\mathcal{V}_u^0$ such that $\omega_{v'} \mid \mathcal{R} = \omega$, then

$$\|v - u\| = \inf\{\|z - u\| : \omega_z \mid \mathcal{R} = \omega\} = \|v' - u\|,$$

and $v' = V'Hu$ for some partial isometry V' in $\mathcal{R}'$ with Hu $(= v)$ in its initial space. Hence

$$\operatorname{Re}\langle V'Hu, u\rangle = Re\langle v', u\rangle = \operatorname{Re}\langle v, u\rangle = \langle Hu, u\rangle,$$

and the inequality of (2) is equality in the present case. It follows that

$$\langle V'Hu, u\rangle = |\langle V'Hu, u\rangle| = \operatorname{Re}\langle V'Hu, u\rangle = \langle Hu, u\rangle,$$

so that

$$\langle V'H^{\frac{1}{2}}u, H^{\frac{1}{2}}u\rangle = \|V'H^{\frac{1}{2}}u\|\,\|H^{\frac{1}{2}}u\| = \|H^{\frac{1}{2}}u\|^2.$$

Thus $V'H^{1/2}u = H^{1/2}u$ and

$$v' = V'Hu = V'H^{\frac{1}{2}}(H^{\frac{1}{2}}u) = H^{\frac{1}{2}}V'H^{\frac{1}{2}}u = Hu = v.$$

Combining Theorems 11 and 14 yields a far reaching non-commutative Radon-Nikodým result. In effect, the normal state ω is "absolutely continuous" with respect to $\omega_u \mid \mathcal{R}$, and $\omega = \omega_{Hu} \mid \mathcal{R}$, where $H^* = H \geq 0$ and $H \,\eta\, \mathcal{R}$. Loosely, $\omega(A) = \omega_u(HAH)$ for each A in $\mathcal{R}$ (although HAH is not bounded, in general, so that, in fact $\omega(HAH)$ is not defined). Thus H^2 is the Radon-Nikodým derivative of ω with respect to ω_u. In the non-commutative context, "HAH" rather than "AH^2" is the appropriate formulation.

References.

1. H. Araki, *Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodým theorem with a chain rule*, Pac. J. Math. **50**(1974), 309-354.

2. H. Araki, *Positive cones for von Neumann algebras*, in "Operator Algebras and Applications" (Proc. of Symposia in Pure Math. Vol. 38 Part 2, R. Kadison, ed., 1980), pp. 5-15. Amer. Math. Soc., Providence, 1982.

3. A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann*, Ann. Inst. Fourier., Grenoble **24**(1974), 121-155.

4. K. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren*, Math. Ann. **109**(1934), 465-487.

5. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37**(1975), 271-283.

6. R. Kadison and J. Ringrose, "Fundamentals of the Theory of Operator Algebras," Vols. I, II. Academic Press, New York, Orlando, 1983, 1986.

7. F. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. **37**(1936), 116-229.

8. M. Takesaki, "Tomita's Theory of Modular Hilbert Algebras and Its Applications," Lecture Notes in Mathematics, Vol. 128. Springer-Verlag, Heidelberg, 1970.

9. M. Tomita, *Standard forms of von Neumann algebras*, Fifth Functional Analysis Symposium of the Math. Soc. of Japan, Sendai, 1967.

Mathematics Department
University of Pennsylvania