THE WEYL THEOREM AND BLOCK DECOMPOSITIONS

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Dedicated to Marshall H. Stone on his eighty fifth birthday

1 INTRODUCTION

In [9; Satz VI], Weyl proves that each bounded, self-adjoint operator can be "perturbed" by the addition of a self-adjoint compact operator to yield a self-adjoint operator that is diagonalized by some orthonormal basis. Von Neumann [8] gives a simpler proof and sharpens this result. In answer to a question of Halmos [2], Berg [1] (see also [3]) proves the same result for normal operators.

In a sweeping extension of these results, Zsido [10], using techniques developed by Halmos [4], when introducing the important concept of quasitriangularity, and methods of von Neumann algebra theory, proves the corresponding result for countably generated commutative C*-subalgebras of a countably decomposable, infinite, semi-finite factor. Among other results, Zsido shows that each self-adjoint operator in a countably decomposable factor of type I_∞ or II_∞ is the sum of a diagonal operator $\Sigma_{n=1}^{\infty} a_n E_n$ and a self-adjoint operator C in the (unique) proper, norm-closed ideal generated by the finite projections, where each E_n is one-dimensional in the I_∞ case and of trace 1 relative to a given normal, semi-finite tracial weight in the II_∞ case. Kaftal [7] has extended some of these results to include normal operators.

In the next section, we present an extension of the Weyl theorem that is stronger than the Zsido extension, in one way, and weaker in another. The proof is simpler than most arguments that yield the Weyl theorem. It deals with the possibility of special block decompositions of the self-adjoint operators in these factors and makes use of the "block diagonalization" theorem of [5]. Both this result (Theorem A) and the Zsido result yield the Weyl theorem at once in the classical (I_∞) case.

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2 BLOCK DECOMPOSITION AND WEYL'S THEOREM IN SEMI-FINITE FACTORS

By using [5], we show directly that each self-adjoint operator has a block decomposition with blocks of arbitrarily prescribed small dimension, the "off-diagonal" matrix lying in the (unique) proper norm-closed ideal.

Theorem A.

Suppose M is an infinite countably decomposable, semi-finite factor and F is the (unique, proper) norm-closed, two-sided ideal, the norm closure of the ideal I of operators with finite range projection. Let F be a non-zero finite projection and H a self-adjoint operator in M. Then there is an orthogonal family of projections $\{G_1, G_2, \ldots\}$ in M such that $\Sigma G_j = I$, $G_j \precsim F$ $(j = 1,2,\ldots)$, and $\Sigma_{j \neq k} G_j H G_k$ is an operator in F.

Proof.

We use the spectral projections for H and iterated bisection of $[-||H||, ||H||]$ to construct projections $\{E_{jk}\}$, where $j \in \{0,1,2, \ldots\}$ and $k \in \{1,\ldots,2^j\}$, such that $\Sigma_{k=1}^{2^j} E_{jk} = I$. Let $\{E_1, E_2, \ldots\}$ be an orthogonal family of finite projections in M with sum I. Let F_0 be 0 and F_j be

$$\cup\{R(E_{jk}E_h) : k \in \{1,\ldots,2^j\},\ h \in \{1,\ldots,j\}\} .$$

Since E_{jk} is the sum of $E_{j+1k'}$ for certain k', we have that $F_j \leq F_{j+1}$.

Let a_{jk} be the midpoint of the interval for which E_{jk} is the spectral projection and let A_j be $\Sigma_{k=1}^{2^j} a_{jk}E_{jk}$. Then $||H-A_j|| \leq 2^{-j}||H||$. Let F_{jk} be

$$\cup\{R(E_{jk}E_h) : h \in \{1,\ldots,j\}\} .$$

Then $F_{jk} \leq E_{jk}$ for each k, $\{F_{jk} : k \in \{1,2,\ldots,2^j\}\}$ is an orthogonal family since $\{E_{jk} : k \in \{1,2,\ldots,2^j\}\}$ is, and $F_j = \Sigma_{k=1}^{2^j} F_{jk}$.

As $E_h = (\Sigma_{k=1}^{2^j} E_{jk})E_h$, we have that $\Sigma_{h=1}^{j} E_h \leq F_j$. Thus $\cup_{j=1}^{\infty} F_j = I$, and $\{F_j\}$ is strong-operator convergent to I. Since $R(E_{jk}E_h) \sim R((E_{jk}E_h)^*) = R(E_hE_{jk}) \leq E_h$ [6; Proposition 6.1.6], and E_h is finite, each $R(E_{jk}E_h)$ is finite. From [6; Theorem 6.3.8], each

F_j is finite. Let M_j be $F_j - F_{j-1}$ $(j = 1,2,\ldots)$. We show that M_{j+1} commutes with A_j . To see this, note that if $j \le j'$, then $E_{j'k} \le E_{jk'}$ for some k' , whence $F_{j'k} \le E_{j'k} \le E_{jk'}$. It follows that $F_{j'k}$ commutes with A_j as does $F_{j'}$. Hence M_{j+1} $(=F_{j+1} - F_j)$ commutes with A_j . We conclude that

$$||HM_j - M_jH|| \le ||(H-A_{j-1})M_j|| + ||M_j(A_{j-1}-H)|| \le 2^{-(j-2)}||H|| .$$

Thus

$$\begin{aligned} ||F_j[(HM_j-M_jH)M_j-M_j(HM_j-M_jH)]F_j|| &= ||\sum_{k=1}^{j-1}(M_kHM_j+M_jHM_k)|| \\ &\le 2||HM_j-M_jH|| \\ &\le 2^{-(j-3)}||H|| . \end{aligned}$$

Since $\Sigma_{k=1}^{j-1}(M_kHM_j + M_jHM_k)$ $(=C_j)$ has a finite range projection (from [6; Theorem 6.3.8]) and $||C_j|| \le 2^{-(j-3)}||H||$, we have that $\Sigma_{j=2}^{\infty} C_j$ converges in norm to an operator $\Sigma_{j\neq k} M_jHM_k$ in $\mathcal{F}$.

We can find a finite orthogonal family of equivalent projections with sum M_1 and with each projection in the family equivalent to a subprojection of F . (If $M_1 \precsim F$, we may use $\{M_1\}$ as this family.) Let $\{G_{jk}^{(1)} : j,k = 1,\ldots,n_1\}$ be a self-adjoint system of matrix units with $\{G_{jj}^{(1)}\}$ the orthogonal family (cf. [6; Lemma 6.6.4]). From [5], there is a unitary operator U_1 in $M_1\mathcal{M}M_1$ (acting on $M_1(\mathcal{H})$) such that $U_1M_1HM_1U_1^*$ has a diagonal matrix relative to $\{G_{jk}^{(1)}\}$; that is

$$U_1M_1HM_1U_1^* = \sum_{j=1}^{n_1} G_{jj}^{(1)}\, U_1M_1HM_1U_1^*\, G_{jj}^{(1)} .$$

Equivalently,

$$M_1HM_1 = \sum_{j=1}^{n_1} U_1^*G_{jj}^{(1)}\, U_1M_1HM_1U_1^*\, G_{jj}^{(1)}U_1 .$$

We now repeat this construction for each of $M_2, M_3, \ldots,$ producing unitary elements $U_2, U_3, \ldots,$ in $M_2\mathcal{M}M_2$, $M_3\mathcal{M}M_3, \ldots,$ and matrix unit systems $\{G_{jk}^{(2)}\}$, $\{G_{jk}^{(3)}\}, \ldots,$ such that

$$M_hHM_h = \sum_{j=1}^{n_h} U_h^*G_{jj}^{(h)}U_hM_hHM_hU_h^*G_{jj}^{(h)}U_h \ .$$

With H the Hilbert space on which M acts and x a vector in $M_h(H)$, let Ux be U_hx. Then U defines a unitary operator in M and U commutes with each M_h. Moreover, for h in $\{1,2,\ldots\}$,

$$M_hHM_h = \sum_{j=1}^{n_h} U^*G_{jj}^{(h)}UM_hHM_hU^*G_{jj}^{(h)}U \ .$$

We enumerate $U^*G_{11}^{(1)}U$, $U^*G_{22}^{(1)}U,\ldots,U^*G_{n_1n_1}^{(1)}U$, $U^*G_{11}^{(2)}U,\ldots$ as $G_1,G_2,\ldots$. Thus

$$M_1 = G_1 +\ldots+ G_{n_1},\ M_2 = G_{n_1+1} +\ldots+ G_{n_1+n_2},\ \ldots \ .$$

It follows that $\Sigma_{h=1}^{\infty} M_hHM_h = \Sigma_{j=1}^{\infty} G_jHG_j$ and that the compact operator $\Sigma_{j\neq k} M_jHM_k$ coincides with $\Sigma_{j\neq k} G_jHG_k$, for the apparently missing terms G_jHG_k, where $j \neq k$ and G_j,G_k are subprojections of the same M_h, are all 0 since G_j and G_k are distinct principal units of the same matrix unit systems that diagonalize the operators M_hHM_h in these cases. □

Corollary B.

With H a separable Hilbert space and H a self-adjoint element in $B(H)$, there is an orthonormal basis for H relative to which the matrix for H with the diagonal entries replaced by 0 is compact.

Proof.

Let F be a one-dimensional projection in Theorem A. Then each G_j is either 0 or one dimensional. For our orthonormal basis, we choose an orthonormal basis for each $G_j(H)$ and use their union. □

Once we are reconciled to the loss of diagonalizability of self-adjoint "matrices" when we pass from finite to infinite, we note that, at any rate, the spectral theorem allows us to find an orthonormal basis relative to which all off-diagonal terms are small. Indeed, with ε positive and $\{E_1,\ldots,E_n\}$ a finite orthogonal family of spectral projections for H with sum I such that $||H-\Sigma\lambda_jE_j|| < \varepsilon$, where $\lambda_1,\ldots,\lambda_n$ are points of the spectrum of H, the union of orthonormal bases for $E_1(H),\ldots,E_n(H)$ provides us with an orthonormal basis relative to which the matrix of $H-\Sigma\lambda_j E_j$ has each entry of absolute value less than ε and $\Sigma\lambda_j E_j$ has a diagonal matrix. Thus each off-diagonal entry of H has absolute value less than ε . (In fact, the operator whose matrix has 0 at diagonal entries and the off-diagonal entries of H has norm less than 2ε , for all diagonal entries of $H-\Sigma\lambda_j E_j$ have absolute value less than ε , whence the diagonal matrix with these diagonal entries has norm less than ε as does $H-\Sigma\lambda_j E_j$.) Arranging for smallness of the off-diagonal matrix in the sense of compactness, as in Corollary B, is striking. By starting with M_r , r sufficiently large, in place of M_1 in the proof of Theorem A (so that our series estimates begin with 2^r), we can arrange that the off-diagonal matrix has small norm as well as being compact.

Some simple considerations allow us to draw the matrix representation result of Corollary B directly from the Weyl theorem. If we know that $H = D + C$, where D is a diagonal relative to the orthonormal basis $\{e_n\}$ and C is compact, then $\{\|Ce_n\|\}$ tends to 0 since $\{e_n\}$ converges weakly to 0 , and compact operators convert weakly convergent sequences to norm convergent sequences [6; Exercise 2.8.20]. Thus $\{\langle Ce_n, e_n\rangle\}$ converges to 0 . The diagonal entries for the matrix of C relative to $\{e_n\}$ are $\{\langle Ce_n, e_n\rangle\}$. It follows that the diagonal operator with diagonal entries the diagonal of C is compact [6; Exercise 2.8.26]. Hence the difference of C and this diagonal operator is compact. That difference is the off-diagonal matrix obtained from the matrix of H relative to $\{e_n\}$.

With some sharpening of these techniques, we can see that the "block diagonal" matrix formed from C is compact for all "sizes" of blocks. We make this precise in the following proposition.

<u>Proposition C.</u>

If C is a compact operator on the separable Hilbert space $\mathcal{H}$ and $\{E_n\}$ is an orthogonal family of projections, then $\Sigma_n E_n C E_n$ is compact.

<u>Proof.</u>

Let α be the linear mapping of $\mathcal{B}(\mathcal{H})$ into itself that assigns $\Sigma E_n T E_n$ to T . Since

$$\|\Sigma E_n T E_n\| = \sup_n\{\|E_n T E_n\|\} \le \sup_n\{\|E_n\|\ \|T\|\ \|E_n\|\} \le \|T\|,$$

we have that $\|\alpha\| \le 1$. Suppose that F is a one-dimensional projection and x is a unit vector in its range. Then E_nFE_n is self-adjoint with range spanned by E_nx. If $E_nFE_n \neq 0$, then E_nFE_n is a positive multiple of the one-dimensional projection with range spanned by E_nx. If y is a unit vector in the range of E_n, then

$$E_nFE_ny = \langle y,x\rangle E_nx = ||E_nx||^2\langle y,||E_nx||^{-1}E_nx\rangle ||E_nx||^{-1}E_nx ,$$

whence that multiple (and the norm of E_nFE_n) is $||E_nx||^2$. Since $\Sigma||E_nx||^2 < \infty$, we have that $\alpha(F)$ is the norm limit of the operators $\Sigma_{n=1}^{m} E_nFE_n$ $(m = 1,2,\ldots)$, each of which has finite-dimensional range. Thus $\alpha(F)$ is compact.

If T has finite-dimensional range, it is a linear combination of one-dimensional projections (not necessarily mutually orthogonal) and $\alpha(T)$ is compact. Since α is bounded, we have that $\alpha(C)$ $(= \Sigma E_nCE_n)$ is compact when C is compact. □

The argument of Proposition C relies on the fact that $\{||E_nFE_n||\}$ tends to 0. This need not hold in a factor of Type II_∞ when F is replaced by a projection of (relative) dimension 1, even when each E_n has relative dimension 1. To see this, let $F = \Sigma_{n=1}^{\infty} F_n$, where F_n is a subprojection of E_n of dimension 2^{-n}. Then $||E_nFE_n|| = 1$ for all n. Of course $\Sigma E_nFE_n = \Sigma F_n = F$, in this case, so that ΣE_nFE_n is in the unique, proper, norm-closed ideal of the factor. These comments raise some doubts about the validity of the assertion of Proposition C as formulated for (countably decomposable) factors of Type II_∞. By using more sophisticated techniques, we prove that the assertion of Proposition C is valid for all countably decomposable semi-finite factors. (Although Proposition D subsumes Proposition C, it was our purpose to argue the assertion of Proposition C in relatively basic terms.)

Proposition D.

Let $\mathcal{M}$ be an infinite, countably decomposable, semi-finite factor, $\mathcal{I}$ the two-sided ideal in $\mathcal{M}$ consisting of all operators in $\mathcal{M}$ whose range projection is finite, and $\mathcal{F}$ be the norm closure of $\mathcal{I}$. If $\{E_1,E_2,\ldots\}$ is an orthogonal family of projections in $\mathcal{M}$ and $A \in \mathcal{F}$, then $\Sigma_{j=1}^{\infty} E_jAE_j \in \mathcal{F}$.

Proof.

Suppose, first, that A is a finite projection F in $\mathcal{M}$. Let ρ be the unique, normal, semi-finite tracial weight on $\mathcal{M}$ such that $\rho(F) = 1$ (cf. [6; §8.5]). Then

$$\begin{aligned}\rho\Big(\sum_{j=1}^{\infty} E_jFE_j\Big) &= \sum_{j=1}^{\infty} \rho(E_jFE_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \rho(E_jFE_j)\\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \rho(FE_jF) = \lim_{n\to\infty} \rho\Big(F\Big(\sum_{j=1}^{n} E_j\Big)F\Big)\\ &\le \lim_{n\to\infty} \rho(F) = \rho(F) .\end{aligned}$$

As noted in [6; Proposition 8.5.1], $\mathcal{M}_\rho$ is a two-sided ideal. Since $\mathcal{M}$ is infinite and semi-finite, $I \notin \mathcal{M}_\rho$ (cf. [6; Theorem 8.5.7]), whence $\mathcal{M}_\rho$ is a proper, two-sided ideal. Hence $\mathcal{M}_\rho \subseteq \mathcal{F}$, and $\Sigma_{j=1}^{\infty} E_jFE_j \in \mathcal{F}$. □

Using Proposition D, we can extend Corollary B to infinite, countably decomposable, semi-finite factors and prove it in sharpened form.

Corollary E.

Let $\mathcal{M}$ be an infinite, countably decomposable, semi-finite factor, $\mathcal{D}$ a numerical dimension function on the set of projections in $\mathcal{M}$, $\{1',2',\ldots\}$ an infinite sequence of positive integers (not necessarily distinct), and H a self-adjoint operator in $\mathcal{M}$. Then there is an orthogonal family $\{E_n\}$ of projections in $\mathcal{M}$ with sum I such that $n' \le \mathcal{D}(E_n) < n'+1$ and an orthogonal family $\{E_{1n},E_{2n},\ldots,E_{n'n}\}$ of equivalent projections in $\mathcal{M}$ with sum E_n such that $E_nHE_n = \Sigma_{j=1}^{n'} E_{jn}HE_{jn}$, and $\Sigma_{n\neq m}E_nHE_m$ is in the unique, proper, norm-closed ideal $\mathcal{F}$ of $\mathcal{M}$.

Proof.

With F a projection of dimension 1, there are projections $G_1,G_2,\ldots$ as described in the statement of Theorem A. Since $\Sigma\mathcal{D}(G_j) = \mathcal{D}(\Sigma G_j) = \mathcal{D}(I) = \infty$ and $\mathcal{D}(G_j) \le 1$, we can find an orthogonal family of projections $\{E_n\}$, with sum I, such that $n' \le \mathcal{D}(E_n) < n'+1$, for some positive integer n', and each E_n is the sum of a subset of $\{G_1,G_2,\ldots\}$. By choice of $\{G_j\}$, $\Sigma_{j\neq k}\, G_jHG_k \in \mathcal{F}$. From Proposition D, $\Sigma_{h=1}^{\infty} E_h(\Sigma_{j\neq k}G_jHG_k)E_h \in \mathcal{F}$. Since each E_n is a sum of certain G_j,

$$\sum_{j\neq k} G_jHG_k - \sum_{h=1}^{\infty} E_h\Big(\sum_{j\neq k} G_jHG_k\Big)E_h = \sum_{n\neq m} E_nHE_m \in \mathcal{F} .$$

As $n' \le \mathcal{D}(E_n) < n'+1$, we can find an orthogonal family $\{F_{1n}, F_{2n}, \ldots, F_{n'n}\}$ of equivalent projections in M with sum E_n. Using [5], as in the proof of Theorem A, we construct a unitary operator U in M that commutes with each E_n and satisfies

$$E_nHE_n = \sum_{j=1}^{n'} U^*F_{jn}UE_nHE_nU^*F_{jn}U \quad .$$

We complete the proof by choosing $U^*F_{jn}U$ as E_{jn}. □

If we take $\mathcal{B}(H)$ as M, in the preceding corollary, we conclude that, for each bounded, self-adjoint operator, we can find an orthonormal basis relative to which the matrix of that operator has an arbitrarily preassigned system of diagonal blocks of finite size in diagonal form and the matrix, with its diagonal entries replaced by zero, is compact.

Remark F.

If the self-adjoint operator H on the separable Hilbert space H is represented as a matrix whose associated off-diagonal matrix is compact, then the essential spectrum and the essential norm of H (that is, the spectrum and norm of the image of H in the Calkin algebra) can be read from the diagonal of this matrix. Let $\mathbb{F}$ be the set of diagonal entries that appear at a finite number of diagonal positions, $\mathbb{S}$ be the closure of the set of all diagonal entries, and $sp_e(H)$ be $\mathbb{S} \setminus \mathbb{F}$. Then $sp_e(H)$ is the essential spectrum of H and $\sup\{|\lambda| : \lambda \in sp_e(H)\}$ $(= ||H||_e)$ is the essential norm of H. If $||H||_e = 0$, then H is compact. In any case, $||H||_e$ is the (minimum) distance from H to the ideal of compact operators on H. If $||H||_e \neq 0$, the special matrix representation described provides a ready means for constructing (many) compact approximants to H realizing this distance. For general bounded operators on H, the polar decomposition can be used, in conjunction with the preceding construction, to produce compact approximants. Similar comments apply to countably decomposable II_∞ factors and their unique, proper, norm-closed ideals. □

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