NOTES ON THE CANONICAL ANTICOMMUTATION RELATIONS

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<u>1. Introduction.</u> Since the late 1920s and early 1930s, when Dirac and Von Neumann phrased the basic assumptions of quantum mechanics in the formalism of Hilbert spaces, it has been accepted procedure to identify the observables of a physical system with self-adjoint operators on a Hilbert space. Computation with the mathematical model requires that we consider functions and, in particular, polynomials in these observables. These functions occur, for example, when we try to describe the Hamiltonian of the system. The family of self-adjoint operators representing our observables must possess some algebraic structure. One may simply assume that each self-adjoint operator represents some observable - this works reasonably well if the system being studied has finitely many degrees of freedom and we have no need to consider the mathematical model of it in other than irreducible representations. That assumption is not adequate for systems with infinitely many degrees of freedom - the study of quantized fields and quantum statistical mechanics (after passing to the thermodynamical limit) requires other models.

In this article, some of the models that have come to be useful for studying systems with infinitely many degrees of freedom will be described along with some of the powerful techniques and results that have been developed during the more than fifty years that this subject has been studied. The canonical commutation and anticommutation relations, associated with infinite systems of particles satisfying Bose-Einstein and Fermi-Dirac statistics, respectively, and their representations by operators on Hilbert spaces is a recurring theme in the investigation of such systems. A particular operator algebra is remarkably suited to the analysis of the representations of the canonical anticommutation relations. We shall describe this connection. The results we discuss will be illustrated by applying them to this algebra and producing information about representations of the canonical anticommutation relations. All of these results are to be found in the text and exercises of [6] (specific reference to results in [6] will be made where appropriate).

2. Notation and preliminaries. Our Hilbert space $\mathcal{H}$ has scalar field $\mathbb{C}$, the complex numbers. The inner product of two vectors x and y in $\mathcal{H}$ is denoted by $\langle x,y \rangle$. It is linear in x and conjugate linear in y. The length or norm of a vector x is $\langle x,x \rangle^{\frac{1}{2}}$ and is denoted by $\|x\|$. The bound or norm of a (continuous) linear transformation T of $\mathcal{H}$ into itself (bounded operator) is ($\sup\{ \|Tx\| : \|x\| \leq 1, x \in \mathcal{H} \}$) denoted by $\|T\|$. The family of all bounded operators on $\mathcal{H}$ is denoted by $\mathfrak{B}(\mathcal{H})$. The adjoint of an operator T in $\mathfrak{B}(\mathcal{H})$ is denoted by T^* (and characterized by the equality, $\langle Tx,y \rangle = \langle x,T^*y \rangle$ for all x and y in $\mathcal{H}$).

A family $\mathfrak{F}$ of operators in $\mathfrak{B}(\mathcal{H})$ is said to be self-adjoint when $\mathfrak{F} = \mathfrak{F}^*$ $(= \{ T^* ; T \in \mathfrak{F}\})$. A subset $\mathfrak{A}$ of $\mathfrak{B}(\mathcal{H})$ that contains each linear combination $aT+S$ and product TS of operators T and S in $\mathfrak{A}$ and is self-adjoint ($T^* \in \mathfrak{A}$ if $T \in \mathfrak{A}$) is said to be a (self-adjoint) operator algebra. If $\mathfrak{A}$ is a self-adjoint operator algebra such that $T \in \mathfrak{A}$ when $\|T-T_n\| \to 0$ and each $T_n \in \mathfrak{A}$, we say that $\mathfrak{A}$ is a C*-algebra. We assume that our operator algebras contain the identity element I (for each x in $\mathcal{H}$, $Ix = x$) If $\mathfrak{A}$ is a self-adjoint operator algebra such that $T \in \mathfrak{A}$ when $\|(T-T_n)x\| \to 0$ for each x in $\mathcal{H}$ and each $T_n \in \mathfrak{A}$, we say that $\mathfrak{A}$ is a von Neumann algebra. A factor is a von Neumann algebra $\mathfrak{M}$ whose center ($\{ T \in \mathfrak{M} : TS = ST$ for all S in $\mathfrak{M}\}$) consists of scalar multiples of I.

3. Matricial operator algebras. The algebra $\mathfrak{B}(\mathcal{H})$ is a factor (hence, a von Neumann algebra and a C*-algebra). If $\mathcal{H}$ is n-dimensional with n finite, then $\mathfrak{B}(\mathcal{H})$ is isomorphic to $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices.

A class of C*-algebras that has come to be useful for models in quantum physics was introduced by Glimm [2]. These are the matricial C*-algebras. Given a sequence of positive integers $r(1), r(2), \cdots$, and an infinite dimensional Hilbert space $\mathcal{H}$, one can construct a family $\{\mathfrak{A}_j\}$ of C*-subalgebras $\mathfrak{A}_j$ of $\mathfrak{B}(\mathcal{H})$ such that each $\mathfrak{A}_j$ contains I, $A_jA_k = A_kA_j$ when $A_j \in \mathfrak{A}_j, A_k \in \mathfrak{A}_k$ and $j \neq k$ (we say that $\mathfrak{A}_j$ and $\mathfrak{A}_k$ commute, in this case), and $\mathfrak{A}_j$ is isomorphic to $M_{r(j)}(\mathbb{C})$ $(j = 1,2,\cdots)$. The norm closure of the algebra generated by $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$, is a matricial C*-algebra. Glimm shows (see [6:Section 12.1]):

THEOREM Two matricial C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, generated by $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ and $\mathfrak{B}_1, \mathfrak{B}_2, \cdots$, with orders $r(1), r(2), \cdots$ and $s(1), s(2), \cdots$, respectively, are isomorphic if and only if each prime power p^m that divides some product $r(1)\cdots r(j)$ also divides some product $s(1)\cdots s(k)$.

If each $r(j)$ is 2 and each $s(j)$ is 3, then $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic. If each $r(j)$ is 2 and $s(1) = 2$, $s(2) = 4$, $s(3) = 8$, $\cdots$, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. The case where each $r(j)$ is 2 gives rise to the <u>CAR</u> <u>algebra</u>, which is of special interest in quantum physics. The representations of the CAR algebra and those of canonical anticommutation relations are very closely related.

<u>4. The canonical anticommutation relations.</u> The system of relations

$$C_jC_k + C_kC_j = 0 \qquad (j,k = 1,2,\cdots)$$

$$C_jC_k^* + C_k^*C_j = 0 \qquad (j \neq k)$$

$$C_jC_j^* + C_j^*C_j = I \qquad (j = 1,2,\cdots)$$

in the infinite set of variables $C_1, C_2, \cdots$, is called the <u>canonical</u> <u>anticommutation</u> <u>relations</u>. A set of operators $C_1^o, C_2^o, \cdots$, on a Hilbert space, that satisfies the canonical anticommutation relations is said to be a <u>representation</u> of the canonical anticommutation relations.

A representation of a C*-algebra $\mathfrak{A}$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathcal{H})$ that preserves adjoints for some Hilbert space $\mathcal{H}$. The connection between the representations of the canonical anticommutation relations and representations of the CAR algebra is established with the aid of <u>Pauli</u> <u>spin</u> <u>matrices</u>. We write σ_x, σ_y, σ_z, for the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively. With $\mathfrak{A}$ the CAR algebra generated by commuting subalgebras $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$, each isomorphic to $M_2(\mathbb{C})$, we identify $\mathfrak{A}_n$ with $M_2(\mathbb{C})$ and write $\sigma_x^{(n)}$, $\sigma_y^{(n)}$, $\sigma_z^{(n)}$, for the Pauli matrices in $\mathfrak{A}_n$. Let A_n be $\sigma_z^{(1)}\cdots\sigma_z^{(n-1)}(\sigma_x^{(n)} - \sigma_y^{(n)})/2$. In this notation, the following theorem describes the way the representations of the canonical anticommutation relations are tied to those of the CAR algebra.

THEOREM The elements $A_1, A_2, \cdots$, in $\mathfrak{A}$ satisfy the canonical anticommutation relations and generate $\mathfrak{A}$ as a C*-algebra. If φ is a representation of $\mathfrak{A}$ on the Hilbert space $\mathscr{H}$, then $\varphi(A_1), \varphi(A_2), \cdots$, is a representation of the canonical anticommutation relations (on $\mathscr{H}$). If $C_1, C_2, \cdots$, is a representation of the canonical anticommutation relations on a Hilbert space $\mathscr{H}$, then there is a unique representation φ of the CAR algebra $\mathfrak{A}$ on $\mathscr{H}$ such that $\varphi(A_1) = C_1$, $\varphi(A_2) = C_2$, $\cdots$.

5. Some irreducible representations. There are two basic structures needed for the characterization of representations of C*-algebras [4]. If $\mathfrak{A}$ is a C*-algebra and φ is a representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$, then $\varphi(\mathfrak{A})$ is a self-adjoint operator algebra on $\mathscr{H}$. (It is, in fact, a C*-algebra - that is, closed with respect to taking limits relative to the operator norm [6: 4.1.9].) If we adjoin to $\varphi(\mathfrak{A})$ all the operators in $\mathfrak{B}(\mathscr{H})$ that are limits of operators in $\varphi(\mathfrak{A})$ on vectors in $\mathscr{H}$ (as described in Section 2), the resulting family $\varphi(\mathfrak{A})^-$ is a von Neumann algebra. The combination of the "type decomposition" for $\varphi(\mathfrak{A})^-$ and that for $\varphi(\mathfrak{A})'$, the von Neumann algebra consisting of those elements of $\mathfrak{B}(\mathscr{H})$ that commute with every element of $\varphi(\mathfrak{A})$ (and, hence, of $\varphi(\mathfrak{A})^-$), is one of the basic structures involved. It need not concern us, for the present, since we consider irreducible representations in this section. When φ is irreducible, $\varphi(\mathfrak{A})^- = \mathfrak{B}(\mathscr{H})$ (equivalently, $\varphi(\mathfrak{A})'$ consists of scalar multiples of I). (We may take either of these conditions as our definition of irreducibility.) If $\mathfrak{A}$ is the CAR algebra, then $\mathscr{H}$ is necessarily separable when φ is irreducible. (This is true, more generally, when $\mathfrak{A}$ has a countable number of generators as a C*-algebra.) In this way, the considerations of the type decompositions of $\varphi(\mathfrak{A})^-$ and $\varphi(\mathfrak{A})'$ disappear.

The second basic structure needed is measure-theoretic in nature. While it has a general description [4], it usually appears in more convenient special forms in special cases. We shall take advantage of such special structure for our construction of inequivalent irreducible representations of the CAR algebra.

The CAR algebra $\mathfrak{A}$ is generated by an infinite commuting family of self-adjoint subalgebras $\{\mathfrak{A}_n\}_{n=1,2,\cdots}$ with each $\mathfrak{A}_n$ containing the identity element I of $\mathfrak{A}$ and * isomorphic to $M_2(\mathbb{C})$. Choose matrix units $\{E^n_{jk}\}_{j,k=1,2}$ for each $\mathfrak{A}_n$ such that

$(E^n_{jk})^* = E^n_{kj}$. (For example, we may choose for E^n_{jk} the element of $\mathfrak{A}_n$ corresponding to the matrix in $M_2(\mathbb{C})$ with 1 in row j and column k and 0 at all other entries.) The subalgebra $\mathfrak{B}_n$ of generated by $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is * isomorphic to $M_{2^n}(\mathbb{C})$. The set of all products $E^1_{j(1)k(1)} \cdots E^n_{j(n)k(n)}$ is a (self-adjoint) system $\{F^n_{jk} : j,k = 1,\cdots, 2^n\}$ of $(2^n \times 2^n)$ matrix units for $\mathfrak{B}_n$. As n varies these systems of matrix units fulfill certain compatibility conditions:

$$F^n_{jk} = \sum_{h=1}^{r} F^m_{(j-1)r+h,(k-1)r+h} \quad (j,k = 1,\cdots,2^n)$$

where $n \le m$ and $r = 2^{m-n}$. It is reasonably clear that such a compatible family of (self-adjoint, $2^n \times 2^n$, $n = 1,2,\cdots$) matrix unit systems acting on a Hilbert space $\mathcal{H}$ generate the CAR algebra $\mathfrak{A}$ and, thereby, give rise to a representation of $\mathfrak{A}$ and of the canonical anticommutation relations.

We construct our irreducible representations of the CAR by describing such systems of matrix units on the Hilbert space $\mathcal{H}$ $L_2(S, \mathcal{S}, m)$ ($= L_2$), where S is the half-open interval $[0,1)$, $\mathcal{S}$ is the σ-algebra of Borel subsets of S and m is a (σ-finite) positive measure on $\mathcal{S}$. Let D be the subset of $[0,1)$ consisting of rationals with denominator some power of 2 (the "dyadic rationals"). Provided with addition modulo 1, S is a group and D is a subgroup. For each d in D, let g_d be translation by d (modulo 1) on S so that $\{g_d : d \in D\}$ is a group G of transformations of S. Assume that our measure m has been chosen invariant under each element of G. Define F^n_{jk} $(j,k = 1,\cdots,2^n)$ acting on L_2 by

$$(F^n_{jk}f)(s) = \begin{cases} f(s+2^{-n}(k-j)) & (2^{-n}(j-1) \le s < 2^{-n}j) \\ 0 & \text{elsewhere on } S \end{cases}$$

where $f \in L_2$. Computations show that $\{F^n_{jk} : j,k = 1,\cdots,2^n\}$ is a self-adjoint system of $2^n \times 2^n$ matrix units and that these systems form a compatible family as n varies. With h a bounded measurable function on S, let M_h denote the " multiplication" operator on L_2 that transforms f (in L_2) to the product hf. Further computation shows that F^n_{jj} is M_h where h is the characteristic function of $[2^{-n}(j-1), 2^{-n}j)$. It follows that the only operators commuting with all F^n_{jk} are those M_h such that h is (almost everywhere) invariant under each element of G. The condition that such invariant functions are constant (almost everywhere) is equivalent to the condition that G act ergodically on S (relative to m - see [6 : 8.6.6]). Thus the representation π_m of the CAR on

$L_2(S, \mathcal{S}, m)$ is irreducible if and only if G acts ergodically on S.

We say that a point s of S is an "atom" for m when $m(\{s\}) > 0$. A point s in S is an atom for m if and only if the multiplication operator corresponding to the characteristic function of the one-point set $\{s\}$ is a non-zero projection P(m) on L_2. For each n, there is a unique integer j in $1, \cdots, 2^n$ such that $2^{-n}(j-1) \leq s < 2^{-n}j$.

The multiplication operator corresponding to the characteristic function of $[2^{-n}(j-1), 2^{-n}j)$ is a projection in $\pi_m(\mathfrak{A})$ (the algebra on L_2 generated by all F^n_{jk}) and corresponds to a projection P_n in $\mathfrak{A}$ (in fact, to the matrix unit E^n_{jj}). Now $\wedge_n \pi_m(P_n) = P(m)$. Thus s is an atom for m if and only if $\wedge_n \pi_m(P_n) \neq 0$. If there is a unitary operator U such that $U^* \pi_m(A) U = \pi_{m'}(A)$ for each A in $\mathfrak{A}$ (that is, if π_m and $\pi_{m'}$ are unitarily equivalent for our two G-invariant measures m and m'), then

$$U^*P(m)U = U^* \wedge_n \pi_m(P_n)U = \wedge_n U^* \pi_m(P_n)U = \wedge_n \pi_{m'}(P_n) = P(m') .$$

Of course, then, $P(m) \neq 0$ if and only if $P(m') \neq 0$. Thus m and m' have the same set of atoms when π_m and $\pi_{m'}$ are unitarily equivalent.

For specific examples, choose a point s in S and let G(s) be its "orbit" under G. Let m_s be the measure that assigns to each Borel set, as measure, the number of points of G(s) it contains. Since G(s) is an orbit, m_s is G-invariant and the atoms of m_s are precisely the points of G(s). Now G is a countable group and S has the cardinality of the continuum. Two orbits either coincide or are disjoint. Thus there are a continuum of disjoint orbits and so a continuum of measures m_s with mutually disjoint sets of atoms. Each of these continuum measures is ergodic under G and therefore gives rise to an irreducible representation of the canonical anticommutation relations. No two of these representations are unitarily equivalent. (See [6 : pp. 759-766] for details.)

6. <u>States and representations.</u> The most effective method for producing representations of C*-algebras involves the <u>GNS construction</u> for a <u>state</u> of that algebra. A linear functional ρ on a C*-algebra $\mathfrak{A}$ is said to be a <u>state</u> of $\mathfrak{A}$ when $\rho(A) \geq 0$ for each positive operator A in $\mathfrak{A}$ and $\rho(I) = 1$. If $\mathfrak{A}$ acts on the Hilbert space $\mathcal{H}$ and x is a unit vector in $\mathcal{H}$, then the mapping $A \rightarrow \langle Ax, x \rangle$ provides us with an example of a state.

We denote this state by ω_x when $\mathfrak{A}$ is $\mathfrak{B}(\mathcal{H})$ and $\omega_x|\mathfrak{A}$ otherwise. We refer to it as a <u>vector state</u> of $\mathfrak{A}$. The state ρ is said to be <u>faithful</u> when $\rho(A) > 0$ if $A > 0$. With ρ a faithful state of $\mathfrak{A}$, the mapping $(A,B) \to \rho(B^*A)$ defines a (positive definite) inner product $\langle\ ,\ \rangle_\rho$ on $\mathfrak{A}$ relative to which it has a Hilbert space completion $\mathcal{H}_\rho$. With $\pi_\rho(A)(B)$ defined as AB, $\pi_\rho(A)$ is a linear operator on $\mathfrak{A}$ and

$$\langle\pi_\rho(A)(B),\ \pi_\rho(A)(B)\rangle_\rho = \langle AB,AB\rangle_\rho = \rho(B^*A^*AB) .$$

Now $A^*A \le \|A\|^2 I$ so that $B^*A^*AB \le \|A\|^2 B^*B$, and

$$\rho(B^*A^*AB) \le \|A\|^2 \rho(B^*B) = \|A\|^2 \langle B,B\rangle_\rho .$$

Thus $\|\pi_\rho(A)B\|_\rho^2 \le \|A\|^2\|B\|_\rho^2$ and $\|\pi_\rho(A)\| \le \|A\|$. It follows that $\pi_\rho(A)$ extends (uniquely) to a bounded linear operator, we denote again by $\pi_\rho(A)$, on $\mathcal{H}_\rho$. It is easy to check that π_ρ is a representation of $\mathfrak{A}$ on $\mathcal{H}_\rho$. We call π_ρ the GNS (Gelfand-Neumark-Segal) representation for ρ . This construction has an extension to general states (not necessarily faithful) that need not concern us. Note that $\rho(A) = \langle\pi_\rho(A)(I),I\rangle_\rho$ for each A in $\mathfrak{A}$, so that ρ becomes a vector state when "transported" to $\pi_\rho(\mathfrak{A})$. Note too that $\{\pi_\rho(A)(I) : A \in \mathfrak{A}\} = \mathfrak{A}$ and $\mathfrak{A}$ is dense in $\mathcal{H}_\rho$. The "vector" I in $\mathfrak{A}$ is said to be <u>cyclic</u> under $\pi_\rho(\mathfrak{A})$ and π_ρ is said to be a <u>cyclic representation</u> of $\mathfrak{A}$. With ρ a faithful state of $\mathfrak{A}$, π_ρ is a faithful representation of $\mathfrak{A}$ - that is, $\pi_\rho(A) = 0$ only if $A = 0$ (for $\pi_\rho(A)(I) = A$).

The class of states of the CAR algebra $\mathfrak{A}$ known as <u>product states</u> is of special interest in quantum statistical mechanics. Suppose that $\mathfrak{A}$ is generated by the commuting family $\{\mathfrak{A}_r\}$ with each $\mathfrak{A}_r$ * isomorphic to $M_2(\mathbb{C})$. Choose a self-adjoint system of 2x2 matrix units $\{E_{jk}\}$ in $\mathfrak{A}_r$. Then each A in $\mathfrak{A}_r$ has the form $\Sigma\, c_{jk}E_{jk}$ with c_{jk} a complex number. Define $\rho_r(A)$ to be $a_r c_{11}+(1-a_r)c_{22}$, where $0 < a_r \le \frac{1}{2}$. Then ρ_r is a state of $\mathfrak{A}_r$. With A_j in $\mathfrak{A}_j$, define $\rho(A_1\cdots A_n)$ to be $\rho_1(A_1)\cdots\rho_n(A_n)$ It is not difficult to show that ρ extends to a state ρ of $\mathfrak{A}$, the <u>product state</u> $\otimes\, \rho_r$. While far from apparent, it the case that the GNS representation π_ρ for ρ has the following properties: $\pi_\rho(\mathfrak{A})^-$ is a factor on $\mathcal{H}_\rho$, and with x_ρ the unit vector in $\mathcal{H}_\rho$ corresponding to I (so that $\rho(A) = \langle\pi_\rho(A)x_\rho, x_\rho\rangle_\rho$ for each A in $\mathfrak{A}$ and $\pi_\rho(\mathfrak{A})x_\rho$ is dense in $\mathcal{H}_\rho$), $\omega_{x_\rho}|\pi_\rho(\mathfrak{A})^-$ is faithful. With $\pi_\rho(\mathfrak{A})^-$ a factor, we say that ρ is a factor state of $\mathfrak{A}$. Our program in the final sections is to study these factor representations of the canonical anticommutation relations arising from the product states as described. For this purpose, we develop some of the essential theory of factors.

7. Types of factors. An initial separation of factors into types that are algebraically distinct (non-isomorphic) can be effected by studying their lattices of projections. If $\mathfrak{M}$ is a factor and E is a minimal projection in $\mathfrak{M}$ (that is, $E \neq 0$ and if $0 \leq F \leq E$ with F a projection in $\mathfrak{M}$, then $F = 0$ or $F = E$), then $\mathfrak{M}$ is isomorphic to $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In particular, I in $\mathfrak{M}$ is the sum of an orthogonal family of minimal projections in $\mathfrak{M}$. If n is the (possibly infinite) cardinal number of that family of minimal projections, then $\mathcal{H}$ has dimension n. In this case, we say that $\mathfrak{M}$ is of type I_n.

Factors need not have minimal projections. Let G be a (discrete) group all of whose conjugacy classes (other than that of the group identity) are infinite. Let $\mathcal{H}$ be $l_2(G)$ (square-summable, complex-valued functions on G with the inner product $\langle f,h\rangle = \Sigma_{g \in G} f(g)\overline{h(g)}$). For each g in G, define $(L_g f)(g')$ to be $f(g^{-1}g')$ for each g' in G, where $f \in \mathcal{H}$. Then each L_g is a unitary operator on $\mathcal{H}$ and $\{L_g : g \in G\}$ generates a factor $\mathfrak{M}$. If x_0 is the element of $\mathcal{H}$ that takes the value 1 at the group identity and 0 at all other elements of G, then $\omega_{x_0}|\mathfrak{M}$ is a state τ of $\mathfrak{M}$ with very special properties. Most importantly, $\tau(AB) = \tau(BA)$ (an easy computation). We say that τ is a tracial state on $\mathfrak{M}$ (although we may happen on it in many different ways). It is also the case that $\mathfrak{M}$ has no minimal projections. If it did, $\mathfrak{M}$ would be isomorphic to $\mathfrak{B}(\mathcal{H})$ and I would be the sum of n minimal projections. The value of τ at all minimal projections is the same positive number b (easily seen) and $\tau(I)(= 1)$ would be nb - from which, n is finite. Thus $\mathcal{H}$, $\mathfrak{B}(\mathcal{H})$, and $\mathfrak{M}$, would all have finite linear dimension. But $\{L_g\}$ is an infinite linearly independent family in $\mathfrak{M}$ (not difficult). We call factors with no minimal projections and a tracial state, factors of type II_1. Specific examples are obtained by choosing for G the free group on n (> 1) generators or the group of those permutations of the integers that move at most a finite number of integers. The factors of type II_1 obtained from these two groups can be shown to be non-isomorphic. Factors of type II_1 need not be isomorphic to one another.

A factor $\mathfrak{M}$ may have no minimal projections and may have no tracial state but possess a family of projections $\{E_a\}$ with the following properties: $\Sigma E_a = I$, $\{A : A \in \mathfrak{M}, E_a A E_a = A\}$ $(= \mathfrak{M}_a)$ has a tracial state τ_a ($\mathfrak{M}_a$ is necessarily a factor). In this case, with H a positive element of $\mathfrak{M}$, $\Sigma\, \tau_a(H)$ $(= \tau(H))$ is in $[0,\infty]$. The mapping τ of $\mathfrak{M}^+$, the positive elements in $\mathfrak{M}$, into $[0,\infty]$ is a <u>tracial weight</u> on $\mathfrak{M}$ - that is, $\tau(H+K) = \tau(H) + \tau(K)$, $\tau(aH) = a\,\tau(H)$ when $a > 0$, and $\tau(A^*A) = \tau(AA^*)$ for each A in $\mathfrak{M}$. In addition, τ is <u>semi-finite</u> - that is, each T in $\mathfrak{M}$ is the limit (on vectors) of linear combinations of elements in $\mathfrak{M}^+$ at which τ takes finite values. Finally, τ is normal - that is there is a family of vectors $\{x_b\}$ such that $\tau(H) = \Sigma\langle Hx_b, x_b\rangle$ for each H in $\mathfrak{M}^+$. We say that $\mathfrak{M}$ is a factor of type II_∞ when it has no minimal projections, has no tracial state but has a non-zero normal semi-finite tracial weight.

Specific examples of factors of type II_∞ arise from specific examples of factors of type II_1. If $\mathfrak{M}$ is a factor of type II_1 acting on a Hilbert space $\mathcal{H}$ and $\mathcal{K}$ is the countable (Hilbert-space) direct sum of $\mathcal{H}$ with itself, then each element of $\mathfrak{B}(\mathcal{K})$ corresponds to an infinite matrix all of whose entries lie in $\mathfrak{B}(\mathcal{H})$. The elements in $\mathfrak{B}(\mathcal{K})$ whose matrix representations have all entries in $\mathfrak{M}$ form a factor of type II_∞. Moreover each factor of type II_∞ has this form (namely, infinite matrices with entries in a factor of type II_1).

Finally, there are the factors that possess no non-zero normal semi-finite tracial weights. These are the factors of type III. They play the dominant role in the operator algebra formulation of quantum field theory and quantum statistical mechanics. Specific examples were first obtained from an ergodic-theoretic construction [7]. We won't describe these, since we shall be constructing examples by other means.

The type classification of factors provides us with a means of distinguishing among factor representations of the canonical anticommutation relations. We say that a factor representation π of a C*-algebra $\mathfrak{A}$ is of type I_n, II_1, II_∞, or III, when the factor $\pi(\mathfrak{A})^-$ has the corresponding type. With π_ρ the GNS representation for the state ρ of $\mathfrak{A}$ we say that ρ is of type I_n, II_1, II_∞, or III, when π_ρ has the corresponding type.

8. Modular theory and the T-invariant. A deep result of Tomita's (see [6 : 9.29]) associates with each faithful state ω of a von Neumann algebra $\mathcal{R}$ some structure (the modular structure of $\mathcal{R}$ and ω) that will be critical in helping us distinguish factor representations of the canonical anticommutation relations. It is easiest to describe this modular structure after using the GNS representation π_ω of $\mathcal{R}$ for ω. Through this representation, we may assume that $\mathcal{R}$ acts on a Hilbert space $\mathcal{H}$ and that $\omega(A) = \langle Au,u\rangle$ for each A in $\mathcal{R}$ where u is a unit vector in $\mathcal{H}$ such that $\mathcal{R}u$ is dense in $\mathcal{H}$ (we say that u is generating for $\mathcal{H}$) and $T = 0$ when $Tu = 0$ and $T \in \mathcal{R}$ (we say u is separating for $\mathcal{R}$). The mapping S_0 that assigns A^*u to Au for each A in $\mathcal{R}$ is a conjugate-linear operator on the dense domain $\mathcal{R}u$. Its adjoint contains the conjugate-linear operator F_0 that assigns A'^*u to $A'u$, where A' is an operator in the commutant $\mathcal{R}'$ of $\mathcal{R}$ (those bounded operators commuting with all operators in $\mathcal{R}$). Now $\mathcal{R}'u$ is dense in $\mathcal{H}$ (this follows from the fact that u is separating for $\mathcal{R}$), so that S_0 has a closure S. Let Δ be S^*S. Then S has a polar decomposition $J\Delta^{\frac{1}{2}}$, where J is a conjugate-linear isometry of $\mathcal{H}$ onto itself. From the fact that $S_0 = S_0^{-1}$, it follows that $J = J^{-1} = J^*$. Tomita's main result states that $J\mathcal{R}J = \mathcal{R}'$ and $\Delta^{it}\mathcal{R}\Delta^{-it} = \mathcal{R}$ for each real t. In particular, with $\sigma_t(A)$ defined to be $\Delta^{it}A\Delta^{-it}$ for each A in $\mathcal{R}$, we have that σ_t is a * automorphism of $\mathcal{R}$ for each real t. Moreover, $\sigma_{t+t'} = \sigma_t\sigma_{t'}$ for each pair of real numbers t and t'. We refer to the one-parameter group of * automorphisms $t \to \sigma_t$ of $\mathcal{R}$ as the modular automorphism group for ω (or u).

The state ω and the one-parameter automorphism group $\{\sigma_t\}$ are interrelated by a condition introduced into the infinite-system formulation of quantum statistical mechanics to describe equilibrium states by Haag, Hugenholtz, Winnink [3,7]. It results from the construction of σ_t that for each pair of elements A and B in $\mathcal{R}$ there is a complex-valued function f defined, continuous, and bounded on the strip $\{z \in \mathbb{C} : 0 \le \operatorname{Im} z \le 1\}$ $(= \Omega)$, and analytic on the interior of that strip, such that

$$f(t) = \omega(\sigma_t(A)B), \qquad f(t+i) = \omega(B\sigma_t(A)) \qquad (t \in \mathbb{R}).$$

When a one-parameter group of automorphisms $\{\sigma_t\}$ and a state ω of a von Neumann algebra $\mathcal{R}$ fulfill this condition, we say that the group satisfies the modular condition relative to ω. It follows from this condition (together with some complex function theory) that $\omega(\sigma_t(A)) = \omega(A)$ for each A in $\mathfrak{R}$ and all real t. (We say that ω is invariant under σ_t.) Using the assumption that ω is faithful, the modular condition yields that $\sigma_t(H) = H$ for some H in $\mathfrak{R}$ and all real t, if and only if $\omega(AH) = \omega(HA)$ for all A in $\mathfrak{R}$. An element such as H is said to lie in the <u>centralizer</u> of ω. (See [6 : 9.2.13 and 9.2.14].) In addition, there is a <u>unique</u> (continuous) one-parameter group of * automorphisms of $\mathfrak{R}$ (the <u>modular group</u>) that satisfies the modular condition relative to a given faithful normal state ω of $\mathfrak{R}$ [6 :9.2.16]. A useful extension of this last result [6 : 9.2.17] asserts that it suffices to find our complex-valued function f satisfying the continuity, boundedness, and analyticity condition just for A and B in a self-adjoint subalgebra of $\mathfrak{R}$ provided each element of $\mathfrak{R}$ is a limit (on vectors) of elements of this subalgebra.

The relation between the modular automorphisms corresponding to two faithful normal states of $\mathfrak{R}$ is established in [1]. If $\{\alpha_t\}$, $\{\beta_t\}$ are the corresponding modular automorphism groups, there is, for each t, a unitary operator U_t <u>in $\mathfrak{R}$</u> such that, for each A in $\mathfrak{R}$, $\alpha_t(A) = U_t \beta_t(A) U_t^*$, $U_{s+t} = U_s \beta_s(U_t)$, and $U_t x \to x$ for each x in $\mathcal{H}$ as $t \to 0$. (The mapping $t \to U_t$ is the <u>Connes cocycle</u> relating α_t and β_t.) In particular, α_t is an inner * automorphism of $\mathfrak{R}$ (that is, there is a unitary operator V in $\mathfrak{R}$ such that $\alpha_t(A) = VAV^*$ for each A in $\mathfrak{R}$) if and only if β_t is inner. Thus the set $T(\mathfrak{R})$ of real numbers t such that σ_t is inner for the modular automorphism group $\{\sigma_t\}$ of $\mathfrak{R}$ relative to a faithful normal state of $\mathfrak{R}$ is the same for all faithful normal states of $\mathfrak{R}$. It follows that this set (which is trivially seen to be a subgroup of $\mathbb{R}$) is an (isomorphism) invariant for $\mathfrak{R}$. It is the <u>T-invariant</u> [1], and will help us to distinguish certain factor representations of the canonical anti-commutation relations. (See [6 : 13.1.9].)

<u>9. Some general properties of the T-invariant.</u> If $\mathfrak{M}$ is a factor not of type III, a sequence of Radon-Nikodym results (in the non-commutative setting of operator algebras), involving the normal tracial weight, leads to the conclusion that $T(\mathfrak{M}) = \mathbb{R}$, that is, each modular automorphism group of $\mathfrak{M}$ consists entirely of inner automorphisms. The T-invariant is not at all sensitive to factors of types other than III. (See [6 : 9.2.21].) In the case of a type III factor $\mathfrak{M}$ acting on a separable Hilbert space, almost the reverse situation prevails. A measure-theoretic,

cohomological argument shows that all the automorphisms of a one-parameter group can be inner only if the group is implemented by a one-parameter group of unitary operators in $\mathfrak{M}$ ([5] - see also [6 : 14.4.3-14.4.10]). In this event, the factor will have a non-zero, normal, tracial weight and cannot be of type III [6 : 9.2.21]. Thus, for a factor $\mathfrak{M}$ of type III acting on a separable Hilbert space, $T(\mathfrak{M})$ must be different from $\mathbb{R}$. Indeed, complicated measure- and group-theoretic considerations show that $T(\mathfrak{M})$ must have Lebesgue measure 0.

As to what subgroups of $\mathbb{R}$ appear as $T(\mathfrak{M})$ for some factor $\mathfrak{M}$ acting on a separable Hilbert space, it is known that each countable subgroup of $\mathbb{R}$ arises in this way. Suppose G is an arbitrary subgroup of $\mathbb{R}$ (perhaps $\mathbb{R}$ itself) considered as a discrete group. With ω a faithful state of a factor $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$ and $\{\sigma_t\}$ the corresponding modular automorphism group, a special construction allows us to realize G as $T(\mathfrak{N})$ for some factor $\mathfrak{N}$. To effect this construction, we introduce the Hilbert space direct sum $\Sigma \oplus \mathcal{H}_t$ $(= \mathcal{K})$ of copies $\mathcal{H}_t$ of $\mathcal{H}$ (one for each real t). The elements of $\mathcal{K}$ are functions x on $\mathbb{R}$ such that $x(t) \in \mathcal{H}_t$ for each real t (and $\Sigma \|x_t\|^2 < \infty$). We define operators $\Phi(A)$ on $\mathcal{K}$ for each A in $\mathfrak{M}$ by $(\Phi(A)x)(t) = Ax(t)$ and $V(t)$ by $V(t)x(s) = \Delta^{it}x(s-t)$, where $\Delta^{it}A\ \Delta^{-it} = \sigma_t(A)$ for each A in $\mathfrak{M}$. Then the von Neumann algebra $\mathfrak{R}$ generated by $\{\Phi(A), V(t) : A \in \mathfrak{M}, t \in \mathbb{R}\}$ is called the <u>crossed-product</u> of $\mathfrak{M}$ by the automorphism group $\{\sigma_t\}$. Let $\mathfrak{N}$ be the von Neumann subalgebra of $\mathfrak{R}$ generated by $\Phi(\mathfrak{M})$ and $\{V(t) : t \in G\}$. (Then $\mathfrak{N}$ is isomorphic to the crossed-product of $\mathfrak{M}$ by the automorphism group $\{\sigma_t : t \in G\}$.) If σ_t is an outer automorphism of $\mathfrak{M}$ for t in G different from the identity, then $\mathfrak{N}$ is a factor [6:13.1.5] Relative to appropriately defined (normal) states of $\mathfrak{R}$ and $\mathfrak{N}$, $\{V(t) : t \in \mathbb{R}\}$ implements the modular automorphism groups of $\mathfrak{R}$ and $\mathfrak{N}$. (See [6 : 14.4.19].)

In the next section, we construct certain factor representations of the canonical anticommutation relations and compute the the T-invariant for the factors arising. Among these is a factor $\mathfrak{M}$ (of type III) for which $T(\mathfrak{M}) = \{0\}$ (no σ_t is inner other than the identity automorphism). If we perform the foregoing construction with this factor as $\mathfrak{M}$, then $\mathfrak{R}$ becomes a factor of type III and $T(\mathfrak{R}) = \mathbb{R}$ - but $\mathcal{K}$ is <u>not separable</u>. In this same case, $T(\mathfrak{N}) = G$, the restriction of $\mathfrak{N}$ to $\Sigma_{t \in G} \oplus \mathcal{H}_t$ is a * isomorphism, and $\Sigma_{t \in G} \oplus \mathcal{M}_t$ is separable.

10. Some factor representations of the CAR. With $\mathfrak{A}$ the CAR algebra and ρ the product state of $\mathfrak{A}$ determined by the sequence of numbers $\{a_r\}$ in $(0,\frac{1}{2}]$, as described in Section 6, we noted that π_ρ is a factor representation $\mathfrak{A}$. We introduce b_r as $\log\ (a_r^{-1}(1-a_r))$. Then:

(1) $T(\pi_\rho(\mathfrak{A})^-) = \{t \in \mathbb{R} : \sum_{r=1}^{\infty} [1 - |a_r^{1+it} + (1-a_r)^{1+it}|] < \infty\}$

$$\{t \in \mathbb{R} : \sum_{r=1}^{\infty} e^{-b_r} \sin^2(\tfrac{1}{2}b_r t) < \infty\}$$

(2) If $a_r \to \frac{1}{2}$ and $b_r^2 \to \infty$, then $\pi_\rho(\mathfrak{A})^-$ is a factor of type II_1 and $T(\pi_\rho(\mathfrak{A})^-) = \mathbb{R}$.

(3) If $\Sigma a_r < \infty$ (equivalently, $\Sigma e^{-b_r} < \infty$), then $\pi_\rho(\mathfrak{A})^-$ is a factor of type I_∞.

(4) If $a_r \to 0$ and $\Sigma a_r = \infty$, then $\pi_\rho(\mathfrak{A})^-$ is a factor of type III.

(5) If $a_r = (r+1)^{-1}$, then $T(\pi_\rho(\mathfrak{A})^-) = \{0\}$. In this case, $\pi_\rho(\mathfrak{A})^-$ serves as the factor $\mathfrak{m}$ we needed in Section 9.

With a_r as in (5), b_r is $\log r$. If we alter the a_r (equivalently, the b_r) slightly, the T-invariant can change significantly. Let $[x]$ denote the largest integer not exceeding x.

(6) If $b_r = [\log r]$ $(r = 3,4,\cdots)$, then

$$T(\pi_\rho(\mathfrak{A})^-) = \{0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots\}.$$

(7) If $b_r = n!$ when $[e^{n!}] < r \le [e^{(n+1)!}]$, then $T(\pi_\rho(\mathfrak{A})^-)$ contains each rational multiple of 2π (but is not $\mathbb{R}$).

(See [6 : 13.1.15,13.4.9-13.4.15] for details.)

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