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Algebras of unbounded functions and operators

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1. Introduction and preliminaries

In [7] M.H. Stone introduces the concept of what has become known as an *extremely disconnected* compact Hausdorff space. These are the compact Hausdorff spaces with the property that the closure of each open subset is open as well as closed. Such closed and open subsets are said to be *clopen*. Stone describes the main result, relating such spaces to their algebras of continuous functions in [7] and presents the detailed analysis and arguments in [9]. He notes that X is an extremely disconnected compact Hausdorff space if and only if $C(X, \mathbb{R})$, the algebra of continuous, real-valued functions on X, is a boundedly complete lattice (that is, each subset of $C(X, \mathbb{R})$ that has an upper bound in $C(X, \mathbb{R})$, relative to the pointwise ordering, has a least upper bound). At the same time he states the equivalent condition that each bounded Baire function differs from a continuous function on a *meager* set (that is, a subset of a countable union of closed nowhere-dense subsets of X – also called *a set of the first category*).

Again in [7], Stone alludes to the possibility of dealing with "unbounded continuous" functions on an extremely disconnected compact Hausdorff space X. In [9; Theorem 9], he proves that each finite Baire function differs from the quotient of two continuous functions on a meager set, where the denominator vanishes on a nowhere-dense subset of X. Fell and Kelley [1] continue Stone's work. They extend one of Stone's results slightly; by showing that each Borel function on X with values in a compact metric space differs from a continuous function on X on a meager subset of X. Their purpose is to use the Riemann sphere as the compact metric space; so that they can deal with (unbounded) complex-valued functions. They note, "Let $\bar{\mathscr{C}}$ be the set of all continuous functions on X to the complex sphere which are ∞ only on a non-dense set. $\bar{\mathscr{C}}$ is an algebra, if we agree that fg and $f+g$ are those continuous functions which agree with the product and sum save on a set of Cat I." (Their note is a sketch account – a fuller report would establish uniqueness of the associated continuous function and deal with those that assume the value ∞ only on a nowhere-dense subset.) They continue Stone's program, describing a Borel function calculus for normal operators and noting that "The algebra $\bar{\mathscr{C}}$ is isomorphic to an algebra $\bar{\mathscr{A}}$ ... ." Their starting point is an abelian von Neumann algebra $\mathscr{A}$ and an isomorphism of $\mathscr{A}$ with $C(X)$, the algebra of complex-valued continuous functions on an extremely disconnected

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compact Hausdorff space X. They describe a process for extending this isomorphism to $\bar{\mathscr{C}}$ and mapping $\bar{\mathscr{C}}$ onto an unspecified algebra $\bar{\mathscr{A}}$ of (unbounded) normal operators containing $\mathscr{A}$. (We shall see that the image is, in fact, the algebra of operators "affiliated" with $\mathscr{A}$ - to be described later.) This extended isomorphism amounts to the introduction of an unbounded-Borel-function calculus for the normal operators in $\mathscr{A}$ (and $\bar{\mathscr{A}}$), on the one hand, and a simultaneous spectral resolution of those operators on the other. This was, of course, a very large part of Stone's point in 1940.

In a broad sense, the purpose of this article is to supply the next logical steps in Stone's program: a fuller understanding of the structure of the algebra of unbounded continuous functions on X, a fuller understanding of the algebra of unbounded normal operators containing the initial abelian von Neumann algebra, and detailed understanding of the relation between the two. To state, more specifically, what is done, and by way of establishing basic definitions and preliminary notation, we define the principal structures to be studied.

1.1 Definition. Let X be an extremely disconnected compact Hausdorff space. A (finite) complex-valued function f defined and continuous on $X \backslash Z$, where Z is a closed nowhere-dense subset of X, is said to be a *normal function* (*on* X) when, given a point p in Z and a positive number n, there is an open set $\mathcal{O}$ in X containing p such that $n \leq |f(q)|$ for each q in $\mathcal{O} \backslash Z$. If f is real-valued, we say that f is a *self-adjoint function* (*on* X). We denote by $\mathscr{N}(X)$ and $\mathscr{S}(X)$, respectively, the family of normal functions on X and the family of self-adjoint functions on X. ■

It follows from this definition that the functions in $\mathscr{N}(X)$ are those continuous mappings of X into the Riemann sphere that assume the value ∞ on a (closed) nowhere-dense subset of X. In §2 we show that $\mathscr{N}(X)$ and $\mathscr{S}(X)$ are algebras (over the complex numbers, $\mathbb{C}$, in the first case, and over the real numbers, $\mathbb{R}$, in the second). To pass, as Fell and Kelley do in [1], through their extension of Stone's result concerning the association of a continuous function on X with a Borel function, would represent a shortening of the argument (though not a significant shortening if careful and detailed arguments are given); but this route encounters some disadvantages when our special purposes are taken into consideration. It does not give us a close enough view of how the normal functions combine to produce the normal functions that are their sum and product. (We shall see that the sum and product is the pointwise sum and product at points where both functions are finite - so that this sum and this product are no distant relation to the normal functions that give rise to them.) It masks the fact that the existence of these algebras is a truly bizarre and remarkable phenomenon (stemming from the equally bizarre and remarkable properties of the extremely disconnected compact Hausdorff spaces). To illustrate this last point more fully, we consider the (pointwise) algebraic structure on the family of continuous mappings of the unit interval [0, 1] into the Riemann sphere (as above). If $f(x)$ is x^{-1}

$+\sin(x^{-1})$ and $g(x)$ is $-x^{-1}$, what value (in the Riemann sphere) can be assign to $f+g$ at 0 in order to arrive at a continuous mapping? Why are we not embarrassed in the same way when we deal with an *extremely disconnected* compact Hausdorff space? The extraordinary topological properties of such spaces shield us from the difficulties so easily fabricated on a space such as $[0, 1]$. It is not enough to have total-disconnectedness for a genuine algebraic structure to exist as is illustrated by the space $\left\{0, \frac{1}{n} : n=1, 2, \dots\right\}$ where the restrictions of the functions f and g, defined above, lead to the same problems. We shall see (Corollary 2.7), for example, that if f is a self-adjoint function on an extremely disconnected compact Hausdorff space X then at no point of X does f take (both) arbitrarily large positive and negative values.

In the definition that follows, we describe the parallel operator-algebraic constructs.

1.2 Definition. Let $\mathscr{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. We say that a closed densely defined operator T is *affiliated* with $\mathscr{A}$ (and we write $T\eta\mathscr{A}$, in this case) when $U^*TU=T$ for each unitary operator U commuting with $\mathscr{A}$. When $\mathscr{A}$ is abelian, we denote by $\mathscr{N}(\mathscr{A})$ the family of operators affiliated with $\mathscr{A}$ and by $\mathscr{S}(\mathscr{A})$ the family of self-adjoint operators affiliated with $\mathscr{A}$. ■

In §3 we shall place an algebraic structure on $\mathscr{S}(\mathscr{A})$ (and $\mathscr{N}(\mathscr{A})$) in which sum and product involve passing to the closure of the ordinary sum and product. We shall see (Lemma 3.2 and Theorem 3.3(i)) that each finite collection of operators in $\mathscr{N}(\mathscr{A})$ have a common core. (A *core* for a closed densely defined operator A is a linear submanifold of $\mathscr{H}$ such that the graph of the restriction A_0 of A to this submanifold is dense in the graph of A - that is, A_0 has closure $\bar{A}_0$ equal to A.) As in the case of $\mathscr{N}(X)$, where the function sum of two normal functions may not be normal but will have a (unique) normal extension, the sum $A+B$ of two operators A and B in $\mathscr{N}(\mathscr{A})$ may not be closed but will have a (unique) closed extension $A \hat{+} B\eta\mathscr{A}$. Similarly AB will have a (unique) closed extension $A \hat{\cdot} B\eta\mathscr{A}$ and $A\hat{\cdot}B=B\hat{\cdot}A$. In addition $A^*\hat{\cdot}A=A^*A=A\hat{\cdot}A^*=AA^*$ for each A in $\mathscr{N}(\mathscr{A})$. (At the heart of the spectral theory of unbounded operators is the fact that A^*A and AA^* are self-adjoint for each closed densely defined operator A.) In this case ($AA^*=A^*A$), we say that A is *normal* (in obvious analogy with bounded operators). Thus each operator affiliated with an abelian von Neumann algebra is normal. We shall see, too, that each normal operator is affiliated with an abelian von Neumann algebra (Theorem 3.6).

In §4 we shall extend the isomorphism between $\mathscr{A}$ and $C(X)$ to an isomorphism between $\mathscr{N}(\mathscr{A})$ and $\mathscr{N}(X)$. (This extension carries $\mathscr{S}(\mathscr{A})$ onto $\mathscr{S}(X)$.) Thus $\mathscr{N}(X)$ is isomorphic to the *full* algebra of operators affiliated with $\mathscr{A}$. ("Affiliation" was known but not too well understood at the time [1] appeared.)

In §5 we characterize $\mathscr{S}(X)$ as a vector lattice. This requires an "unbounded-function" Stone-Weierstrass theorem, which has been proved in §2 (Theorem 2.13).

We note, with gratitude, the support of the National Science Foundation (USA) during the preparation of this article. The techniques described here have been used as the basis for the presentation of the spectral resolution of unbounded normal operators and the introduction of a Borel function calculus for such operators in [3; §5.6]. The reader who wishes a detailed account of those applications of this theory can pursue the study there. Much of what appears here has been affected by that presentation. It is a pleasure to record our debt to John Ringrose for many illuminating conversations on these topics.

2. The algebra of functions

We assume throughout this section that X is an extremely disconnected compact Hausdorff space. If f is a self-adjoint function defined on $X\backslash Z$, we can distinguish three subsets of Z: the subset Z_+ of those points p such that $f(q)$ tends to $+\infty$, as q in $X\backslash Z$ tends to p; the subset Z_- of those p such that $f(q)$ tends to $-\infty$; and the subset $Z_\pm$ of all other points in Z. If p_0 is a limit point of Z_+ then $p_0 \in Z$. Moreover, $p_0 \notin Z_-$ for otherwise some open set $\mathcal{O}$ containing p_0 meets $X\backslash Z$ in points at which f is negative. But $\mathcal{O}$ contains a point of Z_+ and hence points of $X\backslash Z$ (since Z is nowhere dense) at which f is large - a contradiction. Similarly Z_- has no limit points in Z_+. Thus if $Z_\pm$ is empty, both Z_+ and Z_- are closed in X. We shall see (in Corollary 2.7) that $Z_\pm$ is empty. For the moment, we note some obvious facts concerning this in the remark that follows.

2.1 Remark. If f is normal and defined on $X\backslash Z$, then $|f|$ is self-adjoint (defined on $X\backslash Z$) and $Z=Z_+$. If $f\geq 0$ on $X\backslash Z$, $Z=Z_+$. If $f\leq 0$ on $X\backslash Z$, $Z=Z_-$. In either case ($f\geq 0$ or $f\leq 0$), $Z_\pm=\emptyset$.

2.2 Lemma. *If f is defined and continuous on $X\backslash Z$, where Z is a closed nowhere-dense subset of X, the following conditions are equivalent:*

(i) *f is normal;*

(ii) *$\{q: q\in X\backslash Z; |f(q)|<n\}^- \subseteq X\backslash Z$ for each positive integer n;*

(iii) *there is an increasing sequence $\{X_n\}$ of clopen subsets of X such that $X_n\subseteq X\backslash Z$ for each n, $X\backslash\bigcup_{n=1}^\infty X_n$ is nowhere dense in X, and, for each positive a, there is an m such that $a\leq|f(q)|$ when $q\in(X\backslash Z)\backslash X_m$;*

(iv) *there is a countable family $\{Y_n\}$ of mutually disjoint clopen subsets of X such that $Y_n\subseteq X\backslash Z$ for each n, $X\backslash\bigcup_{n=1}^\infty Y_n$ is nowhere dense in X, and, for each positive a, there is an m such that $a\leq|f(q)|$ when $q\in(X\backslash Z)\backslash\bigcup_{n=1}^m Y_n$.*

Proof. If f is normal and $p\in Z$ then there is an open set $\mathcal{O}$ containing p such that $n<|f(q)|$ if $q\in\mathcal{O}\cap(X\backslash Z)$. Thus p is not a limit point of $\{q: q\in X\backslash Z; |f(q)|<n\}$, and (ii) follows.

In any event, $\{q: q \in X\backslash Z; |f(q)| < n\}$ is open in $X\backslash Z$, hence, in X; since f is continuous on $X\backslash Z$ and $X\backslash Z$ is open in X. Thus the closure, X_n, of this set is clopen. Given (ii), $\bigcup_{n=1}^{\infty} X_n = X\backslash Z$ so that $X\backslash\bigcup_{n=1}^{\infty} X_n = Z$, which is nowhere dense in X. Moreover if $q \in (X\backslash Z)\backslash X_n$ then $n \le |f(q)|$ and (iii) follows.

Given (iii), (iv) follows by letting Y_1 be X_1 and Y_n be $X_n \backslash X_{n-1}$ for $n = 2, 3, \ldots$.

Given (iv), if $p \in Z$ then $X\backslash\bigcup_{j=1}^{n} Y_n$ is an open set $\mathcal{O}$ containing p such that, for each q in $(X\backslash Z)\cap\mathcal{O}$, $a \le |f(q)|$. Thus f is normal. ∎

Note that if f is normal and $\mathcal{U}$ is a bounded open subset of $\mathbb{C}$ (say, $|z| \le n$ for z in $\mathcal{U}$) then

(1) $\quad f^{-1}(\mathcal{U})^{-} \subseteq \{q: q \in X\backslash Z; |f(q)| < n\}^{-} \subseteq X\backslash Z.$

We could replace (ii) of the preceding lemma by:

(ii)′ $\quad f^{-1}(\mathcal{U})^{-} \subseteq X\backslash Z$ for each open bounded $\mathcal{U}$ in $\mathbb{C}$.

We note for the lemma that follows, that $Z \cup Z'$ is a (closed) nowhere-dense subset of X when Z and Z' are (from the Baire Category Theorem).

2.3 Lemma. *If the normal function f defined on $X\backslash Z$ and g defined on $X\backslash Z'$ coincide on an everywhere-dense subset X_0 of $X\backslash(Z\cup Z')$, then $Z = Z'$ and f and g coincide on $X\backslash Z$.*

Proof. If $p \in Z$ and a positive a is given, there is an open set $\mathcal{O}$ containing p such that $a \le |f(q)|$ if $q \in (X\backslash Z)\cap\mathcal{O}$. Since X_0 is dense in $X\backslash(Z\cup Z')$ which, in turn, is dense in X, $\mathcal{O}\cap X_0$ contains some point q. We have $a \le |f(q)| = |g(q)|$; so that $|g|$ assumes large values near p. Thus $p \in Z'$, and $Z \subseteq Z'$. Symmetrically $Z' \subseteq Z$, and $Z = Z'$. As f and g coincide on the dense subset X_0 of $X\backslash Z$ and are continuous on $X\backslash Z$, they coincide on $X\backslash Z$. ∎

The preceding lemma assures us that a function defined and continuous on a dense subset of X has *at most* one normal extension.

2.4 Lemma. *If $\{X_n\}$ is a family of mutually disjoint clopen subsets of X and f_n is a continuous function on X_n, there is a continuous function f on X, vanishing on $X\backslash(\bigcup_{n=1}^{\infty} X_n)^{-}$, whose restriction to X_n is f_n for each n if and only if $\{\|f_n\|\}$ is bounded. If Y_n is an ascending sequence of clopen subsets of X and g is a function defined on $\bigcup_{n=1}^{\infty} Y_n$ and continuous and bounded there, then there is a function h in $C(X)$ vanishing on $X\backslash(\bigcup_{n=1}^{\infty} Y_n)^{-}$ and equal to g on $\bigcup_{n=1}^{\infty} Y_n$.*

Proof. If there is an f with the properties described, $\|f\|$ is a bound for $\{\|f_n\|\}$. Suppose, now, that $\{\|f_n\|\}$ is bounded. Then $\{\|(\mathrm{Re} f_n)_+\|\}$, $\{\|(\mathrm{Re} f_n)_-\|\}$, $\{\|(\mathrm{Im} f_n)_+\|\}$ and $\{\|(\mathrm{Im} f_n)_-\|\}$ are bounded. If the first assertion of the lemma is established when all $f_n \ge 0$, then we can find functions $(\mathrm{Re} f)_+$, $(\mathrm{Re} f)_-$, $(\mathrm{Im} f)_+$, $(\mathrm{Im} f)_-$ in $C(X)$ whose restrictions to X_n are $(\mathrm{Re} f_n)_+$, $(\mathrm{Re} f_n)_-$, $(\mathrm{Im} f_n)_+$, $(\mathrm{Im} f_n)_-$, respectively, and that vanish on $X\backslash(\bigcup_{n=1}^{\infty} X_n)^{-}$. Thus $(\mathrm{Re} f)_+ - (\mathrm{Re} f)_- + i[(\mathrm{Im} f)_+ - (\mathrm{Im} f)_-]$ is the desired function f. We assume (as we may without loss of generality) that $f_n \geq 0$ for each n. Let h_n be the function in $C(X)$ equal to f_j on X_j for $j=1,\ldots,n$ and vanishing on $X\setminus\bigcup_{j=1}^{n} X_j$. Then $\{h_n\}$ is an increasing sequence of functions in $C(X)$ bounded above by an upper bound for $\{\|f_n\|\}$. The least upper bound f in $C(X)$ of $\{h_n\}$ has the desired properties; for the function equal to f on $(\bigcup_{n=1}^{\infty} X_n)^-$ and vanishing on $X\setminus(\bigcup_{n=1}^{\infty} X_n)^-$ is an upper bound for $\{h_n\}$ in $C(X)$; so that f vanishes on $X\setminus(\bigcup_{n=1}^{\infty} X_n)^-$. If $\tilde{f}_n$ is the function (in $C(X)$) equal to f_n on X_n and vanishing on $X\setminus X_n$ then $\tilde{f}_n \leq h_n \leq f$ and $f-(g_n-\tilde{f}_n)$ is an upper bound in $C(X)$ for $\{h_n\}$, where g_n is the function (in $C(X)$) equal to f on X_n and vanishing on $X\setminus X_n$. Thus $g_n=\tilde{f}_n$, and the restriction of f to X_n is f_n.

The last assertion of the lemma follows by letting X_1 be Y_1, X_n be $Y_n - Y_{n-1}$ for $n=2,3,\ldots,f_n$ be the restriction of g to X_n, and applying what we have just proved. ■

An obvious modification of the preceding argument establishes the preceding lemma in the case where $\{X_n\}$ is replaced by an arbitrary family of mutually disjoint clopen sets. We will not have need for the more general result thus obtained.

2.5 Lemma. *The function f defined on a subset of X has a normal extension if and only if its real and imaginary parts have normal extensions.*

Proof. If $\mathrm{Re} f$ and $\mathrm{Im} f$ have normal extensions g and h to $X\setminus Z'$ and $X\setminus Z''$, respectively, then $g+ih$ (defined on $X\setminus(Z'\cup Z'')$) extends f and it is normal (since $|(g+ih)(q)|^2 = |g(q)|^2 + |h(q)|^2$ for q in $X\setminus(Z'\cup Z'')$).

If f has a normal extension the real and imaginary parts of that extension extend the real and imaginary parts of f. We assume (as we may without loss of generality) that f is normal and defined on $X\setminus Z$. We denote by $R_{n,m}$ the open rectangle in $\mathbb{C}$ with vertices $(-n,-m)$, $(n,-m)$, (n,m) and $(-n,m)$. From Lemma 2.2 (ii) (and the comment following it), $f^{-1}(R_{n,m})^-$ is a clopen subset $X_{n,m}$ of X contained in $X\setminus Z$. Since $|(\mathrm{Re} f)(q)| \leq n$ for q in $X_{n,m}$, Lemma 2.4 applies and $\mathrm{Re} f|\bigcup_{m=1}^{\infty} X_{n,m}$ has a continuous extension g_n to $(\bigcup_{m=1}^{\infty} X_{n,m})^-$ $(=X_n)$ vanishing on $X\setminus X_n$. If $n \leq n'$ then $X_n \subseteq X_{n'}$ and $g_{n'}$ agrees with g_n on $\bigcup_{m=1}^{\infty} X_{n,m}$ (both are equal to $\mathrm{Re} f$ there) so that $g_{n'}$ and g_n agree on X_n. Note, too, that $X\setminus\bigcup_{n=1}^{\infty} X_n$ $(=Z') \subseteq Z$. If g is defined to be g_n on X_n for each n, then g is defined on $X\setminus Z'$ and $n \leq |g(q)|$ for q in $(X\setminus Z)\setminus X_n$. Lemma 2.2 (iii) applies and g is a normal extension of $\mathrm{Re} f$. Similarly $\mathrm{Im} f$ has a normal extension. ■

2.6 Lemma. *If f and g are normal functions defined on $X\setminus Z$ and $X\setminus Z'$, respectively, then $f+g$ defined on $X\setminus(Z\cup Z')$ has a normal extension $f \dotplus g$.*

Proof. From Lemma 2.5, $\mathrm{Re} f$, $\mathrm{Re} g$, $\mathrm{Im} f$, and $\mathrm{Im} g$, have normal extensions. Now $\mathrm{Re}(f+g) = \mathrm{Re} f + \mathrm{Re} g$ and $\mathrm{Im}(f+g) = \mathrm{Im} f + \mathrm{Im} g$. Again, from Lemma 2.5, $f+g$ has a normal extension if $\mathrm{Re}(f+g)$ and $\mathrm{Im}(f+g)$ do. We may assume without loss of generality that f and g are real valued. With this assumption, we define the sets:

$$X_n = f^{-1}[(-1, n+1)]^- \backslash f^{-1}[(-1, n)]^-$$
$$Y_n = g^{-1}[(-1, n+1)]^- \backslash g^{-1}[(-1, n)]^-$$

for $n = 0, 1, 2, \ldots$; and

$$X_{-n} = f^{-1}[(-n, 0)]^- \backslash f^{-1}[(-n+1, 0)]^-$$
$$Y_{-n} = g^{-1}[(-n, 0)]^- \backslash g^{-1}[(-n+1, 0)]^-$$

for $n = 2, 3, \ldots$. Let X_{-1} and Y_{-1} be $f^{-1}[(-1, 0)]^-$ and $g^{-1}[(-1, 0)]^-$, respectively. Then $\{X_n\}_{n \in \mathbb{Z}}$ and $\{Y_n\}_{n \in \mathbb{Z}}$ are families of mutually disjoint clopen sets in X. Moreover,

$$X \backslash Z = \bigcup_{n \in \mathbb{Z}} X_n, \qquad X \backslash Z' = \bigcup_{n \in \mathbb{Z}} Y_n$$

and for q in X_n or Y_n,

$$n \leq f(q) \leq n+1 \quad \text{or} \quad n \leq g(q) \leq n+1,$$

respectively. Thus if $q \in X_n \cap Y_m (= V_{n,m})$

(2) $\qquad n + m \leq (f+g)(q) \leq n + m + 2$

and $\{V_{n,m} : (n, m) \in \mathbb{Z} \times \mathbb{Z}\}$ is a family of mutually disjoint clopen subsets of X with union dense in X. For $j = 0, 1, 2, \ldots$, let U_j be $\bigcup_{|n+m|=j} V_{n,m}$ and note that, with q in U_j,

$$(f+g)(q) \in [j, j+2] \cup [-j, -j+2],$$

from (2); so that $j - 2 \leq |(f+g)(q)| \leq j + 2$. It follows from Lemma 2.4 that there is a function h_j in $C(X)$ equal to $f+g$ on U_j and vanishing on $X \backslash U_j^-$. Since

$$\bigcup_{j=0}^{\infty} U_j = \bigcup_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} V_{n,m} = (\bigcup_{n \in \mathbb{Z}} X_n) \cap (\bigcup_{n \in \mathbb{Z}} Y_n);$$

we have

$$Z'' = X \backslash \bigcup_{j=0}^{\infty} U_j^- \subseteq Z \cup Z',$$

so that Z'' is a closed nowhere-dense subset of X. If $f \hat{+} g$ is the function defined on $X \backslash Z''$ as h_j on U_j^- for $j = 0, 1, \ldots$, then, since $\{U_j^-\}$ is a family of mutually disjoint clopen subsets of X and $j - 2 \leq |(f \hat{+} g)(q)|$ for q in U_j^-, Lemma 2.2 (iv) applies and $f \hat{+} g$ is self-adjoint (normal). Moreover, $f \hat{+} g$ is an extension of $f + g$. ■

2.7 Corollary. *If f is a self-adjoint function defined on $X \backslash Z$ then $Z = Z_+ \cup Z_-$ (and $Z_{\pm} = \emptyset$).*

Proof. As noted in Remark 2.1, $|f|$ is self-adjoint. From Lemma 2.6, then, $|f| + f$ has a self-adjoint extension. Now $|f| + f$ is defined on $X \backslash Z$. If p is a point of $Z_{\pm}$ then each open set containing p contains points at which $|f| + f$ assumes arbitrarily large (positive) values and points at which it is 0. In this case $|f|+f$ can have no normal extension. Thus $Z_{\pm}=\emptyset$, $Z=Z_{+}\cup Z_{-}$, and, as noted before Remark 2.1, both Z_{+} and Z_{-} are closed. ∎

2.8 Corollary. *If f is a self-adjoint function defined on $X\backslash Z$ then the functions $\frac{1}{2}(|f|+f)$ and $\frac{1}{2}(|f|-f)$ have self-adjoint extensions f_{+} and f_{-}.*

Proof. Since $|f|$ is self-adjoint, from Remark 2.1, and $-f$ is self-adjoint, Lemma 2.6 applies to yield f_{+} and f_{-}. We note that f_{+} is defined on $X\backslash Z_{+}$, is 0 on Z_{-} and at points q of X such that $f(q)\leq 0$, and $f_{+}(q)=f(q)$ if $f(q)\geq 0$. Similarly, f_{-} is defined on $X\backslash Z_{-}$, is 0 on Z_{+} and at points q of X such that $f(q)\geq 0$, and $f_{-}(q)=-f(q)$ if $f(q)\leq 0$. Note, too, that $f(q)=f_{+}(q)-f_{-}(q)$ for q in $X\backslash Z$. ∎

2.9 Lemma. *If f is a self-adjoint function defined on $X\backslash Z$ then $\exp f$ defined on $X\backslash Z_{+}$ is self-adjoint, where $(\exp f)(q)=\exp f(q)$ for each q in $X\backslash Z$ and $(\exp f)(p)=0$ for p in Z_{-}.*

If $0\leq f(q)$ for each q in $X\backslash Z$ and Z_0, the set of points of $X\backslash Z$ at which f takes the value 0, is nowhere dense, then $\log f$ defined on $X\backslash(Z\cup Z_0)$ is self-adjoint, where $(\log f)(q)=\log(f(q))$ for q in $X\backslash(Z\cup Z_0)$.

Proof. From Corollary 2.7, $Z=Z_{+}\cup Z_{-}$ and Z_{+} is closed in X. With p in Z_{-} and a positive a given, there is some open set $\mathcal{O}$ containing p such that if $q\in X\backslash Z$, $f(q)\leq -a$; and no point of Z_{+} is in $\mathcal{O}$. Thus $\exp f(q)\leq \exp -a$ for each q in $\mathcal{O}$. (At points q of Z_{-}, $\exp f(q)$ is 0, by definition.) Thus $\exp f$ is continuous on $X\backslash Z_{+}$.

If $p\in Z_{+}$, since Z_{-} is closed some open set $\mathcal{O}$ containing p contains no points of Z_{-} and is such that if $q\in X\backslash Z$ then $f(q)$ (hence $\exp f(q)$) is large. Thus $\exp f$ defined on $X\backslash Z_{+}$ is self-adjoint.

If $0\leq f(q)$ and Z_0 is nowhere dense, $\log f$ is continuous on $X\backslash(Z\cup Z_0)$. In this case, Z is Z_{+} for $\log f$ (and f) and Z_0 is Z_{-} for $\log f$. Thus $\log f$ defined on $X\backslash(Z\cup Z_0)$ is self-adjoint. ∎

2.10 Lemma. *If f defined on $X\backslash Z$ and g defined on $X\backslash Z'$ are normal then $f\cdot g$ defined on $X\backslash(Z\cup Z')$ has a normal extension $f\hat{\cdot}\, g$.*

Proof. We note, first that the interior X_0 of $f^{-1}(0)$ is clopen (possibly empty); for $f^{-1}(0)$ is closed in $X\backslash Z$ (by continuity of f on $X\backslash Z$) and $f^{-1}(0)\subseteq f^{-1}(\{z:|z|<1\})^{-}\subseteq X\backslash Z$, from Lemma 2.2 (ii). Since $f^{-1}(\{z:|z|<1\})^{-}$ is closed in X, $f^{-1}(0)$ is closed in X. Thus $X_0^{-}\subseteq f^{-1}(0)$. As X_0^{-} is open (as well as closed) in X, $X_0^{-}=X_0$. Similarly the interior Y_0 of $g^{-1}(0)$ is clopen in X.

Let $\tilde{X}$ be $X\backslash(X_0\cup Y_0)$ and $\tilde{f},\tilde{g}$ be the restrictions of f and g to $\tilde{X}\backslash\tilde{Z}$ and $\tilde{X}\backslash\tilde{Z}'$, respectively, where $\tilde{Z}=\tilde{X}\cap Z$ and $\tilde{Z}'=\tilde{X}\cap Z'$. Since $\tilde{Z}$ and $\tilde{Z}'$ are closed nowhere-dense subsets of $\tilde{X}$, $\tilde{f}$ and $\tilde{g}$ are normal. By choice of X_0 and Y_0, $\tilde{f}^{-1}(0)$ and $\tilde{g}^{-1}(0)$ are closed nowhere-dense subsets of $\tilde{X}$. If $\tilde{f}\cdot\tilde{g}$ defined on $\tilde{X}\backslash(\tilde{Z}\cup\tilde{Z}')$ has a normal extension $\tilde{f}\hat{\cdot}\,\tilde{g}$ defined on $\tilde{X}\backslash Z''$ then $f\hat{\cdot}\, g$ defined as $\tilde{f}\hat{\cdot}\,\tilde{g}$ on $\tilde{X}\backslash Z''$ and 0 on $X_0\cup Y_0$ is a normal extension of $f\cdot g$. To see this observe that $\tilde{X}$ is a clopen subset of X whose complement in X is $X_0 \cup Y_0$; so that Z'' is a closed nowhere-dense subset of X (as well as of $\tilde{X}$). Moreover, $f \hat{\cdot} g$ is continuous on $X \backslash Z''$ since it is continuous on $\tilde{X} \backslash Z''$ and on $X_0 \cup Y_0$; and $|f \hat{\cdot} g|$ tends to ∞ at points of Z''. Thus $f \hat{\cdot} g$ is a normal extension of $f \cdot g$. We assume (as we may without loss of generality) that $f^{-1}(0)$ and $g^{-1}(0)$ are nowhere dense in X.

If $\mathrm{Re} f \cdot \mathrm{Re} g$ and $-\mathrm{Im} f \cdot \mathrm{Im} g$ have self-adjoint extensions then the sum of these extensions has a self-adjoint extension (from Lemma 2.6) which is a self-adjoint extension of $\mathrm{Re}(f \cdot g)$. Similarly, if $\mathrm{Re} f \cdot \mathrm{Im} g$ and $\mathrm{Im} f \cdot \mathrm{Re} g$ have self-adjoint extensions, then $\mathrm{Im}(f \cdot g)$ has a self-adjoint extension. From Lemma 2.5, $f \cdot g$ has a normal extension in this case. On the other hand, again from Lemma 2.5, $\mathrm{Re} f$, $\mathrm{Re} g$, $\mathrm{Im} f$, $\mathrm{Im} g$ have self-adjoint extensions. We assume (as we may without loss of generality) that f and g are real valued.

From Corollary 2.8, $f = f_+ - f_-$ on $X \backslash Z$ where $f_+ (\geq 0)$ and $f_- (\geq 0)$ are normal; and $g = g_+ - g_-$ on $X \backslash Z'$ where $g_+ (\geq 0)$ and $g_- (\geq 0)$ are normal. Now $f \cdot g = f_+ g_+ + f_- g_- - (f_+ g_- + f_- g_+)$ on $X \backslash (Z \cup Z')$. If each of $f_+ g_+$, $f_- g_-$, $f_+ g_-$ and $f_- g_+$ have self-adjoint extensions then $f \cdot g$ has. We assume (as we may without loss of generality) that $f \geq 0$ on $X \backslash Z$, $g \geq 0$ on $X \backslash Z'$, and (using our first reduction) that $f^{-1}(0)$ and $g^{-1}(0)$ are nowhere dense.

Under these assumptions Lemma 2.9 applies, so that $\log f$ defined on $X \backslash (Z \cup Z_0)$ and $\log g$ defined on $X \backslash (Z' \cup Z'_0)$ are self-adjoint, where $Z_0 = f^{-1}(0)$ and $Z'_0 = g^{-1}(0)$. Moreover Z_0 and Z'_0 are the sets of points at which $\log f$ and $\log g$ tend to $-\infty$. From Lemma 2.6, $\log f + \log g$ defined on $X \backslash (Z \cup Z' \cup Z_0 \cup Z'_0)$ has a self-adjoint extension $\log f \hat{+} \log g$ defined on $X \backslash Z''$. Applying Lemma 2.9 again, $\exp(\log f \hat{+} \log g) (= f \hat{\cdot} g)$ defined on $X \backslash Z''_+$ is self-adjoint. For q in $X \backslash (Z \cup Z' \cup Z_0 \cup Z'_0)$, we have $(f \hat{\cdot} g)(q) = \exp(\log f \hat{+} \log g)(q) = (f \cdot g)\, q)$. If $q \in (Z_0 \cup Z'_0) \cap (X \backslash (Z \cup Z'))$ then $(f \cdot g)(q) = 0$ and $f \cdot g$ is continuous at q. Thus, on some open set $\mathcal{O}$ in X containing q, $f \cdot g$ assumes small values. Since $Z \cup Z' \cup Z_0 \cup Z'_0$ is nowhere dense in X (under the present assumptions), $\mathcal{O}$ contains points of $X \backslash (Z \cup Z' \cup Z_0 \cup Z'_0)$ and at such points $f \hat{\cdot} g$ has the same value as $f \cdot g$. It follows that $q \notin Z''_+$, that $f \hat{\cdot} g$ is continuous at q and that $0 = (f \hat{\cdot} g) q) = (f \cdot g)(q)$. Thus $f \hat{\cdot} g$ is self-adjoint extension of $f \cdot g$ defined on $X \backslash (Z \cup Z')$. ∎

Combining Lemmas 2.6 and 2.10 with the obvious fact that a scalar multiple of a normal function is normal (with the same set of definition except when the scalar is 0, in which case the unique normal extension is the function 0), we are in a position to define our algebra of normal functions.

2.11 Theorem. *The sets $\mathcal{N}(X)$ and $\mathcal{S}(X)$ of normal, respectively, self-adjoint functions on X provided with the operations $\hat{+}$, $\hat{\cdot}$ and multiplication by scalars (complex, respectively, real) are (associative, commutative) algebras with the constant function 1 as unit and $C(X)$ as a subalgebra.*

Proof. To establish such laws as

$$f \hat{\cdot} (g \hat{\cdot} h) = (f \hat{\cdot} g) \hat{\cdot} h$$

note that $f \hat{\cdot} (g \hat{\cdot} h)$ and $(f \hat{\cdot} g) \hat{\cdot} h$ agree on $X \backslash (Z \cup Z' \cup Z'')$ with $f \cdot g \cdot h$ and apply Lemma 2.3. ∎

Although caution must be exercised, since the functions in $\mathscr{S}(X)$ have varying domains of definition, the notion of "positive element" that suggests itself endows $\mathscr{S}(X)$ with a partial ordering relative to which it becomes a *boundedly complete lattice* in the sense described in the following theorem.

2.12 Theorem. *The set $\mathscr{S}(X)_+$ of functions f in $\mathscr{S}(X)$ such that $f(q) \geq 0$ for each q in $X \backslash Z$, where f is defined on $X \backslash Z$, is a proper cone in $\mathscr{S}(X)$. If $\{f_a\}$ is a net of elements of $\mathscr{S}(X)$ increasing, relative to the partial ordering on $\mathscr{S}(X)$ with positive cone $\mathscr{S}(X)_+$, and $\{f_a\}$ is bounded above by some element f of $\mathscr{S}(X)$, then $\{f_a\}$ has a least upper bound f_0 in $\mathscr{S}(X)$.*

Proof. If $f \in \mathscr{S}(X)_+$ and a is positive real number then $af \in \mathscr{S}(X)_+$. If f and g are in $\mathscr{S}(X)_+$, f is defined on $X \backslash Z$ and g is defined on $X \backslash Z'$, then $Z = Z_+$, $Z' = Z'_+$ and $f+g$ defined on $X \backslash (Z \cup Z')$ is in $\mathscr{S}(X)$. Thus $f \hat{+} g$ is $f+g$ defined on $X \backslash (Z \cup Z')$ and $f \hat{+} g \in \mathscr{S}(X)_+$. If both f and $-f$ are in $\mathscr{S}(X)_+$ then $f(q) = 0$ for q in $X \backslash Z$; so that $Z = \emptyset$ and $f = 0$. Thus $\mathscr{S}(X)_+$ is a proper cone in $\mathscr{S}(X)$ and induces a partial ordering on $\mathscr{S}(X)$. We adopt all the usual notational conventions for this ordering (e.g., $f \geq g$ or $g \leq f$ when $f \hat{-} g \in \mathscr{S}(X)_+$).

Note that if h in $\mathscr{S}(X)$ is such that $h(q) \geq 0$ for each q in a dense subset of the set on which h is defined then $h \in \mathscr{S}(X)_+$ (from continuity of h on its domain). With f and g in $\mathscr{S}(X)$, we denote the functions $\frac{1}{2}(f \hat{+} g \hat{+} |f \hat{-} g|)$ and $\frac{1}{2}(f \hat{+} g \hat{-} |f \hat{-} g|)$ by $f \vee g$ and $f \wedge g$ respectively. It follows from the comment just noted that $f \vee g$ and $f \wedge g$ are the least upper and greatest lower bounds of f and g in $\mathscr{S}(X)$; so that $\mathscr{S}(X)$ is a lattice relative to the given partial ordering.

If f and g in $\mathscr{S}(X)$ are defined on $X \backslash Z$ and $X \backslash Z'$, respectively, and $f \leq g$ then either $q \in X \backslash (Z \cup Z')$ and $f(q) \leq g(q)$, or $q \in Z_-$, or $q \in Z'_+$. Thus, if we think of f as taking the value $-\infty$ on Z_- and $+\infty$ on Z_+ and interpret "$f(q) \leq g(q)$" in the obvious way for points q of $Z \cup Z'$, then $f \leq g$ if and only if $f(q) \leq g(q)$ for each q in X. Moreover, $(f \vee g)(q) = \max\{f(q), g(q)\}$ and $(f \wedge g)(q) = \min\{f(q), g(q)\}$ for each q in X. Suppose, now, that $\{f_a\}$ is an increasing sequence of elements of $\mathscr{S}(X)$ bounded above by f in $\mathscr{S}(X)$. We may assume, without loss of generality, that $\{f_a\}$ has a first (least) element $f_{a'}$, for it will suffice to find a least upper bound in $\mathscr{S}(X)$ for the cofinal net of elements of $\{f_a\}$ following $f_{a'}$. If g is a least upper bound in $\mathscr{S}(X)$ for the increasing net $\{f_a \hat{-} f_{a'} \hat{+} 1\}$ then $g \hat{+} f_{a'} \hat{-} 1$ is the least upper bound in $\mathscr{S}(X)$ for $\{f_a\}$ (since $\mathscr{S}(X)$ is a partially ordered vector space relative to the partial ordering defined by $\mathscr{S}(X)_+$). Thus we may assume, without loss of generality, that $1 \leq f_a$ for each a. If f_a is defined on $X \backslash Z_a$ then g_a, defined at q in $X \backslash Z_a$ as $f_a(q)^{-1}$ and at p in Z_a as 0, lies in $C(X)$ and $f_a \hat{\cdot} g_a = 1$. Similarly, g defined as 0 on Z and $1/f$ on $X \backslash Z$ lies in $C(X)$ and $f \hat{\cdot} g = 1$. Since $\{g_a\}$ is a decreasing net in $C(X)$ bounded below by g, $\{g_a\}$ has a greatest lower bound g_0 in $C(X)$. Now, $g^{-1}(0) = Z$ and $0 \leq g \leq g_0$; so that $(Z_0 =) g_0^{-1}(0)$ is a closed nowhere-dense subset of X. If f_0 is defined as $1/g_0$ on $X \backslash Z_0$ then f_0 is the least upper bound of $\{f_a\}$ in $\mathscr{S}(X)$. ■

The fact that the partial ordering of $\mathscr{S}(X)$ permits interpreting $f \leq g$ in a pointwise sense on all of X underscores the extraordinary fact that a (sensible) addition and multiplication can be defined on $\mathscr{S}(X)$. What pointwise sense is to be made of $f+g$ at a point q where f "takes the value $+\infty$" and g "takes the value $-\infty$"?

In the theorem that follows, we prove a Stone-Weierstrass type of result for the algebra $\mathscr{S}(X)$. If $\mathscr{S}_0$ were assumed to be a boundedly complete sublattice of $C(X,\mathbb{R})$ (rather than $\mathscr{S}(X)$) in the result that follows (in the sense that each increasing net in $\mathscr{S}_0$ that is bounded above by some constant has a least upper bound relative to $C(X,\mathbb{R})$ that lies in $\mathscr{S}_0$) then the first paragraph of the following proof shows that $\mathscr{S}_0 = C(X,\mathbb{R})$ from which, X is extremely disconnected.

2.13 Theorem. *If $\mathscr{S}_0$ is a boundedly complete sublattice of $\mathscr{S}(X)$ such that*

$$\{(f(p_1), f(p_2)) : f \in \mathscr{S}_0\} = \mathbb{R}^2$$

for each pair of points p_1, p_2 in X then $\mathscr{S}_0 = \mathscr{S}(X)$.

Proof. Given f in $C(X,\mathbb{R})$, ε positive, and p_0 in X, choose an element f_p in $\mathscr{S}_0$ such that $f_p(p_0) = f(p_0)$ and $f_p(p) = f(p)$, for each p in X. Let N_p be an open set in X containing p such that $f_p(q) < f(q) + \varepsilon$ for q in N_p. Select a finite subcovering of the covering $\{N_p\}$ of X. Let g_{p_0} be the lattice intersection of the functions f_p corresponding to the sets N_p of this finite subcovering. Then $g_{p_0} \in \mathscr{S}_0$, $g_{p_0}(p_0) = f(p_0)$, and $g_{p_0}(q) < f(q) + \varepsilon$, for each q in X (in the extended sense that $g_{p_0}(q)$ may be $-\infty$). For each p in X, choose an open set $\mathcal{O}_p$ containing p such that we have $f(q) - \varepsilon < g_p(q)$ for q in $\mathcal{O}_p$. Select a finite subcovering of the covering $\{\mathcal{O}_p\}$ of X. Let g be the lattice union of the functions g_p corresponding to the sets $\mathcal{O}_p$ in this finite subcovering. Then $g \in \mathscr{S}_0$ and $f - \varepsilon < g < f + \varepsilon$. Hence $g \in \mathscr{S}_0$, $g \in C(X,\mathbb{R})$ and $\|f - g\| < \varepsilon$. Choosing g_n in $\mathscr{S}_0 \cap C(X,\mathbb{R})$ such that

$$\left\| g_n - \left(f - \frac{1}{n}\right) \right\| < \frac{1}{2n},$$

we have,

$$f - \frac{3}{2n} \leq g_n \leq f - \frac{1}{2n} < f.$$

If $f_n = g_1 \vee \ldots \vee g_n$ then $f_n \in \mathscr{S}_0$ and $\{f_n\}$ has f as its least upper bound. Thus $C(X,\mathbb{R}) \subseteq \mathscr{S}_0$.

With h in $\mathscr{S}(X)$, let f_m be the function (in $C(X)$) equal to h on $h^{-1}((-n,m))^-$ and 0 on $X \backslash h^{-1}((-n,m))^-$, where n and m are positive integers. Let h_n be the least upper bound in $\mathscr{S}(X)$ of the increasing sequence $\{f_m\}$. (Note, for this, that $\{f_m\}$ is bounded above by $h \vee 0$ in $\mathscr{S}(X)$.) Then h_n is in $\mathscr{S}_0$ and is equal to h at all points at which h takes values greater than $-n$. The greatest lower bound in $\mathscr{S}(X)$ of the sequence $\{h_n\}$ (bounded below by $h \wedge 0$) is h. (Note that if $n < n'$, the function g_m in $C(X)$ entering in the determination of $h_{n'}$ is less than or equal to f_m; so that h_n is an upper bound for $\{g_m\}$ and $h_{n'} \le h_n$.) Thus $h \in \mathscr{S}_0$, and $\mathscr{S}_0 = \mathscr{S}(X)$. ■

We conclude this section with a development of the "spectral resolution" of a function in $\mathscr{S}(X)$ and a result on automorphisms.

2.14 Definition. A family $\{e_\lambda\colon \lambda \in \mathbb{R}\}$ of characteristic functions e_λ of clopen subsets X_λ of X is said to be a *resolution of the identity* when it satisfies:

(i) $e_\lambda \le e_{\lambda'}$ if $\lambda \le \lambda'$;
(ii) e_λ is the greatest lower bound of $\{e_{\lambda'}\colon \lambda < \lambda'\}$ in $C(X)$ (we write: $e_\lambda = \bigwedge_{\lambda < \lambda'} e_{\lambda'}$);
(iii) $0 = \bigwedge_{\lambda \in \mathbb{R}} e_\lambda$ and $1 = \bigvee_{\lambda \in \mathbb{R}} e_\lambda$.

If there are numbers λ and λ' such that $e_\lambda = 0$ and $e_{\lambda'} = 1$ then $\{e_\lambda\}$ is said to be a *bounded* resolution of the identity. Otherwise $\{e_\lambda\}$ is said to be an *unbounded* resolution. ■

If $\{X_a\}$ is a family of clopen subsets of X, $\bigvee_a X_a$ denotes the smallest clopen set containing all X_a and $\bigwedge_a X_a$ denotes the largest clopen set contained in all X_a. Since $\bigcup_a X_a$ is open and $\bigcap_a X_a$ is closed, $\bigvee_a X_a$ is the closure of $\bigcup_a X_a$ and $\bigwedge_a X_a$ is the interior of $\bigcap_a X_a$. If $\{e_\lambda\}$ is a resolution of the identity and e_λ is the characteristic function of X_λ then $X_\lambda \subseteq X_{\lambda'}$ if $\lambda \le \lambda'$, $X_\lambda = \bigwedge_{\lambda < \lambda'} X_{\lambda'}$, $\emptyset = \bigwedge_{\lambda \in \mathbb{R}} X_\lambda$ and $X = \bigvee_{\lambda \in \mathbb{R}} X_\lambda$, from (i), (ii) and (iii) of Definition 2.14.

If f is a self-adjoint function defined on $X \backslash Z$, we denote by $X_\lambda(f)$ the interior of $(U_\lambda =) f^{-1}(\{\lambda'\colon \lambda' \le \lambda\}) \cup Z_-$. We show that U_λ is closed from which it will follow that $X_\lambda(f)$ is clopen. If $p \in U_\lambda^-$ and $p \in X \backslash Z$ then $p \in f^{-1}(\{\lambda'\colon \lambda' \le \lambda\})$ since f is continuous on $X \backslash Z$. If $p \in Z$ then $p \in Z_-$; for if p were in Z_+, some open set $\mathcal{O}$ containing p would contain no points of Z_- (since Z_- is closed in X) and no points of $X \backslash Z$ at which f takes values less than $\lambda + 1$ (since f tends to $+\infty$ at p on $X \backslash Z$). Thus $\mathcal{O}$ would not meet U_λ contradicting the choice of p in U_λ^-. Thus $p \in U_\lambda$ and U_λ is closed in X. If we include the points of Z_- (at which f "takes the value" $-\infty$) when we speak of "the set of points of X at which f takes values not exceeding λ," then this set becomes U_λ and $X_\lambda(f)$ can be characterized as the largest clopen set on which f takes values not exceeding λ. We denote by $e_\lambda(f)$ the characteristic function of $X_\lambda(f)$.

2.15 Theorem. *If f is a self-adjoint function defined on $X \backslash Z$ then $\{e_\lambda(f)\}$ is a resolution of the identity, bounded if and only if $f \in C(X)$. For each resolution of the identity $\{e_\lambda\}$ there is a self-adjoint function g such that $e_\lambda = e_\lambda(g)$ for each λ. If, in addition,*

(3) $\qquad \lambda e_{\lambda,\lambda'} \le f \hat{\cdot} e_{\lambda,\lambda'} \le \lambda' e_{\lambda,\lambda'}$,

where $\lambda \le \lambda'$ and $e_{\lambda,\lambda'} = e_{\lambda'} - e_\lambda$, then $f = g$.

Proof. From the characterization of $X_\lambda(f)$ as the largest clopen set on which f takes values not exceeding λ (in the extended sense), one has $e_\lambda(f) \le e_{\lambda'}(f)$ when $\lambda \le \lambda'$. Condition (ii) of Definition 2.14 also follows easily from this characterization; for $\bigwedge_{\lambda<\lambda'} X_{\lambda'}(f)$ is a clopen set on which f takes values not exceeding λ (in the extended sense), by continuity of f on $X \backslash Z$, and is therefore a subset of $X_\lambda(f)$, the largest such clopen set. As $X_\lambda(f) \subseteq X_{\lambda'}(f)$ when $\lambda < \lambda'$, we have that $X_\lambda(f) = \bigwedge_{\lambda<\lambda'} X_{\lambda'}(f)$ and $e_\lambda(f) = \bigwedge_{\lambda<\lambda'} e_{\lambda'}(f)$. Since $X \backslash Z$ is contained in $\bigcup_{\lambda\in\mathbb{R}} X_\lambda(f)$ and Z is nowhere dense in X;

$$X = (X \backslash Z)^- \subseteq \Big(\bigcup_{\lambda\in\mathbb{R}} X_\lambda(f)\Big)^- = \bigvee_{\lambda\in\mathbb{R}} X_\lambda(f).$$

As the clopen set $\bigwedge_{\lambda\in\mathbb{R}} X_\lambda(f)$ is contained in the nowhere-dense set Z; $\bigwedge_{\lambda\in\mathbb{R}} X_\lambda(f) = \emptyset$. Hence $0 = \bigwedge_{\lambda\in\mathbb{R}} e_\lambda(f)$, $1 = \bigvee_{\lambda\in\mathbb{R}} e_\lambda(f)$ and $\{e_\lambda(f)\}$ is a resolution of the identity.

If $e_\lambda(f) = 0$ and $e_{\lambda'}(f) = 1$ for some λ and λ' then $X_\lambda = \emptyset$ and $X_{\lambda'} = X$. By continuity of f on the open set $X \backslash Z$, it follows, now, that $\lambda \le f(q)$ for each q in $X \backslash Z$, and $Z_- = \emptyset$. Moreover $f(q) \le \lambda'$ for each q in $X (= X_{\lambda'}(f))$; so that $Z_+ = \emptyset$, $f \in C(X)$ and $\lambda \le f \le \lambda'$. If, on the other hand, we are given that $\lambda < f(q) \le \lambda'$ for each q in X and $f \in C(X)$, then $X_{\lambda'}(f) = X$ and $X_\lambda(f) = \emptyset$. Thus $e_{\lambda'}(f) = 1$, $e_\lambda(f) = 0$, and $\{e_\lambda(f)\}$ is a bounded resolution of the identity.

If $\{e_\lambda\}$ is a resolution of the identity then $\bigvee_{n=1}^\infty X_n = X$ and $\bigwedge_{n=1}^\infty X_{-n} = \emptyset$, where e_λ is the characteristic function of X_λ. Thus $\bigvee_{n=1}^\infty X_{-n,n} = X$, where $X_{\lambda,\lambda'} = X_{\lambda'} \backslash X_\lambda$; and $X \backslash \bigcup_{n=1}^\infty X_{-n,n} (= Z')$ is a closed nowhere-dense subset of X. If $q \in X \backslash Z'$ then $q \in X_n \backslash X_{-n}$ for some n; so that $\{\lambda' : q \in X_{\lambda'}\}$ is not empty and is bounded below. Let $g(q) (= \lambda)$ be its greatest lower bound. If $p \in X_{\lambda''} \backslash X_{\lambda'}$, where $\lambda' < \lambda < \lambda''$, then $p \in X \backslash Z'$ (since $X_{\lambda'',\lambda'} \subseteq X_{-n,n}$ for large n) and $\lambda' \le g(p) \le \lambda''$. Thus $|g(p) - g(q)| \le |\lambda'' - \lambda'|$. With λ'' chosen near λ', $X_{\lambda'',\lambda'}$ is a clopen set containing q on which g takes values near $g(q)$. Thus g is continuous on $X \backslash Z'$. Since $n \le |g(q)|$ if $q \in (X \backslash Z') \backslash X_{-n,n}$, Lemma 2.2 (iii) applies and g is self-adjoint (as defined on $X \backslash Z'$). Now $g(q) \le \lambda$ if $q \in X_\lambda \cap (X \backslash Z')$ so that $X_\lambda \cap Z'_+ = \emptyset$. If $p \in Z'_-$ then each open set $\mathcal{O}$ containing p meets $X \backslash Z'$ in a point q such that $g(q) < \lambda$. Thus $q \in X_\lambda$ and p is a limit point of X_λ. Since X_λ is closed, $p \in X_\lambda$ and $Z'_- \subseteq X_\lambda$ for each λ. If Y is a clopen set on which g takes values not exceeding λ (in the extended sense - so that points of Z'_- may lie in Y), then $Y \subseteq X_{\lambda'}$ when $\lambda < \lambda'$. Since $\bigwedge_{\lambda<\lambda'} X_{\lambda'} = X_\lambda$ and Y is clopen, $Y \subseteq X_\lambda$. Thus X_λ is the largest clopen set on which g takes values not exceeding λ (in the extended sense). It follows that $X_\lambda = X_\lambda(g)$ and $e_\lambda = e_\lambda(g)$ for each λ.

We complete the proof by showing that $Z = Z'$ and $g = f$ if $\{e_\lambda\}$ satisfies (3). From (3), $X_{-n,n} \subseteq X \backslash Z$; for if $p \in X_{-n,n}$ then the clopen (in particular, open) set $X_{-n,n}$ is one on which $|f|$ does not assume arbitrarily large values so that $p \notin Z$. Thus $Z \subseteq Z'$. If $q \in X \backslash Z'$ and $g(q) = \lambda''$ then $q \in X_{\lambda,\lambda'}$ when $\lambda < \lambda'' < \lambda'$. In this case, $q \in X \backslash Z$ and

$$\lambda = \lambda e_{\lambda,\lambda'}(q) \le (f \hat{\cdot}\, e_{\lambda,\lambda'})(q) = (f \cdot e_{\lambda,\lambda'})(q) = f(q) \le \lambda' e_{\lambda,\lambda'}(q) = \lambda'.$$

Since this holds for all λ and λ' such that $\lambda < \lambda'' < \lambda'$, we have that $f(q) = \lambda'' = g(q)$. Thus f and g coincide on $X \backslash Z'$. Lemma 2.3 applies and $Z = Z'$, $f = g$. ■

2.16 Theorem. *The mapping α is an automorphism of $\mathscr{S}(X)$ if and only if there is a homeomorphism η of X onto itself such that $\alpha(f) = f \circ \eta$ for each f in $\mathscr{S}(X)$.*

Proof. If $f \in \mathscr{S}(X)_+$ then $f = h^2$ with h in $\mathscr{S}(X)_+$ and $\alpha(f) = \alpha(h)^2$. Thus $\alpha(f) \in \mathscr{S}(X)_+$. Applying this to α^{-1}, we see that α maps $\mathscr{S}(X)_+$ onto itself so that α is an order isomorphism of $\mathscr{S}(X)$ onto itself. Now $f \in C(X)$ if and only if $-a \cdot 1 \leq f \leq a \cdot 1$ for some positive real number a. Thus α maps $C(X)$ onto itself and there is a homeomorphism η such that $\alpha(f) = f \circ \eta$ for each f in $C(X)$. Let β be the automorphism of $\mathscr{S}(X)$ induced by η. Then $\alpha \circ \beta^{-1}$ is an automorphism γ of $\mathscr{S}(X)$ that leaves each f in $C(X)$ fixed. We complete the proof by showing that γ is the identity mapping on $\mathscr{S}(X)$.

With h in $\mathscr{S}(X)_+$, the proof of Theorem 2.13 assures us that there is an increasing sequence $\{f_n\}$ of functions f_n in $C(X)$ with h as its least upper bound. Thus $\gamma(h)$ is the least upper bound of $\{\gamma(f_n)\}$. As $\gamma(f_n) = f_n$, we have $\gamma(h) = h$. Since each g in $\mathscr{S}(X)$ is the difference of functions, $g \vee 0$ and $-(g \wedge 0)$, in $\mathscr{S}(X)_+$, we have $\gamma(g) = g$. ■

3. The algebra of operators

We assume throughout this section, that $\mathscr{A}$ is an abelian von Neumann algebra acting on the complex Hilbert space $\mathscr{H}$. We begin with some preparatory material concerning extensions of unbounded operators. For arbitrary operators on $\mathscr{H}$, the following simple facts are easily verified.

(1) If $A \subseteq B$ and $C \subseteq D$ then $A + C \subseteq B + D$.

(2) If $A \subseteq B$ then $CA \subseteq CB$ and $AC \subseteq BC$.

(3) $(A + B)C = AC + BC$, $CA + CB \subseteq C(A + B)$.

In connection with the last assertion of (3), note that, in general, we do not have equality. This is illustrated by choosing C to be densely (but not *everywhere*) defined, A to be I and B to be $-I$. Then $C(A + B)$ is 0 but $CA + CB$ is $0|\mathscr{D}(C)$ (that is, the restriction of 0 to the domain of C). It follows from these rules that if $CA \subseteq AC$ for each C in some family $\mathscr{F}$ then $TA \subseteq AT$ for each sum T of products of operators in $\mathscr{F}$. We cannot speak of the "algebra" generated by $\mathscr{F}$, for, as we have just noted, a distributive law fails. However, if $\mathscr{F}$ consists of everywhere defined operators (in particular, of operators in $\mathscr{B}(\mathscr{H})$), we can speak of this algebra.

We may add to (1), (2), (3) another easily proved rule.

(4) If $\{T_a\}$ is a net of operators in $\mathscr{B}(\mathscr{H})$) tending to T in the strong-operator topology and $T_a A \subseteq BT_a$ for each a, where B is closed, then $TA \subseteq BT$.

To see this, suppose $x \in \mathscr{D}(A)$. Then $T_a x \in \mathscr{D}(B)$, and $BT_a x = T_a Ax \to TAx$. Now, $T_a x \to Tx$. As B is closed $Tx \in \mathscr{D}(B)$ and $BTx = TAx$, from which (4) follows.

Combining the results of this discussion, we have the following lemma.

3.1 Lemma. *If A is a closed operator acting on the Hilbert space $\mathscr{H}$ and $CA \subseteq AC$ for each C in a self-adjoint subset $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$ then $TA \subseteq AT$ for each T in the von Neumann algebra generated by $\mathscr{F}$.*

If A is a closed operator and E is a projection on $\mathscr{H}$ such that $EA \subseteq AE$ and AE is a bounded everywhere-defined operator on $\mathscr{H}$, we say that E is a *bounding* projection for A. If $\{E_n\}$ is an increasing sequence of projections each of which is bounding for A and $\bigvee_{n=1}^{\infty} E_n = I$, we say that $\{E_n\}$ is a *bounding sequence* for A.

3.2 Lemma. *If E is a bounding projection for a closed densely-defined operator A on the Hilbert space $\mathscr{H}$ then E is bounding for A^*, A^*A and AA^*; and $(AE)^* = A^*E$. If $\{E_n\}$ is a bounding sequence for A then $\bigcup_{n=1}^{\infty} E_n(\mathscr{H})$ is a core for each of A, A^*, A^*A and AA^*.*

Proof. Note that EA is preclosed, densely defined and bounded, since $EA \subseteq AE$ and AE is bounded. Thus EA has closure AE and $(EA)^* = (AE)^*$ from general theory. If $x \in E(\mathscr{H})$ and $y \in \mathscr{D}(A)$ then $\langle Ay, x\rangle = \langle y, (EA)^* x\rangle$; so that $x \in \mathscr{D}(A^*)$ and $A^*x = (EA)^*x$. It follows that $A^*E = (EA)^*E$. But $(I-E)\overline{EA} = 0$ so that $(EA)^*E = (EA)^* = A^*E$. Now $EA^* \subseteq (AE)^* = (EA)^* = A^*E$; and E is bounding for A^*. Since $EA^*A \subseteq A^*EAE = A^*AE$, E is bounding for A^*A and, similarly, for AA^*.

It follows that $\{E_n\}$ is a bounding sequence for A^*, A^*A, and AA^*, as well as A. If $x \in \mathscr{D}(A)$ then $E_n x \to x$, $E_n x \in \mathscr{D}(A)$ and $AE_n x = E_n Ax \to Ax$. Thus $\bigcup_{n=1}^{\infty} E_n(\mathscr{D}(A))$ is a core for A. Since $E_n(\mathscr{H}) \subseteq \mathscr{D}(A)$; $\bigcup_{n=1}^{\infty} E_n(\mathscr{H})$ is a core for A (and A^*, A^*A, AA^* as well). ■

It is convenient to introduce, at this point, the connection (more fully explored in the next section) between the self-adjoint operators affiliated with $\mathscr{A}$ and the functions in $\mathscr{S}(X)$. Suppose $A \eta \mathscr{A}$ and $A = A^*$. From the basic theory of self-adjoint operators, $A + iI$ and $A - iI$ are one-to-one linear mappings of $\mathscr{D}(A)$ onto $\mathscr{H}$ whose inverses, T_+ and T_-, are (everywhere-defined) linear operators with bound not exceeding 1 and (0) as null space. Moreover, $T_+ = T_-^*$. Since T_+ and T_- are bounded and affiliated with $\mathscr{A}$, they lie in $\mathscr{A}$. Each increasing net of self-operators in $\mathscr{A}$ bounded above has a least upper bound so that, if $\mathscr{A}$ is isomorphic to $C(X)$ with X compact Hausdorff then X is extremely disconnected. Let g_+ and g_- be the functions in $C(X)$ corresponding to T_+ and T_-. Then $g_+ = \bar{g}_-$ so that g_+ and g_- are 0 on the same closed subset Z of X. If the interior Z_0 of Z is non-null its characteristic function e_0 corresponds to a projection E_0 in $\mathscr{A}$ such that $T_+E_0 = 0$, since $g_+e_0 = 0$. As this contradicts the fact that T_+ has null space (0), Z is nowhere dense in X. Let h_+ and h_- be the reciprocals of g_+ and g_- on $X \backslash Z$. Then h_+ and h_- are complex conjugates of one another so that $\frac{1}{2}(h_+ + h_-)(=h)$ is a real-valued continuous function on $X \backslash Z$. From the definition of T_+ and T_-, we have

$$AT_-T_+ = T_+ + iT_-T_+ \quad \text{and} \quad AT_+T_- = T_- - iT_+T_-.$$

Since $T_-T_+ = T_+T_-$ it follows that

(5) $\quad 2iT_+T_- = T_- - T_+$

and

(6) $\quad AT_+T_- = \frac{1}{2}(T_+ + T_-).$

From (5), we have $2ig_+g_- = g_- - g_+$ so that

(7) $\quad (h(q)+i)^{-1} = g_+(q) \quad \text{and} \quad (h(q)-i)^{-1} = g_-(q)$

for q in $X\backslash Z$. Thus, with p in Z and q in $X\backslash Z$ near p, $g_+(q)$ is near 0 and $|h(q)|$ is large. Thus h is a self-adjoint function as defined on $X\backslash Z$. In a sense that will be made precise in the next section, h "represents" A in $\mathscr{S}(X)$. For present purposes, we use the spectral resolution $\{e_\lambda(h)\}$ of h to produce the spectral resolution $\{E_\lambda(A)\}$ of A (see Theorem 2.15). Of course $E_\lambda(A)$ is the projection in $\mathscr{A}$ represented by $e_\lambda(h)$ under the isomorphism of $\mathscr{A}$ with $C(X)$. We shall show that AE is bounded and everywhere defined and that

(8) $\quad \lambda E \le AE \le \lambda' E,$

where $E = E_{\lambda'}(A) - E_\lambda(A)$ and $\lambda \le \lambda'$. Writing e for $e_{\lambda'}(h) - e_\lambda(h)$, we have that e represents E in $C(X)$. Now e is the characteristic function of $X_{\lambda'}(h)\backslash X_\lambda(h)$ which is contained in $X\backslash Z$. Thus $g_+(p)g_-(p) \neq 0$ when $e(p) = 1$. For p in $X\backslash Z$, we have

(9) $\quad h(p) = [\frac{1}{2}(g_+g_-)^{-1}(g_+ + g_-)](p),$

by definition of h. Moreover, there is a k in $C(X)$ such that $kg_+g_- = e$ and $ke = k$ (since g_+g_- is continuous and vanishes nowhere on the clopen set $X_{\lambda'}(h)\backslash X_\lambda(h)$). If K in $\mathscr{A}$ corresponds to k, then

(10) $\quad KT_+T_- = E.$

Since $\lambda \le h(p) \le \lambda'$, if $p \in X_{\lambda'}(h)\backslash X_\lambda(h)$; we have

$$\lambda g_+g_-e \le \tfrac{1}{2}(g_+ + g_-)e \le \lambda' g_+g_-e,$$

from (9), and

$$\begin{aligned}\lambda k g_+g_-e = \lambda e \le \tfrac{1}{2}(g_+ + g_-)ke = \tfrac{1}{2}(g_+ + g_-)k \\ \le \lambda' kg_+g_-e = \lambda' e.\end{aligned}$$

Thus

(11) $\quad \lambda E \le \frac{1}{2}(T_+ + T_-)K \le \lambda' E.$

Combining (6), (10), and (11), we have (8). It follows that AE is bounded and everywhere defined; and, from (6), (9) and (10), that $h \cdot e$ corresponds to AE in $C(X)$.

With these few spectral-theoretic techniques at our disposal, we proceed to the construction of the algebra of operators affiliated with $\mathscr{A}$.

3.3 Theorem. *If $\mathscr{A}$ is an abelian von Neumann algebra acting on the Hilbert space $\mathscr{H}$ and $A, B\eta\mathscr{A}$ then*

(i) *each finite set of operators affiliated with $\mathscr{A}$ have a common bounding sequence in $\mathscr{A}$;*
(ii) *$A+B$ is densely defined and preclosed and its closure $A \hat{+} B\eta\mathscr{A}$;*
(iii) *$A \cdot B$ is densely defined and preclosed and its closure $A \hat{\cdot} B\eta\mathscr{A}$;*
(iv) *$A \hat{\cdot} B = B \hat{\cdot} A$ and $A^*A = AA^*(=A^* \hat{\cdot} A)$;*
(v) *$(aA \hat{+} B)^* = \bar{a}A^* \hat{+} B^*$;*
(vi) *$(A \hat{\cdot} B)^* = B^* \hat{\cdot} A^*$;*
(vii) *if $A \subseteq B$ then $A = B$; if A is symmetric, $A = A^*$;*
(viii) *the family $\mathscr{N}(\mathscr{A})$ of operators affiliated with $\mathscr{A}$ forms a commutative *-algebra (with unit I) under the operations of addition $\hat{+}$ and multiplication $\hat{\cdot}$ described in* (ii) *and* (iii).

Proof. Throughout this argument, U denotes a unitary operator in $\mathscr{A}'$. Since $U^*AU = A$, we have $U^*A^*U = A^*$; and $A^*\eta\mathscr{A}$. At the same time $U^*A^*AU = A^*A$ and $A^*A\eta\mathscr{A}$. If E is a projection in $\mathscr{A}$, $(2E-I)$ is a unitary operator in $\mathscr{A}(\subseteq\mathscr{A}')$; so that $(2E-I)A(2E-I) = A$. Thus $EA \subseteq AE$. From general theory, A^*A is self-adjoint. Let $\{E_\lambda\}$ be its spectral resolution and let F_n be $E_n - E_{-n}$. Then $U^*E_\lambda U = E_\lambda$ and $E_\lambda \in \mathscr{A}$. As A^*AF_n is bounded and everywhere defined, AF_n is everywhere defined and closed, since A is closed and F_n is bounded. The closed graph theorem tells us that AF_n is bounded. (This follows directly, as well, since

$$\|AF_n x\|^2 = \langle F_n x, A^*AF_n x\rangle \leq \|A^*AF_n\| \, \|x\|^2.)$$

As $\{F_n\}$ is an increasing sequence of projections in $\mathscr{A}$ with least upper bound I and $F_nA \subseteq AF_n$; if $x \in \mathscr{D}(A)$, $F_n x \to x$ and $AF_n x = F_nAx \to Ax$. Thus $\bigcup_{n=1}^{\infty} F_n(\mathscr{H})$ is a core for A and $\{F_n\}$ is a bounding sequence in $\mathscr{A}$ for A.

Suppose $\{E_n\}$ is a bounding sequence in $\mathscr{A}$ for $\{A_j\}$, $j = 1, \ldots, m-1$ and $\{F_n\}$ is a bounding sequence in $\mathscr{A}$ for A_m, where $A_j \in \mathscr{A}$. Then $\{E_nF_n\}$ is a bounding sequence in $\mathscr{A}$ for $A_1, \ldots, A_m$. In particular, $\bigcup_{n=1}^{\infty} E_nF_n(\mathscr{H})$ is a common core for $A_1, \ldots, A_m$. It follows that both $A+B$ and A^*+B^* are densely defined. But $A^* + B^* \subseteq (A+B)^*$, so that $(A+B)^*$ is densely defined and $A+B$ is preclosed.

If $\{E_n\}$ is a bounding sequence in $\mathscr{A}$ for A, B, A^* and B^*, then $E_nAB \subseteq AE_nB \subseteq ABE_n$ and $AE_nBE_n \subseteq ABE_n$. As AE_n and BE_n are bounded and defined everywhere, $AE_nBE_n = ABE_n$. Thus $\{E_n\}$ is a bounding sequence for AB and, similarly, for BA and B^*A^*. In particular B^*A^* is densely defined. As $B^*A^* \subseteq (AB)^*$, $(AB)^*$ is densely defined and AB is preclosed. At the same time, $ABE_n = AE_nBE_n = BE_nAE_n = BAE_n$. Thus $A \hat{\cdot} B$ and $B \hat{\cdot} A$ agree on their common core $\bigcup_{n=1}^{\infty} E_n(\mathscr{H})$; and $A \hat{\cdot} B = B \hat{\cdot} A$. As A^*A and AA^* are self-adjoint, $A^*A = A^* \hat{\cdot} A = A \hat{\cdot} A^* = AA^*$. If $x \in \mathscr{D}(A) \cap \mathscr{D}(B)(=\mathscr{D}(A+B))$, $Ux \in \mathscr{D}(A+B)$ and $U^*x \in \mathscr{D}(A+B)$. Thus $U(\mathscr{D}(A+B)) = \mathscr{D}(A+B)$ and $U^*(A+B)U = A+B$. It follows that $U^*(A \hat{+} B)U = A$

$\hat{+} B$ and $A \hat{+} B \eta \mathscr{A}$. If $y \in \mathscr{D}(AB)$, then $y \in \mathscr{D}(B)$ and $By \in \mathscr{D}(A)$. Thus $Uy \in \mathscr{D}(B)$ and $BUy = UBy \in \mathscr{D}(A)$. It follows that $Uy \in \mathscr{D}(AB)$. Since $U^* y \in \mathscr{D}(AB)$, $U(\mathscr{D}(AB)) = \mathscr{D}(AB)$. As $U^* ABUy = ABy$, $U^* A \hat{\cdot} BU = A \hat{\cdot} B$ and $A \hat{\cdot} B \eta \mathscr{A}$.

With $\{E_n\}$ bounding for A and A^*, $E_n A^* \subseteq A^* E_n$ and $E_n A^* \subseteq (AE_n)^*$. Thus $A^* E_n$ and $(AE_n)^*$ are bounded everywhere-defined extensions of the same densely defined operator $E_n A^*$. It follows that $(AE_n)^* = A^* E_n$. Suppose that $\{E_n\}$ is bounding for B, B^*, $aA \hat{+} B$, $(aA \hat{+} B)^*$, $A \hat{\cdot} B$, $(A \hat{\cdot} B)^*$, and $A^* \hat{\cdot} B^* (= B^* \hat{\cdot} A^*)$ as well. Then, from the foregoing,

$$\begin{aligned}(\bar{a} A^* \hat{+} B^*) E_n &= \bar{a} A^* E_n + B^* E_n = \bar{a}(AE_n)^* + (BE_n)^* \\ &= ((aA \hat{+} B) E_n)^* = (aA \hat{+} B)^* E_n\end{aligned}$$

and

$$\begin{aligned}(A \hat{\cdot} B)^* E_n &= ((A \hat{\cdot} B) E_n)^* = (AE_n BE_n)^* = (BE_n)^* (AE_n)^* \\ &= B^* E_n A^* E_n = (B^* \hat{\cdot} A^*) E_n.\end{aligned}$$

Since $(aA \hat{+} B)^*$ and $\bar{a} A^* \hat{+} B^*$ agree on their common core $\bigcup_{n=1}^{\infty} E_n(\mathscr{H})$, they are equal. Similarly $(A \hat{\cdot} B)^* = B^* \hat{\cdot} A^* (= A^* \hat{\cdot} B^*)$.

If A is symmetric, $A \subseteq A^*$, a special case of $A \subseteq B$. If $\{E_n\}$ in $\mathscr{A}$ is bounding for A and B, we have $AE_n = BE_n$ since $AE_n \subseteq BE_n$. Thus A and B agree on their common core $\bigcup_{n=1}^{\infty} E_n(\mathscr{H})$, and $A = B$.

Choosing a common bounding sequence for all operators involved, it is routine to verify such identities as $(A \hat{\cdot} B) \hat{\cdot} C = A \hat{\cdot} (B \hat{\cdot} C)$, and (viii) follows. ■

As noted in (iv) of the preceding theorem, $A^* A = AA^*$ for each A affiliated with an abelian von Neumann algebra $\mathscr{A}$. By analogy with the case of bounded operators, we expect normal operators to be affiliated with abelian von Neumann algebras. With the aid of the lemmas that follow, we shall prove this. We conclude from this that the multiplication operators corresponding to unbounded (complex-valued) measurable functions (finite almost everywhere) are normal. Our first lemma is a condition for an operator to be normal.

3.4 Lemma. *If $\{F_n\}$ is a bounding sequence for the closed operator A on the Hilbert space $\mathscr{H}$ and AF_n is normal for each n, then A is normal.*

Proof. From Lemma 3.2, $(AF_n)^* = A^* F_n$; so that $A^* AF_n = A^* F_n AF_n = (AF_n)^* AF_n = AF_n (AF_n)^* = AF_n A^* F_n = AA^* F_n$. Thus the self-adjoint operators $A^* A$ and AA^* agree on $\bigcup_{n=1}^{\infty} F_n(\mathscr{H})$, a core for each of them. Thus $A^* A = AA^*$. ■

3.5 Lemma. *If $BA \subseteq AB$ and $\mathscr{D}(A) \subseteq \mathscr{D}(B)$, where A is a self-adjoint operator and B is a closed operator on the Hilbert space $\mathscr{H}$, then $E_\lambda B \subseteq BE_\lambda$ for each E_λ in the spectral resolution $\{E_\lambda\}$ of A.*

Proof. We note that $B(A + iI) = BA + iB$ under the present assumptions. From (3), $BA + iB \subseteq B(A + iI)$. Suppose $x \in \mathscr{D}(B(A + iI))$. Then $x \in \mathscr{D}(A)$ and $Ax + ix \in \mathscr{D}(B)$. By assumption $x \in \mathscr{D}(A) \subseteq \mathscr{D}(B)$ so that $Ax \in \mathscr{D}(B)$, as well. Thus $x \in \mathscr{D}(BA+iB)$ and $B(A+iI)x = BAx + iBx$. Hence $B(A+iI) \subseteq BA+iB$ and the stated equality follows. Similarly $B(A-iI) = BA - iB$.

Let T_+ and T_- be the (bounded, everywhere-defined) inverses to $A+iI$ and $A-iI$, respectively. Then, from (1), (2), (3) and the preceding paragraph,

$$\begin{aligned} T_+ B = T_+ B(A+iI)T_+ &= T_+(BA+iB)T_+ \subseteq T_+(AB+iB)T_+ \\ &= T_+(A+iI)BT_+ \subseteq BT_+ . \end{aligned}$$

Similarly $T_- B \subseteq BT_-$. From the discussion preceding Theorem 3.3, $T_+ = T_-^*$ so that Lemma 3.1 applies; and $TB \subseteq BT$ for each T in the von Neumann algebra $\mathscr{A}$ generated by T_+ and T_-. In particular $E_\lambda B \subseteq BE_\lambda$ for each λ. ■

3.6 Theorem. *An operator A is normal if and only if it is affiliated with an abelian von Neumann algebra.*

Proof. Since $AA^*A = A^*AA$ and $\mathscr{D}(A^*A) \subseteq \mathscr{D}(A)$, Lemma 3.5 applies. Thus $E_\lambda A \subseteq AE_\lambda$ for each λ, where $\{E_\lambda\}$ is the spectral resolution of A^*A; and $F_n A \subseteq AF_n$ for each n, where $F_n = E_n - E_{-n}$. In the same way, $A^*A^*A = A^*AA^*$ and $\mathscr{D}(A^*A) = \mathscr{D}(AA^*) \subseteq \mathscr{D}(A^*)$; so that $F_n A^* \subseteq A^* F_n$ for each n. As in the proof of Theorem 3.3, AF_n and A^*F_n are bounded since $A^*AF_n (= AA^*F_n)$ is. Moreover $F_n A^* \subseteq (AF_n)^*$ so that both $(AF_n)^*$ and A^*F_n are bounded extensions of the densely defined $F_n A^*$. Thus $(AF_n)^* = A^*F_n$ (and $(A^*F_n)^* = AF_n$). Note, too, that $AF_n AF_m \subseteq AAF_n$ and $AF_m AF_n \subseteq AAF_n$, when $n \le m$. Since $AF_n AF_m$ and $AF_m AF_n$ are everywhere defined, $AF_n AF_m = AAF_n = AF_m AF_n$. At the same time, $A^*F_m AF_n = A^*AF_n = AA^*F_n = AF_n A^*F_m$. Thus $\{F_n, AF_n, A^*F_n : n = 1, 2, \ldots\}$ generates an abelian von Neumann algebra $\mathscr{A}$.

If $x \in \mathscr{D}(A)$ then $F_n x \to x$ and $AF_n x = F_n Ax \to Ax$; so that $\bigcup_{n=1}^\infty F_n(\mathscr{H})$ is a core $\mathscr{D}_0$ for A. With U a unitary operator in $\mathscr{A}'$ and x in $\mathscr{D}_0$, $AUx = AUF_n x = AF_n Ux = UAF_n x = UAx$. From Remark 3.7, $A \eta \mathscr{A}$ (and $A^* \eta \mathscr{A}$). ■

3.7 Remark. If T is a closed densely defined operator with core $\mathscr{D}_0$ and $TUx = UTx$ for each x in $\mathscr{D}_0$ and each unitary operator U commuting with a von Neumann algebra $\mathscr{R}$, then $T \eta \mathscr{R}$. To see this, note that with y in $\mathscr{D}(T)$, there is a sequence $\{y_n\}$ in $\mathscr{D}_0$ such that $y_n \to y$ and $Ty_n \to Ty$ (since $\mathscr{D}_0$ is a core for T). Now $Uy_n \to Uy$ and $TUy_n = UTy_n \to UTy$. Since T is closed, $Uy \in \mathscr{D}(T)$ and $TUy = UTy$. Thus $\mathscr{D}(T) \subseteq U(\mathscr{D}(T))$. Applied to U^*, we have $\mathscr{D}(T) \subseteq U(\mathscr{D}(T))$; so that $U(\mathscr{D}(T)) = \mathscr{D}(T)$. Hence $\mathscr{D}(U^*TU) = \mathscr{D}(T)$ and $U^*TUy = Ty$ for each y in $\mathscr{D}(T)$. ■

4. The isomorphism

In the process of developing some of the spectral theory for (unbounded) self-adjoint operators through the corresponding theory for such functions in $\mathscr{S}(X)$ (preceding Theorem 3.3), we assigned a function h in $\mathscr{S}(X)$ to a self-adjoint operator A affiliated with an abelian von Neumann algebra $\mathscr{A}$ (where $\mathscr{A}$ is isomorphic to $C(X)$). Our aim, in this section, is to show that this correspondence extends to a *-isomorphism of $\mathscr{N}(\mathscr{A})$ onto $\mathscr{N}(X)$ whose restriction to $\mathscr{A}$ is the given *-isomorphism of $\mathscr{A}$ with $C(X)$. This is effected in the following theorem. The reciprocals of normal functions, appearing in the statement and proof, refer to their inverses in the algebra $\mathscr{N}(X)$.

4.1 Theorem. *If $\mathscr{A}$ is an abelian von Neumann algebra, φ_0 is an isomorphism of $\mathscr{A}$ with $C(X)$ and we define φ on $\mathscr{S}(\mathscr{A})$ by:*

(1) $$2\varphi(H)=\varphi_0([H+iI]^{-1})^{-1}+\varphi_0([H-iI]^{-1})^{-1},$$

then φ is an isomorphism of $\mathscr{S}(\mathscr{A})$ onto $\mathscr{S}(X)$. If we have $A\eta\mathscr{A}$, $A_1=\frac{1}{2}(A+A^*)$, $A_2=\frac{-i}{2}(A-A^*)$, and $\varphi(A)$ is defined to be $\varphi(A_1)+i\varphi(A_2)$, then φ is an isomorphism of $\mathscr{N}(\mathscr{A})$ onto $\mathscr{N}(X)$. The restriction of φ to $\mathscr{A}$ is φ_0.

Proof. We observed (preceding Theorem 3.3) that $\varphi(H)$, as defined in (1), lies in $\mathscr{S}(X)$. If $H\in\mathscr{A}$, $H+iI$ and $H-iI$ have inverses in $\mathscr{A}$; so that $\varphi_0([H+iI]^{-1})^{-1}=\varphi_0(H+iI)$, $\varphi_0([H-iI]^{-1})^{-1}=\varphi_0(H-iI)$, and $\varphi(H)=\varphi_0(H)$. Similarly, with A in $\mathscr{A}$, $\varphi(A)=\varphi_0(A)$.

We prove that φ is an isomorphism. The identities $\varphi(aH\hat{+}K)=a\varphi(H)\hat{+}\varphi(K)$ and $\varphi(H\hat{\cdot}K)=\varphi(H)\hat{\cdot}\varphi(K)$ are proved by reformulating them, using (1), in terms of $\varphi_0([H\pm iI]^{-1})$, $\varphi_0([K\pm iI]^{-1})$, $\varphi_0([aH\hat{+}K\pm iI]^{-1})$, and $\varphi_0([H\hat{\cdot}K\pm iI]^{-1})$, performing the formal algebraic operations (justified in $\mathscr{N}(\mathscr{A})$ and $\mathscr{N}(X)$), and using the corresponding identities for φ_0. Again, these same identities are valid when operators A and B affiliated with $\mathscr{A}$ replace H and K (by virtue of the validity of the usual algebraic operations in $\mathscr{N}(\mathscr{A})$ and $\mathscr{N}(X)$ and these identities, just established, for self-adjoint operators affiliated with $\mathscr{A}$).

If $\varphi(H)=0$ then the normal functions $\varphi_0([H+iI]^{-1})^{-1}$ and $-\varphi_0([H-iI]^{-1})^{-1}$ are equal. Thus $\varphi_0([H+iI]^{-1})$ and $-\varphi_0([H-iI]^{-1})$ are equal. Since φ_0 is an isomorphism, $[H+iI]^{-1}$ and $-[H-iI]^{-1}$ are equal. Thus $H=0$, in this case. It follows that φ is an isomorphism of $\mathscr{N}(\mathscr{A})$ into $\mathscr{N}(X)$.

To see that φ maps $\mathscr{S}(\mathscr{A})$ onto $\mathscr{S}(X)$, note that with h in $\mathscr{S}(X)$, $(h\hat{+}i)^{-1}$ and $(h\hat{-}i)^{-1}$ are in $C(X)$. Choose T_+ and T_- in $\mathscr{A}$ so that $\varphi_0(T_+)=(h\hat{+}i)^{-1}$ and $\varphi_0(T_-)=(h\hat{-}i)^{-1}$. Then $T_-=T_+^*$, since $(h\hat{+}i)^{-1}$ and $(h\hat{-}i)^{-1}$ are complex conjugates of one another. If T_+ has non-zero null space the projection on that null space lies in $\mathscr{A}$ and corresponds to a non-zero characteristic function in $C(X)$ whose product with $(h\hat{+}i)^{-1}$ is 0, contradicting the fact that $(h\hat{+}i)^{-1}$ vanishes on a nowhere-dense set. Thus T_+, and, similarly, T_-, have null space (0). Since $T_-=T_+^*$, both T_+ and T_- are one-to-one mappings of $\mathscr{H}$ onto dense linear manifolds in $\mathscr{H}$. Let B_+ and B_- denote the mappings inverse to T_+ and T_-. If U is a unitary operator commuting with $\mathscr{A}$, U commutes with T_+ and T_-; so that $UB_+U^*=B_+$ and $UB_-U^*=B_-$. Thus $B_+\eta\mathscr{A}$ and $B_-\eta\mathscr{A}$. Since $I=B_+T_+$

$=B_-T_-$, we have $1=\varphi(B_+)\hat{\ }(h\hat{+}i)^{-1}=\varphi(B_-)\hat{\ }(h\hat{-}i)^{-1}$; and $\varphi(B_+)=h\hat{+}i$, $\varphi(B_-)=h\hat{-}i$. Thus $B_+\hat{-}iI$ (and $B_-\hat{+}iI$) in $\mathcal{N}(\mathcal{A})$ are mapped by φ onto h; and φ is an isomorphism of $\mathcal{S}(\mathcal{A})$ onto $\mathcal{S}(X)$. Since the real and imaginary parts of a normal function have self-adjoint extensions, φ is an isomorphism of $\mathcal{N}(\mathcal{A})$ onto $\mathcal{N}(X)$. ■

The spectrum, $sp(T)$, of a closed densely defined operator T on $\mathcal{H}$ is the set of complex numbers z such that $T-zI$ is not a one-to-one mapping of its domain onto $\mathcal{H}$ (with a, necessarily, bounded inverse). If T is normal, from Theorem 3.6, $T\eta\mathcal{A}$ for some abelian von Neumann algebra $\mathcal{A}$. If φ is the isomorphism of $\mathcal{N}(\mathcal{A})$ with $\mathcal{N}(X)$ constructed in the preceding theorem, the range of $\varphi(T)$ is $sp(T)$. To see this, note that $z\notin sp(T)$ if and only if $T\hat{-}zI$ has an inverse in $\mathcal{A}$. Thus $z\notin sp(T)$ if and only if $\varphi(T)\hat{-}z$ has an inverse in $C(X)$. Since $\varphi(T)$ is a normal function on X, $\varphi(T)\hat{-}z$ has an inverse in $C(X)$ if and only if z is not in the range of $\varphi(T)$ (and this inverse is 0 on the closed nowhere-dense subset of X on which $\varphi(T)$ is not defined). It is possible for an element T of $\mathcal{N}(\mathcal{A})$ to have an inverse in $\mathcal{N}(\mathcal{A})$ not in $\mathcal{A}$, so that $0\in sp(T)$.

5. A characterization

In this section, we prove a theorem (5.11) characterizing $\mathcal{S}(X)$ as a vector lattice in the style of the Stone-Krein-Kakutani-Yosida characterization of $C(X,\mathbb{R})$ as a vector lattice [8, 5, 6, 4, 10]. (Compare [2; Section 4] for a discussion of the $C(X,\mathbb{R})$ result and for some of the terminology and constructs we use here.) Two distinctions between the cases of $\mathcal{S}(X)$ and $C(X,\mathbb{R})$ should be noted:

(i) We do not have a norm on $\mathcal{S}(X)$ (nor can we expect to define one).

(ii) Our compact-Hausdorff spaces are extremely disconnected in the case of $\mathcal{S}(X)$.

Let $\mathcal{V}$ be a partially ordered vector space. We recall (see [2; Definition 2.1]) that an element e in the positive cone $\mathcal{V}_+$ of $\mathcal{V}$ is an order unit for $\mathcal{V}$ when, for each v in $\mathcal{V}$, there is a positive real a such that $-ae\leq v\leq ae$. The existence of an order unit amounts to assuming that the elements of $\mathcal{V}$ are "bounded". The ordering on $\mathcal{V}$ is said to be *archimedian* when an element v such that $v\leq ae$ for each positive a must be negative (that is, $v\leq 0$). This condition amounts to assuming that there are no "infinitely small" elements in $\mathcal{V}$. It is the condition that $\mathcal{V}$ has no "radical" (that is, that $\mathcal{V}$ is *semi-simple*). There is a norm associated with the order unit e ($\|v\|$ is defined as $\inf\{a\colon -ae\leq v\leq ae\}$); and among the complete archimedian ordered vector spaces those that are lattices are precisely the ones that are linearly order isomorphic to $C(X,\mathbb{R})$ for some compact Hausdorff space X. The space X is the set of extreme points (the *pure* states) of the family of positive linear functionals on $\mathcal{V}$ that take the value 1 at e (the *states*) topologized by the weak $*$ topology. In case $\mathcal{V}$ is assumed to be a boundedly complete lattice, it is not necessary to assume that $\mathcal{V}$ is norm-complete. An argument similar to the one appearing in the proof of Theorem 5.4 establishes that $\mathscr{V}$ is linearly order isomorphic to $C(X,\mathbf{R})$ with X an extremely disconnected compact-Hausdorff space.

To find an ordered-space characterization of $\mathscr{S}(X)$, we both want and do not want an order-unit assumption. We need many "bounded" elements but do not want to assume that each element is bounded. This is managed with the aid of a *semi-order unit.*

5.1 Definition. An element v of a partially ordered vector space $\mathscr{V}$ is said to be *bounded with respect to an element* e in $\mathscr{V}_+$ (or, simply, *e-bounded*) when there is a positive scalar a such that $-ae \le v \le ae$. When each element of $\mathscr{V}_+$ is the least upper bound of an increasing net of e-bounded elements of $\mathscr{V}$, we say that e is a *semi-order unit* for $\mathscr{V}$. ■

The following lemma will prove useful.

5.2 Lemma. *Let $\mathscr{V}$ be a vector lattice, v_+ be $v \vee 0$, and v_- be $-(v \wedge 0)$ for each v in $\mathscr{V}$. Then $v = v_+ - v_-$. If $\mathscr{V}$ is boundedly complete and e is a semi-order unit for $\mathscr{V}$, then $v = \bigvee_{n=1}^{\infty} v \wedge ne$, for each v in $\mathscr{V}$.*

Proof. For each u in $\mathscr{V}$, the mapping $w \to w+u$ is an order isomorphism of $\mathscr{V}$ onto $\mathscr{V}$ and, therefore, preserves unions and intersections. The mapping $w \to -w$ is an "anti-order isomorphism" of $\mathscr{V}$ onto $\mathscr{V}$ and, therefore, reverses unions and intersections. Thus

$$v = v - (v \wedge 0) + v \wedge 0 = v + (0 \vee -v) + v \wedge 0 = v \vee 0 + v \wedge 0 = v_+ - v_-.$$

If $\mathscr{V}$ is boundedly complete, e is a semi-order unit for $\mathscr{V}$, and $u \in \mathscr{V}_+$, then u is $\bigvee_a u_a$ for some subset $\{u_a\}$ of e-bounded elements of $\mathscr{V}$. For each a we can choose a positive integer n_a so that $u_a \le n_a e$, whence $u_a \le u \wedge n_a e \le u$. If $\{n_a\}$ is an infinite family, it contains arbitrarily large positive integers; and u is $\bigvee_{n=1}^{\infty} u \wedge ne$. If $\{n_a\}$ is finite, it has a largest element n_0; and $u \wedge n_0 e = \bigvee_a u \wedge n_a e = u$. In this case, $0 \le u = u \wedge n_0 e \le n_0 e$; and again, $u = \bigvee_{n=1}^{\infty} u \wedge ne$.

For an arbitrary element v in $\mathscr{V}$,

$$v \wedge ne = (v_+ - v_-) \wedge ne = [v_+ \wedge (ne + v_-)] - v_-.$$

Thus

$$\bigvee_{n=1}^{\infty} v \wedge ne = \bigvee_{n=1}^{\infty} ([v_+ \wedge (ne + v_-)] - v_-) = \left[\bigvee_{n=1}^{\infty} v_+ \wedge (ne + v_-)\right] - v_-$$

$$\ge \left[\bigvee_{n=1}^{\infty} v_+ \wedge ne\right] - v_- = v_+ - v_- = v. \quad ■$$

To characterize $\mathscr{S}(X)$, we want to rule out both infinitely small and infinitely large elements. We rephrase the archimedian condition for this purpose.

5.3 Definition. A partially ordered vector space $\mathscr{V}$ is said to be archimedian ordered when $v \leq 0$ if $v \leq av_0$ for some v_0 in $\mathscr{V}$ and all positive scalars a. ■

This condition follows from the usual archimedian condition on $\mathscr{V}$ when $\mathscr{V}$ has an order unit e; for if $v \leq av_0$ for all positive a, and b is a positive scalar such that $v_0 \leq be$, then $v \leq abe$ for each positive a. From the standard archimedian condition [2; Definition 2.1], $v \leq 0$. In the revised form, the archimedian condition eliminates the possibility that there is an ("infinitely large") element v in $\mathscr{V}$ such that $av_0 \leq v$ for some non-zero v_0 in $\mathscr{V}_+$ and all positive a; for, in this case, $v_0 \leq bv$ for all positive b and $v_0 \leq 0$.

We prove, first, a result characterizing $C(X, \mathbb{R})$ as a vector lattice when X is an extremely disconnected compact-Hausdorff space.

5.4 Theorem. *A boundedly complete archimedian vector lattice $\mathscr{V}$ with order unit e is linearly order isomorphic with $C(X, \mathbb{R})$, where X is an extremely disconnected compact-Hausdorff space and e corresponds to* 1.

Proof. From [2; Theorem 4.1], $\mathscr{V}$ is linearly order isomorphic to a norm-dense linear sublattice of $C(X, \mathbb{R})$ containing the constants, where X is the weak * closure of the space of extreme maximal ideals of $\mathscr{V}$ (effectively, *maximal lattice ideals* or *pure states* of $\mathscr{V}$). The image $\mathscr{V}_0$ of $\mathscr{V}$ in $C(X, \mathbb{R})$ is, therefore, a boundedly complete lattice (and is norm dense in $C(X, \mathbb{R})$). We shall show that the least upper bound f_0 of a subset $\{f_a\}$ of $\mathscr{V}_0$ relative to $\mathscr{V}_0$ is its least upper bound relative to $C(X, \mathbb{R})$. Suppose f is an upper bound for $\{f_a\}$ in $C(X, \mathbb{R})$. We show that $f_0 \leq f$. If p_0 is a point in X corresponding to an extreme maximal ideal of $\mathscr{V}$, then with $\rho(f)$ defined as $f(p_0)$ for f in $C(X, \mathbb{R})$, $\rho | \mathscr{V}_0$ is a positive linear functional on $\mathscr{V}_0$ that has a unique positive extension to $C(X, \mathbb{R})$ (for the extreme points of the set of positive extensions of $\rho | \mathscr{V}_0$ are pure states of $C(X, \mathbb{R})$ and correspond to points of X having the same value on functions in $\mathscr{V}_0$ as p_0; but functions in $\mathscr{V}_0$ separate points of X). Thus

$$f(p_0) = \inf \{g(p_0) : g \in \mathscr{V}_0, f \leq g\}.$$

If $f(p_0) < f_0(p_0)$, there is a g in $\mathscr{V}_0$ such that $f \leq g$ and $g(p_0) < f_0(p_0)$. Since $f_a \leq f \leq g$ for all a, and $g \in \mathscr{V}_0$, and f_0 is the least upper bound of $\{f_a\}$ in $\mathscr{V}_0$, we have that $f_0 \leq g$. In particular, $f_0(p_0) \leq g(p_0)$ – contradicting the choice of g such that $g(p_0) < f_0(p_0)$. It follows that $f_0(p_0) \leq f(p_0)$ for each p_0 in X corresponding to an extreme maximal ideal in $\mathscr{V}$. As these points are dense in X and $f - f_0$ is continuous on $X, f_0 \leq f$. As in the first paragraph of the proof of Theorem 2.13, $\mathscr{V}_0 = C(X, \mathbb{R})$. Hence $C(X, \mathbb{R})$ is a boundedly complete lattice and X is an extremely disconnected compact-Hausdorff space. ■

If f, g and h in $\mathscr{S}(X)$ are defined at p, then

$$\begin{aligned}[(f \vee g) \wedge h](p) &= \min\{(f \vee g)(p), h(p)\} = \min\{\max\{f(p), g(p)\}, h(p)\} \\ &= \max\{\min\{f(p), h(p)\}, \min\{g(p), h(p)\}\} = [(f \wedge h) \vee (g \wedge h)](p).\end{aligned}$$

Thus

(*) $\quad (f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h).$

A "distributive law" stronger than (*) is valid in $\mathscr{S}(X)$:

(**) $\quad g \wedge (\bigvee_a f_a) = \bigvee_a g \wedge f_a.$

To establish (**), note that $g \wedge f_a \le g \wedge f$ for each a, where $f = \bigvee_a f_a$. Thus $h \le g \wedge f$, where $h = \bigvee_a g \wedge f_a$. If $h < g \wedge f$, then there is a clopen set $\mathcal{O}$ in X and a positive constant c such that $h(p) + c \le (g \wedge f)(p)$ for p in $\mathcal{O}$. If $g(p) < f_a(p)$ for some p in $\mathcal{O}$, then $g(q) < f_a(q)$, $g(q) < f(q)$, and $(g \wedge f)(q) = g(q)$ for each q in some clopen subset $\mathcal{O}_0$ of $\mathcal{O}$ containing p. But then, for all q in $\mathcal{O}_0$,

$$(g \wedge f_a)(q) = g(q) \le h(q) \le g(q) - c.$$

Thus $f_a(p) \le g(p)$ for each p in $\mathcal{O}$. Since $f = \bigvee_a f_a$, we have, from this, that $f(p) \le g(p)$ and $(g \wedge f)(p) = f(p)$ for each p in $\mathcal{O}$. Hence

$$f_a(p) = (g \wedge f_a)(p) \le h(p) \le (g \wedge f)(p) - c = f(p) - c$$

for each p in $\mathcal{O}$ – contradicting the fact that $f = \bigvee_a f_a$. Thus $h = g \wedge f$, which is (**).

The condition (**) is a type of (complete) continuity on the operations of intersection and union in $\mathscr{S}(X)$ – in the framework of operator algebras it would be a "normality" or "strong-continuity" condition on the corresponding operations. Weakened forms of (*) and (**) will be needed for our characterization of $\mathscr{S}(X)$.

5.5 Definition. A semi-order unit e for a vector lattice $\mathscr{V}$ is said to be *distributive* when

$$(u \vee v) \wedge ae = (u \wedge ae) \vee (v \wedge ae)$$

for all u and v in $\mathscr{V}$ and all scalars a. If $\mathscr{V}$ is boundedly complete and, for each increasing net $\{v_a\}$ in $\mathscr{V}_+$ with an upper bound in $\mathscr{V}$ and each positive scalar a,

$$ae \wedge (\bigvee_a v_a) = \bigvee_a ae \wedge v_a,$$

we say that e is *completely distributive.* ∎

Note that if e is distributive for $\mathscr{V}$, then

$$\begin{aligned}(u \wedge v) \vee ae &= -[(-[u \wedge v]) \wedge (-ae)] = -[([-u] \vee [-v]) \wedge (-ae)] \\ &= -[([-u] \wedge [-ae]) \vee ([-v] \wedge [-ae])] \\ &= [-([-u] \wedge [-ae])] \wedge [-([-v] \wedge [-ae])] \\ &= (u \vee ae) \wedge (v \vee ae).\end{aligned}$$

5.6 Lemma. *If $\mathscr{V}$ is a boundedly complete archimedian vector lattice with a completely distributive semi-order unit e, there is a linear order isomorphism η of $\mathscr{V}$ into $\mathscr{S}(X)$ for some extremely disconnected compact-Hausdorff space X such that $\eta(e) = 1$ and the least upper bound of $\{\eta(v_a)\}$ relative to $\mathscr{S}(X)$ is $\eta(v)$, where $v = \bigvee_a v_a$.*

Proof. Let $\mathscr{V}_b$ be the family of e-bounded elements in $\mathscr{V}$. Then $\mathscr{V}_b$ is an archimedian vector lattice with order unit e. If $\{v_a\}$ is a subset of $\mathscr{V}_b$ bounded above by ne, then $\{v_a\}$ has a least upper bound v in $\mathscr{V}$. Of course, $v_a \leq v \leq ne$ for each v_a. Thus $v \in \mathscr{V}_b$, and v is the least upper bound of $\{v_a\}$ in $\mathscr{V}_b$. It follows that $\mathscr{V}_b$ is boundedly complete and, from Theorem 5.4, that there is a linear order isomorphism η of $\mathscr{V}_b$ with $C(X, \mathbb{R})$, where X is some extremely disconnected compact-Hausdorff space. From Theorem 2.12, $\mathscr{S}(X)$ is a boundedly complete archimedian vector lattice (as well as an algebra over $\mathbb{R}$). We shall extend η to a linear order isomorphism of $\mathscr{V}$ into $\mathscr{S}(X)$ (with the properties noted in the statement of this lemma).

With v in $\mathscr{V}_+$, let v_n be $v \wedge ne$. Then, if $n \leq m$,

$$v_m \wedge ne = (v \wedge me) \wedge ne = (v \wedge ne) \wedge me = v_n \wedge me = v_n,$$

and, for each p in X,

$$(\eta(v_m) \wedge n)(p) = \min\{\eta(v_m)(p), n\} = \eta(v_n)(p).$$

If $\eta(v_n)(p) < n$, it follows that $\eta(v_m)(p) = \eta(v_n)(p)$. Let $\mathcal{O}$ be the open set $\bigcup_{n=1}^{\infty} \eta(v_n)^{-1}([0, n))$ and define $\eta(v)(p)$, for p in $\mathcal{O}$, to be $\lim \eta(v_n)(p)$ as $n \to \infty$. From the foregoing, $\eta(v)(p)$ is $\eta(v_n)(p)$ when $\eta(v_n)(p) < n$. Thus $\eta(v)$ and $\eta(v_n)$ agree on the open set $\eta(v_n)^{-1}([0, n))$; and $\eta(v)$ is continuous on $\mathcal{O}$. If Z_0 is the closure of the interior of the complement Z of $\mathcal{O}$, then Z_0 is clopen since Z is closed. Let e_0 be the element of $\mathscr{V}_b$ such that $\eta(e_0)$ is the characteristic function of Z_0. If $p \in Z$, then $p \notin \eta(v_n)^{-1}([0, n))$; so that $n \leq \eta(v_n)(p)$ and $n\eta(e_0) \leq \eta(v_n)$. Thus $ne_0 \leq v_n \leq v$, and $e_0 \leq n^{-1}v$ for each positive integer n. Since $\mathscr{V}$ is archimedian, $e_0 \leq 0$; and $e_0 = 0$. Hence $Z_0 = \emptyset$, and Z is nowhere dense. If $p \in Z$, then $m = \eta(v_m)(p)$ for each m; so that $p \in \eta(v_{n+1})^{-1}((n, n+1])$ for each n. Now $\eta(v_{n+1})^{-1}((n, n+1])$ is open in X; and if $q \in \eta(v_{n+1})^{-1}((n, n+1]) \setminus Z$, then $q \in \mathcal{O}$ and $n < \eta(v_{n+1})(q)$. Hence $\eta(v)(q) > n$, and $\eta(v) \in \mathscr{S}(X)$.

From the definition of $\eta(v)$ and Lemma 5.2, we have:

$$\eta(v) = \eta\left(\bigvee_{n=1}^{\infty} v_n\right) = \bigvee_{n=1}^{\infty} \eta(v_n). \tag{1}$$

Suppose u and v are in $\mathscr{V}_+$. Since $u_n + v_n \leq (u+v)_{2n}$; we have that $\eta(u_n) + \eta(v_n) \leq \eta((u+v)_{2n}) \leq \eta(u+v)$. Thus

$$\begin{aligned}\bigvee_{n=1}^{\infty} (\eta(u_n) + \eta(v_n)) &= \bigvee_{n=1}^{\infty} \eta(u_n) \,\hat{+}\, \bigvee_{n=1}^{\infty} \eta(v_n)\\ &= \eta(u) \,\hat{+}\, \eta(v) \leq \eta(u+v).\end{aligned}$$

If we assume, in addition, that $u \in \mathscr{V}_b$, say $u \leq n_0 e$, then with m equal to the larger of n and n_0, we have

$$(u+v)_n \leq (u+v) \wedge (u + me) = u + (v \wedge me) = u_m + v_m.$$

Again, $\eta((u+v)_n) \le \eta(u_m) + \eta(v_m) \le \eta(u) \hat{+} \eta(v)$ for each n; so that $\eta(u+v) \le \eta(u) \hat{+} \eta(v)$. Thus $\eta(u+v) = \eta(u) \hat{+} \eta(v)$ when u and v are in $\mathscr{V}_+$ and $u \in \mathscr{V}_b$. For arbitrary u and v in $\mathscr{V}_+$, we have $\eta(u_n + v) = \eta(u_n) \hat{+} \eta(v)$. Now $\bigvee_{n=1}^{\infty} (\eta(u_n) \hat{+} \eta(v)) = \eta(u) \hat{+} \eta(v)$; while $u_n + v = (u+v) \wedge (ne + v) \ge (u+v)_n$. Thus $\eta(u+v) \le \bigvee_{n=1}^{\infty} \eta(u_n + v) = \eta(u) \hat{+} \eta(v)$; and

$$\eta(u+v) = \eta(u) \hat{+} \eta(v)$$

when u and v are in $\mathscr{V}_+$. It follows, now, that η extends (uniquely) to a linear mapping (we denote, again, by η) of the linear span $\mathscr{V}$ of $\mathscr{V}_+$ into $\mathscr{S}(X)$. Since η maps $\mathscr{V}_+$ into $\mathscr{S}(X)_+$, η is order preserving.

We show, next, that

$$(2) \qquad \eta(u \wedge v) = \eta(u) \wedge \eta(v), \qquad \eta(u \vee v) = \eta(u) \vee \eta(v)$$

when u and v are in $\mathscr{V}_+$. Note for this that

$$(u \wedge v)_n = u \wedge v \wedge ne = (u \wedge ne) \wedge (v \wedge ne) = u_n \wedge v_n.$$

To prove the analogous result for $(u \vee v)_n$, we must make use of our assumption that e is distributive. With this assumption in force,

$$(u \vee v)_n = (u \vee v) \wedge ne = (u \wedge ne) \vee (v \wedge ne) = u_n \vee v_n.$$

If $\eta(u)$ and $\eta(v)$ are defined on $X \backslash Z$ and $X \backslash Z'$, respectively, and $p \in X \backslash (Z \cup Z')$, then

$$(\eta(u) \wedge \eta(v))(p) = \min\{\eta(u)(p), \eta(v)(p)\}$$
$$= \min\{\eta(u_n)(p), \eta(v_n)(p)\},$$

and

$$(\eta(u) \vee \eta(v))(p) = \max\{\eta(u)(p), \eta(v)(p)\}$$
$$= \max\{\eta(u_n)(p), \eta(v_n)(p)\},$$

provided that n is chosen larger than $\eta(u)(p)$ and $\eta(v)(p)$. At the same time,

$$\eta(u \wedge v)(p) = \eta((u \wedge v)_m)(p) = \eta(u_m \wedge v_m)(p)$$
$$= (\eta(u_m) \wedge \eta(v_m))(p) = \min\{\eta(u_m)(p), \eta(v_m)(p)\},$$

and

$$\eta(u \vee v)(p) = \eta((u \vee v)_m)(p) = \eta(u_m \vee v_m)(p)$$
$$= (\eta(u_m) \vee \eta(v_m))(p) = \max\{\eta(u_m)(p), \eta(v_m)(p)\},$$

provided m is larger than $\eta(u \vee v)(p)$ and $\eta(u \wedge v)(p)$. Thus $\eta(u) \wedge \eta(v)$ and $\eta(u \wedge v)$ agree on the dense set $X \backslash (Z \cup Z')$ as do $\eta(u) \vee \eta(v)$ and $\eta(u \vee v)$. From Lemma 2.3, we have (2).

We prove, now, that for each u in $\mathscr{V}$,

$$(3) \qquad \eta(u_+) = \eta(u)_+, \qquad \eta(u_-) = \eta(u)_-.$$

Since η is an order isomorphism of $\mathscr{V}_b$ onto $C(X, \mathbb{R})$, (3) holds when $u \in \mathscr{V}_b$. In general, writing u_n for $(u \wedge ne) \vee (-ne)$, we have,

$$(u_+)_n = [(u \vee 0) \wedge ne] \vee (-ne) = (u \vee 0) \wedge ne = (u \wedge ne) \vee 0$$
$$= [(u \wedge ne) \vee 0] \vee (-ne) = [(u \wedge ne) \vee (-ne)] \vee 0 = (u_n)_+$$

and

$$(u_-)_n = [(-(u \wedge 0)) \wedge ne] \vee (-ne)$$
$$= -[(u \wedge 0) \vee (-ne)] = -[[(u \wedge ne) \wedge 0] \vee (-ne)]$$
$$= ([(u \wedge ne) \vee (-ne)] \wedge 0) = -(u_n \wedge 0) = (u_n)_-.$$

Since $u_n \in \mathscr{V}_b$,

$$(u_+ \wedge u_-)_n = u_+ \wedge u_- \wedge ne = (u_+)_n \wedge (u_-)_n = (u_n)_+ \wedge (u_n)_- = 0.$$

Thus

$$u_+ \wedge u_- = \bigvee_{n=1}^{\infty} (u_+ \wedge u_-)_n = 0.$$

It follows, now, that

(4) $\quad 0 = \eta(u_+ \wedge u_-) = \eta(u_+) \wedge \eta(u_-).$

By construction, $\eta(u) = \eta(u_+) - \eta(u_-)$; and both $\eta(u_+)$ and $\eta(u_-)$ are positive. Combining this last information with (4), we have (3).

Since

$$u \wedge v = u + 0 \wedge (v - u) = u - (v - u)_-$$

and

$$u \vee v = u + 0 \vee (v - u) = u + (v - u)_+,$$

we have that

$$\eta(u \wedge v) = \eta(u) \hat{-} \eta((v-u)_-) = \eta(u) \hat{-} (\eta(v) - \eta(u))_- = \eta(u) \wedge \eta(v)$$

and

$$\eta(u \vee v) = \eta(u) \hat{+} \eta((v-u)_+) = \eta(u) \hat{+} (\eta(v) - \eta(u))_+ = \eta(u) \vee \eta(v).$$

If $u \in \mathscr{V}_+$ and $\eta(u) = 0$, then $\eta(u_n) = 0$ since $0 \le u_n \le u$ and η is order preserving. Thus, from (1),

$$0 = \bigvee_{n=1}^{\infty} \eta(u_n) = \eta\left(\bigvee_{n=1}^{\infty} u_n\right) = \eta(u).$$

Suppose $v \in \mathscr{V}$ and $0 = \eta(v) = \eta(v_+) \hat{-} \eta(v_-)$. Then $\eta(v)_+ = \eta(v_+) = \eta(v_-) = \eta(v)_-$. Since $\eta(v)_+ \wedge \eta(v)_- = 0$, we conclude that $\eta(v_+) = \eta(v)_+ = \eta(v)_- = \eta(v_-) = 0$. Thus $v_+ = v_- = 0$, and $v = v_+ - v_- = 0$.

To this point, we have established that η is a linear order isomorphism of $\mathscr{V}$ onto a sublattice $\mathscr{V}_0$ of $\mathscr{S}(X)$ and that η maps $\mathscr{V}_b$ onto $C(X, \mathbb{R})$. We show, now, that $\mathscr{V}_0$ is boundedly complete in the sense that if $\{\eta(v_a)\}$ is a subset of $\mathscr{V}_0$ that has an upper bound in $\mathscr{V}_0$, then the least upper bound of $\{\eta(v_a)\}$ relative to $\mathscr{S}(X)$ lies in $\mathscr{V}_0$. In fact, we show that if $v = \bigvee_a v_a$, then $\eta(v) = \bigvee_a \eta(v_a)$. Passing to the increasing net of least upper bounds of finite subsets of $\{v_a\}$ containing a given finite subset, we may assume that $\{v_a\}$ is an increasing net in $\mathscr{V}$ with a smallest element $v_{a'}$. Replacing $\{v_a\}$ by $\{v_a - v_{a'}\}$, we may assume that each v_a is in $\mathscr{V}_+$. With this assumption, $v_a \wedge ne \in \mathscr{V}_b$ for each positive integer n; and, since e is completely distributive,

$$\begin{aligned}(\bigvee_a \eta(v_a)) \wedge n1 &= \bigvee_a (\eta(v_a) \wedge n1) = \bigvee_a \eta(v_a \wedge ne) = \eta(\bigvee_a (v_a \wedge ne)) \\ &= \eta((\bigvee_a v_a) \wedge ne) = \eta(\bigvee_a v_a) \wedge n1.\end{aligned}$$

Since the self-adjoint functions $\bigvee_a \eta(v_a)$ and $\eta(\bigvee_a v_a)$ are the least upper bounds in $\mathscr{S}(X)$ of $\{(\bigvee_a \eta(v_a)) \wedge n1\}$ and $\{\eta(\bigvee_a v_a) \wedge n1\}$, respectively, $\eta(v) = \bigvee_a \eta(v_a)$. ■

The hypotheses of the preceding lemma are not sufficient to ensure that there is a linear order isomorphism of $\mathscr{V}$ with $\mathscr{S}(X)$ for some extremely-disconnected compact-Hausdorff space X. If X_0 is a non-null clopen subset of X distinct from X, the set $\mathscr{V}_0$ of functions in $\mathscr{S}(X)$ that are bounded on X_0 satisfy these hypotheses. Clearly $\mathscr{V}_0 \neq \mathscr{S}(X)$. More than this, $\mathscr{V}_0$ is not linearly order isomorphic with any $\mathscr{S}(Y)$. This follows from the (easily proved) fact that $\mathscr{V}_0$ has "quasi-bounded" sequences that are not bounded in $\mathscr{V}_0$ (see Definition 5.9 and Proposition 5.10 following). Although we established in Lemma 5.6 that the image $\mathscr{V}_0$ of $\mathscr{V}$ is a sublattice of $\mathscr{S}(X)$ and each subset of $\mathscr{V}_0$ bounded above by an element of $\mathscr{V}_0$ has its least upper bound relative to $\mathscr{S}(X)$ in $\mathscr{V}_0$, we did not conclude that each subset of $\mathscr{V}_0$ bounded above by an element of $\mathscr{S}(X)$ has an upper bound *in* $\mathscr{V}_0$ (nor can we hope to conclude this, in general, in view of the example just mentioned). Thus Theorem 2.13 is not applicable. The device of quasi-bounded subsets of $\mathscr{V}_+$ allows us to identify *(a priori)* those subsets of $\mathscr{V}_+$ that will have an upper bound in $\mathscr{S}(X)$ (after the isomorphism of Lemma 5.6 is constructed). Preliminary to defining quasi-boundedness, we must introduce the notions of an "idempotent" and "support" in our vector lattices. With the function model in mind, the significance of these definitions becomes clearer.

5.7 Definition. An element e_0 of a vector lattice with semi-order unit e is an $(e-)$ *idempotent* when $ae_0 \wedge e = e_0$ for each real number a greater than 1.

5.8 Definition. The *support* $s(v)$ of a positive element v of a boundedly complete vector lattice with semi-order unit e is $\wedge\{e_0 : e_0 \text{ idempotent}, (e-e_0) \wedge v = 0\}$.

5.9 Definition. A subset $\{v_a\}$ of the positive elements of a boundedly complete vector lattice $\mathscr{V}$ with semi-order unit e is *quasi-bounded* when $\bigvee_{n=1}^{\infty} e_n = e$, where $e_n = s(ne - \bigvee_a v_a \wedge ne)$. When each such subset of $\mathscr{V}$ has an upper bound in $\mathscr{V}$ we say that the lattice is *full*.

5.10 Proposition. *Let X be an extremely disconnected compact-Hausdorff space.*

(i) *A function g in $\mathscr{S}(X)$ is an idempotent if and only if it is the characteristic function of a clopen subset of X.*

(ii) *With f in $\mathscr{S}(X)_+$, $s(f)$ is the characteristic function of the closure X_f of $\{p: p \in X, 0 < f(p)\}$; $s(f)$ is an idempotent.*

(iii) *If $0 \le f \le g$ with f and g in $\mathscr{S}(X)$, then $s(f) \le s(g)$.*

(iv) *A subset of $\mathscr{S}(X)_+$ is quasi-bounded if and only if it has an upper bound in $\mathscr{S}(X)$. In particular, $\mathscr{S}(X)$ is full.*

Proof. (i) Suppose g is defined at p and $0 \ne g(p) \ne 1$. Then $g(p) \ne ag(p)$ for some real a greater than 1. Since $(ag \wedge 1)(p) = \min\{ag(p), 1\} \ne g(p)$, $ag \wedge 1 \ne g$ for this choice of a; and g is not an idempotent. Thus, if g is an idempotent, it takes only the values 0 and 1 where it is defined; and g is the characteristic function of some clopen set.

If g is a characteristic function, then $ag \wedge 1 = g$ for each real a greater than 1; and g is an idempotent.

(ii) If g is an idempotent in $\mathscr{S}(X)$ such that $(1-g) \wedge f = 0$, then $g(p) = 1$ if $f(p) > 0$. Since $s(f)$ is the greatest lower bound in $\mathscr{S}(X)$ of all such idempotents, it is clear that $s(f)$ is the characteristic function of X_f.

(iii) If $0 \le f \le g$, then $X_f \subseteq X_g$ and $s_f \le s_g$.

(iv) If $\{f_a\}$ is a subset of $\mathscr{S}(X)_+$ bounded above by f, then $f_a \wedge n \le f \wedge n$ for each a so that $\bigvee_a f_a \wedge n \le f \wedge n$ and thus $n - f \wedge n \le n - \bigvee_a f_a \wedge n$. From (iii), $s(n - f \wedge n) \le s(n - \bigvee_a f_a \wedge n)$. Now $s(n - f \wedge n)$ is the characteristic function of the closure of the set of points p such that $f(p) < n$. Since $f \in \mathscr{S}(X)$,

$$1 = \bigvee_{n=1}^{\infty} s(n - f \wedge n) \le \bigvee_{n=1}^{\infty} s\left(n - \bigvee_a f_a \wedge n\right).$$

Hence $\{f_a\}$ is quasi-bounded.

Suppose $\{f_a\}$ is a quasi-bounded subset of $\mathscr{S}(X)_+$. If $s(n - \bigvee_a f_a \wedge n)$ is the characteristic function of X_n, then $\bigcup_{n=1}^{\infty} X_n$ is dense in X (since $\{f_a\}$ is quasi-bounded). If $n < f_{a'}(p)$ for some a' and some p in X, then $n < f_{a'}(q)$ for all q in some clopen set Y containing p. Thus $(n - \bigvee_a f_a \wedge n)(q) = 0$ for all q in Y, and $Y \cap X_n = \emptyset$ in particular, $p \notin X_n$. It follows that $f_a(p) \le n$ for each a and all p in X_n. On the other hand, if $n' < n$ and Y' is a clopen set such that $f_a(p') \le n'$ for each a and all p' in Y', then $(\bigvee_a f_a \wedge n)(p') \le n'$ and $s(n - \bigvee_a f_a \wedge n)(p') = 1$ for all p' in Y'. Thus $Y' \subseteq X_n$, in this case. As just noted, for each a and all p in X_{n-1}, $f_a(p) \le n-1$. Hence $X_{n-1} \subseteq X_n$. Let Y_n be $X_n \backslash X_{n-1}$. Define $f(p)$ to be n for p in Y_n, $n = 1, 2, \ldots$. Since $X_0 = \emptyset$, $\{Y_n: n = 1, 2, \ldots\}$ is a disjoint family of clopen sets whose union is dense in X. From Lemma 2.2, $f \in \mathscr{S}(X)$. For p in Y_n, $f_a(p) \le n = f(p)$. Thus, $f_a \le f$ for each a and f is an upper bound for $\{f_a\}$. ■

5.11 Theorem. *If $\mathscr{V}$ is a full boundedly complete archimedian ordered vector lattice with a completely distributive semi-order unit e, there is a linear order isomorphism η of $\mathscr{V}$ onto $\mathscr{S}(X)$, where X is some extremely disconnected compact-Hausdorff* *space. The mapping η carries e onto* 1 *and $\mathscr{V}_b$, the set of e-bounded elements of $\mathscr{V}$, onto $C(X,\mathbb{R})$.*

Proof. From Lemma 5.6, there is a linear order isomorphism η of $\mathscr{V}$ into $\mathscr{S}(X)$ mapping e onto 1 and $\mathscr{V}_b$ onto $C(X,\mathbf{R})$, where X is some extremely disconnected compact-Hausdorff space. In addition, if $\{v_a\}$ has a least upper bound v in $\mathscr{V}$, then $\eta(v)=\bigvee_a \eta(v_a)$. To show that the image $\mathscr{S}_0$ of η is a boundedly complete sublattice of $\mathscr{S}(X)$ in the sense of Theorem 2.13, we must show that if $\{\eta(v_a)\}$ has an upper bound in $\mathscr{S}(X)$, then $\{v_a\}$ is bounded in $\mathscr{V}$. For this, we show that $\{v_a\}$ is quasi-bounded in $\mathscr{V}$ (and then apply the assumption that $\mathscr{V}$ is full). Theorem 2.13 will apply, then, yielding the conclusion that $\mathscr{S}_0$ is $\mathscr{S}(X)$.

Note that e_0 is an idempotent in $\mathscr{V}$ if and only if $\eta(e_0)$ is an idempotent in $\mathscr{S}(X)$, for $\eta(ae_0 \wedge e)=a\eta(e_0)\wedge 1$. Thus the set of idempotents relative to $\mathscr{V}_0$ coincides with the set of idempotents in $\mathscr{S}(X)$ (since $C(X,\mathbf{R})\subseteq\mathscr{V}_0$). If v is a positive element in $\mathscr{V}_0$, we can compute its support $s_0(v)$ relative to $\mathscr{V}_0$ and its support $s(v)$ relative to $\mathscr{S}(X)$. The sets of idempotents whose greatest lower bounds (relative to $\mathscr{V}_0$ and to $\mathscr{S}(X)$) are $s_0(v)$ and $s(v)$ coincide. So do their greatest lower bounds (from the properties of η noted). Thus $s_0(v)=s(v)$. The sets of supports used to determine quasi-boundedness of $\{\eta(v_a)\}$ relative to $\mathscr{V}_0$ and $\mathscr{S}(X)$ are the same. So are the least upper bounds of this set of supports (relative to $\mathscr{V}_0$ and to $\mathscr{S}(X)$). Thus $\{\eta(v_a)\}$ is quasi-bounded relative to one of $\mathscr{V}_0$ and $\mathscr{S}(X)$ if and only if it is quasi-bounded relative to the other. By assumption, $\{\eta(v_a)\}$ is bounded relative to $\mathscr{S}(X)$, and from Proposition 5.10(iv), $\{\eta(v_a)\}$ is quasi-bounded in $\mathscr{S}(X)$. Hence $\{\eta(v_a)\}$ is quasi-bounded in $\mathscr{V}_0$, and $\{v_a\}$ is quasi-bounded in $\mathscr{V}$. ∎

In the (bounded) case where $\mathscr{V}$ has an *order* unit e, it is not necessary to assume that e is distributive to establish the linear order isomorphism of $\mathscr{V}$ with $C(X,\mathbb{R})$. (See Theorem 5.4.) The assumption that e is (finitely) distributive was used to imbed $\mathscr{V}$ as a sublattice of $\mathscr{S}(X)$ (see the proof of Lemma 5.6); although without making this assumption there is a linear order isomorphism of $\mathscr{V}$ into $\mathscr{S}(X)$ such that $\mathscr{V}_b$ maps onto $C(X,\mathbb{R})$.

5.12 Problem. Can $\mathscr{V}_0$ (in the proof of Lemma 5.6) be shown to be a sublattice of $\mathscr{S}(X)$ without the assumption that e is (finitely) distributive?

5.13 Problem. Without the assumption that e is *completely* distributive, can we conclude that the least upper bound of a subset of $\mathscr{V}_0$ that has an upper bound in $\mathscr{V}_0$ coincides with its least upper bound relative to $\mathscr{S}(X)$?

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