Representations of matricial operator algebras

R. V. Kadison

1. Introduction and preliminaries

In [10; Chapter IV], Murray and von Neumann introduce a class of factors of type II_1 which appear as the ultraweak closure $\mathcal{M}$ of the union of an ascending sequence $\{\mathfrak{A}_j\}$ of distinct C^*-algebras, where $\mathfrak{A}_j$ is * isomorphic to the algebra of all $n_j \times n_j$ complex matrices, each containing the same unit element. We call the sequence $\{\mathfrak{A}_j\}$ a *matricial nest*, their union a *matricial operator algebra*, a C^*-algebra in which such a union is norm dense a *uniformly matricial C^*-algebra*, and a von Neumann algebra in which such a union is ultraweakly dense a *matricial von Neumann algebra.* We say that $\{\mathfrak{A}_j\}$ is a *generating* (*matricial*) *nest* for its union and its various closures.

Murray and von Neumann prove that all finite matricial factors are * isomorphic [10; Theorem XII, XIV]. They call such factors 'approximately finite' (the terms 'hyperfinite' and 'approximately finite-dimensional' are also used); and in the process of proving their uniqueness result, they establish various properties of finite matricial factors. In particular, they show that if the finite factor $\mathcal{M}$ has the property that each finite set of operators $A_1, \ldots, A_n$ in $\mathcal{M}$ can be approximated to within a preassigned positive ε in trace norm by operators $B_1, \ldots, B_n$ lying in a finite type I subfactor of $\mathcal{M}$ containing the identity operator I (that is, $B_1, \ldots, B_n$ are such that $[[A_j - B_j]] < \varepsilon$ for all j in $\{1, \ldots, n\}$, where $[[A]]^2 = \tau(A^*A)$ and τ is the (unique) trace on $\mathcal{M}$ normalized so that $\tau(I) = 1$) then $\mathcal{M}$ is matricial. We shall say, in this case, that $\mathcal{M}$ has the (*trace-norm*) *finite approximation property*. More generally, we replace 'trace-norm' by other topologies and refer to this approximation property in these topologies. We apply this terminology to C^*-algebras as well.

In [5] it is proved that a properly-infinite von Neumann algebra that has the * ultrastrong finite approximation property and acts on a separable Hilbert space is matricial with a generating nest $\{\mathfrak{A}_j\}$ such that $\mathfrak{A}_j$ is isomorphic to all $2^j \times 2^j$ complex matrices. In [4], [9], [11; Corollary 5.2], and [11; §6.8], it is proved that the universal enveloping von Neumann algebras (the 'biduals') of all uniformly matricial C^*-algebras are * isomorphic; from which it follows that if a von Neumann algebra acting on a separable Hilbert space has one matricial generating nest it has all possible generating nests. Put in another way, this result states that if a uniformly matricial C^*-algebra has a separable representation on $\mathcal{H}$ then every other uniformly matricial C^*-algebra has a representation on $\mathcal{H}$ with the same commutant. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras such that when one has a (separable, cyclic, factor, etc.) representation on a Hilbert space $\mathcal{H}$ the other has a representation on $\mathcal{H}$ with the same commutant, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are (*separable-*, *cyclic-*, *factor-*, etc.) *isoreductive*. Of course, isomorphic C^*-algebras are isoreductive (in all senses); but isoreductive C^*-algebras need not be isomorphic (as the example of uniformly matricial C^*-algebras illustrates). Glimm introduced the uniformly matricial C^*-algebras (which he called *uniformly hyperfinite*) in [6]. He proves that the prime power divisors of the orders of the algebras in a matricial generating nest determine such algebras up to algebraic isomorphism. He proves, too, that a countably generated C^*-algebra with the (norm) finite approximation property is uniformly matricial. It follows from Glimm's results that, for example, the uniformly matricial algebra with nest $\{\mathfrak{A}_j\}$ where the 'order' of $\mathfrak{A}_j$ is 2^j (this is the CAR algebra) and that where the order of $\mathfrak{A}_j$ is 3^j are not isomorphic. But in a sense that one immediately feels, without its being made precise, all uniformly matricial C^*-algebras are very much alike. The isoreductivity is one important aspect of this 'sameness'.(Indeed, a tentative title for [11] was 'Similarities Between UHF Algebras', where, by 'similarities', Pedersen has in mind 'resemblances'.)

While it is not the subject proper of this article, we propose the study of isoreductivity (in various forms) as an important aspect of the general analysis of the structure of C^*-algebras. There are two broad features of this study that seem especially prominent. The first is the general nature of isoreductivity—including such questions as what forms of isoreductivity imply others or more extensive forms (for example, cyclic isoreductivity is the same as isoreductivity for representations with commutant having countably decomposable centres) and such as whether isoreductive C^*-algebras contain 'common dense approximate models'. The second feature involves establishing various types of isoreductivity for special classes of C^*-algebras. (Are the C^*-algebras generated by the regular representations of the free groups on more than one generator factor-isoreductive? finite-factor-isoreductive? cyclic-isoreductive?) It may be as sensible to develop structural results capable of telling us whether or not two C^*-algebras resemble one another as it is to try to decide whether or not they are algebraically identical (isomorphic). In this connection, compare Rieffel's closely related notion of Morita equivalence [14].

In proving the isoreductivity of uniformly matricial C^*-algebras Elliott, Maréchal and Pedersen pass through Glimm's results on type I C^*-algebras [7] and quote the Murray–von Neumann results for finite matricial factors. Elliott combines a direct quote of Glimm with the result of [5] while Pedersen reconstructs Glimm's techniques of quasi-matrix unit systems in a form suitable for his purposes.

Murray and von Neumann establish their results by what we might call the 'cutting and pasting' techniques of operator algebras (in conjunction with intricate approximations). These techniques are applied to systems of matrix units. A *system* of $n \times n$ *matrix units* is a family $\{E_{jk}\}$ of operators such that $E_{jh}E_{hk} = E_{jk}$, $E_{jh}E_{h'k} = 0$ when $h \neq h'$, and $E_{jk} = E^*_{kj}$, for j, k, h, h' in $\{1, \ldots, n\}$. If $\sum_{j=1}^{n} E_{jj} = I$, we say that $\{E_{jk}\}$ is a *unital* system of $n \times n$ matrix units (for $\mathcal{N}$, if $\{E_{jk}\}$ generates $\mathcal{N}$—so that $\mathcal{N}$ is a factor of type I_n). In general, we use 'unital' to refer to the presence of a unit in the appropriate sense. (So, for example, we refer to a *unital subfactor* when the subfactor contains I.)

By employing these time-honoured techniques of matrix unit approximation and some new strategies, we prove the (cyclic) isoreductivity of uniformly matricial C^*-algebras directly (in Section 4) as well as recapturing the Murray–von Neumann uniqueness result and their finite–approximation–property characterization of finite matricial factors by simpler arguments. In Section 2, we prove some results on finite representations of C^*-algebras with a unique tracial state (that is, a state τ such that $\tau(AB) = \tau(BA)$ for each pair of elements A and B in the algebra); and we note, from these facts, that it suffices, for the uniqueness, to show that all finite matricial von Neumann algebras contain some one uniformly matricial C^*-algebra as an ultraweakly dense subalgebra. Any matricial C^*-algebra will do, but one is extremely well-suited to the needs of our argument. It is the uniformly matricial C^*-algebra (we denote by '$\mathfrak{A}_\infty$') that has a generating nest such that every power of every prime divides some order of an algebra in the nest. This is the point where the strategy changes from that of the Murray–von Neumann argument. A good deal of their technical effort is devoted to 'trimming away the excesses' gathered during their constructions. With our (elementary) representation results in place, and passing through $\mathfrak{A}_\infty$, we need not bother with 'trimming'—on the contrary, we must engage in 'stuffing' (which is technically very easy).

The properly-infinite case is quite different. To begin with, a properly-infinite matricial von Neumann algebra need not be a factor (even if it is of type I—as the direct sum of two inequivalent representations of the *CAR* algebra illustrates).

The major breakthrough of Powers [13] shows that uniqueness is no longer valid (for type III matricial factors); though Connes's classic result [3; Theorem 7.4] establishes uniqueness for matricial factors of type II_∞.

The work of Araki–Woods [1] and Connes [3] parametrizes type III factors and establishes uniqueness for the matricial type III factors in many cases, when the additional parameter is taken into account. The work of Connes [2] employs the Tomita–Takesaki theory [15] in its parametrization of type III factors which extends the scope of the original parametrization of Araki–Woods [1]. At the same time, Connes replaces the internal characterization of finite matricial factors (the finite approximation property) by a deep and powerful 'external' characterization ('injectivity', or the existence of a 'hypertrace') [3; Remark 5.3.4, Theorem 5.1]—and, in the process, he answers virtually all of the older open question about matricial factors. Despite the important structural differences among the infinite matricial factors, the uniformly matricial C^*-algebras are factor-isoreductive—emphasizing the fact that isoreductivity tells us something about the common properties of uniformly matricial C^*-algebras rather than about the von Neumann algebras generated by the images of representations. The failure of general uniqueness in the infinite case denies us the luxury of using only $\mathfrak{A}_\infty$. Our argument, in this case, must produce an arbitrary generating nest. The strategy for this construction is described in the introduction to Section 4, and the techniques are those developed in Lemmas 4.4, 4.5, and 4.6.

In Section 2, we prove the finite factor representation results, mentioned, along with some other preparatory facts. In Section 3, we present a simplified version of Glimm's basic theorems on uniformly matricial C^*-algebras.

We are pleased to record our gratitude to the National Science Foundation (USA) for support of our work. Many of these results and the basic techniques that enter the arguments were developed during the preparation of [8]. Our thanks are due to John Ringrose for many illuminating discussions on this subject.

2. Preparatory results

If $\mathfrak{A}$ is a uniformly matricial C^*-algebra and $\{\mathfrak{A}_n\}$ is a generating nest for $\mathfrak{A}$, each $\mathfrak{A}_n$ has a unique tracial state. If $\mathfrak{A}_n \subseteq \mathfrak{A}_m$ the restriction of the tracial state of $\mathfrak{A}_m$ to $\mathfrak{A}_n$ is the (unique) tracial state of $\mathfrak{A}_n$; so that there is a unique tracial functional τ_0 of norm 1 on $\bigcup_{n=1}^{\infty} \mathfrak{A}_n (= \mathfrak{A}_0)$. Thus τ_0 has a unique bounded extension τ to $\mathfrak{A}$, $\tau(I) = 1 = \|\tau\|$; and τ is a tracial state of $\mathfrak{A}$. If τ' is another tracial state of $\mathfrak{A}$, the restriction of τ' to each $\mathfrak{A}_n$ is the unique tracial state of $\mathfrak{A}_n$. Thus τ and τ' agree on $\mathfrak{A}_0$, and, by norm continuity of $\tau - \tau'$ on $\mathfrak{A}$, $\tau = \tau'$. Each uniformly matricial C^*-algebra has a unique tracial state. Concerning C^*-algebras that admit a unique tracial state, we prove some results that will be of use to us in establishing the (algebraic) uniqueness of the finite, matricial von Neumann algebra.

Proposition 2.1. *If the C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}$ admits at most one tracial state and the weak-operator closure, $\mathfrak{A}^-$, of $\mathfrak{A}$ is a finite von Neumann algebra, then $\mathfrak{A}^-$ is a factor.*

Proof. If τ is the normalized, centre-valued trace on $\mathfrak{A}^-$, P is a non-zero central projection in $\mathfrak{A}^-$, x is a unit vector in the range of P, and y is a unit vector in $\mathcal{H}$, then $\omega_x \circ \tau$ and $\omega_y \circ \tau$ restrict to the unique trace on $\mathfrak{A}$. From the ultraweak continuity of τ on $\mathfrak{A}^-$, $\omega_x \circ \tau$ and $\omega_y \circ \tau$ agree on $\mathfrak{A}^-$. Thus $\langle Py, y\rangle = \langle \tau(P)y, y\rangle = (\omega_y \circ \tau)(P) = (\omega_x \circ \tau)(P) = \langle Px, x\rangle = 1$. Thus $P = I$, and $\mathfrak{A}^-$ is a factor. □

In the theorem that follows, we speak of 'finite representations' of a C^*-algebra $\mathfrak{A}$. These are the representations φ of $\mathfrak{A}$ such that $\varphi(\mathfrak{A})^-$ is a finite von Neumann algebra.

Theorem 2.2. *If the C^*-algebra $\mathfrak{A}$ admits at most one trace then all finite representations of $\mathfrak{A}$ are quasi-equivalent.*

Proof. If φ is a finite representation of $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$, then $\mathfrak{A}$ admits a trace. Let φ_0 be the (cyclic) *GNS* representation of $\mathfrak{A}$ on $\mathcal{H}_0$ corresponding to this trace; and let ψ be the direct sum of φ_0 and φ. Let E_0' be the projection of $\mathcal{H}_0 \oplus \mathcal{H}$ onto $\mathcal{H}_0$; and let x_0 be a unit vector in $\mathcal{H}_0$ such that $[\psi(\mathfrak{A})x_0] = \mathcal{H}_0$ and $\langle ABx_0, x_0\rangle = \langle BAx_0, x_0\rangle$ for all A and B in $\psi(\mathfrak{A})^-$. If P is a central projection in $\psi(\mathfrak{A})^- E_0'$ and P is equivalent to a subprojection E in $\psi(\mathfrak{A})^- E_0'$, then $(P-E)x_0 = 0$; so that $(P-E)E_0 = 0$, where E_0 is the projection in $\psi(\mathfrak{A})^- E_0'$ with range $[E_0'\psi(\mathfrak{A})'E_0'x_0]$. But then $(P-E)$ is orthogonal to the central support, E_0', of E_0 relative to $\psi(\mathfrak{A})^- E_0'$; and $E = P$. Hence $\psi(\mathfrak{A})^- E_0'$ and $\psi(\mathfrak{A})^- C_{E_0'}$ are finite von Neumann algebras (where $C_{E_0'}$ is the central support of E_0' relative to $\psi(\mathfrak{A})'$).

By assumption $\psi(\mathfrak{A})^-(I - E_0')(\cong \varphi(\mathfrak{A})^-)$ and, hence, $\psi(\mathfrak{A})^- C_{I-E_0'}$ are finite. Thus $\psi(\mathfrak{A})^-$ is finite. Since $\psi(\mathfrak{A})$ admits at most one trace; $\psi(\mathfrak{A})^-$ is a factor, from Proposition 2.1. Hence $A \to AE_0'$ and $A \to A(I - E_0')$ are isomorphisms of $\psi(\mathfrak{A})^-$ onto $\psi(\mathfrak{A})^- E_0'(\cong \varphi_0(\mathfrak{A})^-)$ and $\omega(\mathfrak{A})^-(I - E_0')$ $(\cong \varphi(\mathfrak{A})^-)$, respectively; so that φ and φ_0 are quasi-equivalent. □

Corollary 2.3. *All finite representations of a uniformly matricial C^*-algebra are quasi-equivalent.*

Corollary 2.4. *If two finite von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ have ultraweakly-dense, * isomorphic, matricial subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ then $\mathcal{R}_1$ and $\mathcal{R}_2$ are * isomorphic.*

Proof. The identity representation and the * isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ are two finite representations of the uniformly matricial C^*-algebra $\mathfrak{A}_1$. From the preceding corollary, these representations are quasi-equivalent. Thus the given * isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ extends to a * isomorphism of $\mathcal{R}_1$ onto $\mathcal{R}_2$. □

Corollary 2.5. *Each uniformly matricial C^*-algebra $\mathfrak{A}$ has a (faithful) representation as an ultraweakly dense C^*-subalgebra of a factor of type* II_1.

Proof. From the proof of Theorem 2.2, we see that the *GNS* representation corresponding to the (unique) tracial state of $\mathfrak{A}$ is a (faithful) representation of $\mathfrak{A}$ as an ultraweakly dense C^*-subalgebra of a factor of type II_1. □

Lemma 2.6. *A matricial von Neumann algebra, $\mathcal{R}$, has no non-zero, finite-dimensional, central summands; and each unital subfactor $\mathcal{N}_0$ of $\mathcal{R}$ of type I_n is contained in a unital subfactor $\mathcal{N}_1$ of $\mathcal{R}$ of type I_{nm}, where m is a preassigned positive integer.*

Proof. Let Q be a non-zero central projection in $\mathcal{R}$ and $\mathcal{M}$ be a unital subfactor of $\mathcal{R}$ of type I_k. Since $\mathcal{M}$ is simple and contains I, the mapping, $T \to TQ$, is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}Q$ in $\mathcal{R}Q$. Hence $\mathcal{R}Q$ has (linear) dimension at least k^2. There are unital subfactors $\mathcal{M}$ of $\mathcal{R}$ of type I_k with k arbitrarily large, since $\mathcal{R}$ is matricial. Hence $\mathcal{R}Q$ is not finite dimensional.

It follows that there is a unital subfactor $\mathcal{N}$ of $\mathcal{R}$ of type I_k with k any preassigned positive integer. Choose k to be nm; and let $\{E_{jk}\}$ and $\{F_{jk}\}$ be unital systems of $n \times n$ matrix units in $\mathcal{N}$ and for $\mathcal{N}_0$, respectively. If V is a partial isometry in $\mathcal{R}$ such that $V^*V = E_{11}$ and $VV^* = F_{11}$, then $\sum_{j=1}^{n} F_{j1}VE_{1j}$ is a unitary operator U in $\mathcal{R}$; and $UE_{jk}U^* = F_{jk}$. Thus $U\mathcal{N}U^*(= \mathcal{N}_1)$ is a unital subfactor of $\mathcal{R}$ of type I_{nm} containing $\mathcal{N}_0$. □

Lemma 2.7. *A matricial von Neumann algebra, $\mathcal{R}$, with countably-decomposable centre $\mathcal{C}$ is * isomorphic to a cyclic, matricial von Neumann algebra acting on a separable Hilbert space.*

Proof. Suppose $\mathcal{R}$ acts on $\mathcal{H}$. Since $\mathcal{C}$ is countably decomposable; there is a cyclic projection E' in $\mathcal{R}'$ with central support I (see [8; Proposition 5.5.16]). The mapping $T \to TE'$ is a * isomorphism of $\mathcal{R}$ onto $\mathcal{R}E'$ acting on $E'(\mathcal{H})$ $(= \mathcal{H}_0)$. Since $\mathcal{R}$ is matricial, $\mathcal{R}E'$ is matricial, hence, countably generated. As $\mathcal{R}E'$ has a generating vector and a countable generating family, $\mathcal{H}_0$ is separable. □

3. The uniformly matricial C^*-algebras

In this section, we prove Glimm's theorem classifying uniformly matricial C^*-algebras up to * isomorphism. Let $\mathfrak{A}$ be a matricial C^*-algebra and $\{\mathfrak{A}_n\}$ be a generating nest for $\mathfrak{A}$, where $\mathfrak{A}_n$ is a (finite) unital factor of type I_{r_n}. If $q_1^{k_1} \cdots q_m^{k_m}$ is the prime factorization of r_n then (from elementary Wedderburn theory) $\mathfrak{A}_n$ is generated by a family of commuting unital factors, k_j of which are of type I_{q_j} ($j \in \{1, \ldots, m\}$). If we apply this form of decomposition, successively, to $\mathfrak{A}_1$, to $\mathfrak{A}_1' \cap \mathfrak{A}_2$, to $\mathfrak{A}_2' \cap \mathfrak{A}_3, \ldots$, we construct a commuting family of type I subfactors of prime order that generates $\mathfrak{A}$ as a C^*-algebra. These prime order subfactors can be grouped together to form a generating nest whose members are unital factors of types other than $\{r_n\}$; but a 'prime factorization' of the new nest will yield the same primes as before and the same number of unital subfactors of that prime order. (That number may well be infinity.)

If $\mathfrak{A}$ and $\mathfrak{B}$ are uniformly matricial C^*-algebras with generating nests that yield the same prime factorization, subfactors of the same prime order can be put in correspondence (giving rise to a one-to-one correspondence between the total families) and a * isomorphism constructed between corresponding subfactors. Since finite, unital subfactors that commute are (tensor-product) independent, there is a * isomorphism between the subfactors generated by corresponding finite families in each of $\mathfrak{A}$ and $\mathfrak{B}$ that restricts to the given isomorphism on each factor in the family. In this way, we construct generating nests, $\{\mathfrak{A}_n\}$ and $\{\mathfrak{B}_n\}$ for $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and * isomorphisms φ_n (of $\mathfrak{A}_n$ onto $\mathfrak{B}_n$) that are extensions of one another. Since each φ_n is an isometry, the * isomorphism they define of $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ onto $\bigcup_{n=1}^{\infty} \mathfrak{B}_n$ is an isometry, and has a unique isometric extension mapping $\mathfrak{A}$ onto $\mathfrak{B}$ that is a * isomorphism.

From the discussion to this point, we see that the specific numbers r_n appearing as the types of a particular generating nest for $\mathfrak{A}$ are not significant; but that the primes appearing in the factorizations of the numbers r_n, *may* be significant in describing the algebraic structure of $\mathfrak{A}$. We have seen that if $\mathfrak{A}$ and $\mathfrak{B}$ have generating nests with the same prime factorization then $\mathfrak{A}$ and $\mathfrak{B}$ are * isomorphic. Nothing we have seen thus far rules out the possibility that all uniformly matricial C^*-algebras are * isomorphic.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are * isomorphic, uniformly matricial C^*-algebras with generating nests $\{\mathfrak{A}_n\}$ and $\{\mathfrak{B}_n\}$, respectively; and suppose that r divides the order of some $\mathfrak{B}_n$ but of no $\mathfrak{A}_n$. Then $\mathfrak{B}$ contains a unital subfactor of type I_r; and, via the * isomorphism, $\mathfrak{A}$ contains such a subfactor. In the theorem that follows, we show that $\mathfrak{A}$ cannot have a unital subfactor of type I_r when r divides the order of no member of a given generating nest for $\mathfrak{A}$. Glimm's theorem follows directly from this result and our preceding discussion.

Theorem 3.1. *If $\{\mathfrak{A}_n\}$ is a generating nest for a uniformly matricial C^*-algebra $\mathfrak{A}$ and $\mathcal{N}$ is a unital subfactor of $\mathfrak{A}$ of type I_r, then some $\mathfrak{A}_n$ is of type I_{kr}.*

Proof. Let E be a minimal projection in $\mathcal{N}$. Since $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is norm dense in $\mathfrak{A}$; there is an A in some $\mathfrak{A}_n$ such that $\|A-E\|$ is small. Thus $\|A\|$ is near 1; and, replacing A by $\|A\|^{-1}A$, we may assume that $\|A\|=1$. Replacing A by $\frac{1}{2}(A^*+A)$, we may assume that A is self-adjoint. Since $E-A^2=E(E-A)+(E-A)A$; we may assume that A is a positive operator in the unit ball of $\mathfrak{A}_n$. For such an A, $\|A-F\|<\frac{1}{2}$, where F is the spectral projection for A corresponding to $[\frac{1}{2},1]$; so that $\|E-F\|<1$ (and F is a projection in $\mathfrak{A}_n$). Thus $(I-E)\wedge F$ and $(I-F)\wedge E$ are 0. (If there were a unit vector in the range of either, then $\|E-F\|$ would be at least 1.) Hence $1-\tau(E)+\tau(F)=\tau(I-E)+\tau(F)=\tau[(I-E)\wedge F]+\tau[(I-E)\vee F]=\tau[(I-E)\vee F]\leq 1$, where τ is the (unique) normalized tracial state on $\mathfrak{A}$. Thus $\tau(F)\leq\tau(E)$; and, by symmetry, $\tau(E)\leq\tau(F)$. Hence $\tau(E)=\tau(F)$. Now $k/m=\tau(F)=\tau(E)=1/r$, where F is the sum of k minimal projections in $\mathfrak{A}_n$ (of type I_m) and E is a minimal projection in a unital subfactor of type I_r. It follows that $m=rk$. □

It follows from this theorem that if $\mathfrak{A}_0$ is, for example, the uniformly matricial C^*-algebra whose 'prime factorization' contains only unital subfactors of type I_2 (this is the *CAR* algebra) then $\mathfrak{A}_0$ contains no unital subfactor of type I_3. The information contained in the prime factorization of some generating nest for a uniformly matricial C^*-algebra $\mathfrak{A}$ can be assembled in the expression, $2^{n_1}3^{n_2}\ldots$, which we call the *supernatural number* of $\mathfrak{A}$. The exponent n_j of the jth prime p_j in the supernatural number of $\mathfrak{A}$ indicates the number of subfactors of type I_{p_j} appearing in the prime factorization of $\mathfrak{A}$ derived from some generating nest for $\mathfrak{A}$ (as described above). With this terminology, Glimm's result has the following form.

Theorem 3.2. *To each supernatural number there corresponds a uniformly matricial C^*-algebra. Two such algebras are * isomorphic if and only if they have the same supernatural number.*

Proof. The second assertion is a consequence of Theorem 3.1 and the discussion preceding it. The first assertion can be proved by using the argument of the last paragraph of the proof of Lemma 2.6, where $\mathcal{R}$ is replaced by $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ is infinite dimensional. □

We conclude this section by stating Glimm's criterion for a C^*-algebra to be uniformly matricial. (Detailed proofs of lemmas entirely analogous to the norm approximations needed for this result will be given in the next section for the more difficult case of strong *-operator approximation.)

Theorem 3.3. *If the (unital) C^*-algebra $\mathfrak{A}$ is countably generated and if each finite set of operators in $\mathfrak{A}$ can be approximated in norm to a preassigned degree of accuracy by operators in a unital, finite, type* I *subfactor of $\mathfrak{A}$, then $\mathfrak{A}$ is uniformly matricial.*

4. The matricial von Neumann algebras

In this section, we establish our basic results on isoreductivity (Theorem 4.10 and Corollary 4.11) and complete our proof of the Murray–von Neumann uniqueness of finite matricial algebras (Theorem 4.9). The technical lemmas needed for these results involve matrix unit approximations in von Neumann algebras. In the case of matrix approximations in a finite matricial factor, Murray and von Neumann make use of the trace norm, which, after minor rearrangement, amounts to approximation on a single vector—a trace vector. In the case of a general von Neumann algebra, a trace and trace norm are not available; so that more care and some special devices are necessary. Our approximations are performed on prescribed finite sets of vectors; and we must not allow operators chosen at later stages of the arguments to act on vectors yielding vectors then needed in further approximations. We will usually be given a system of $n \times n$ matrix units (or a part thereof) at the outset; and, to avoid the difficulty just mentioned, we will want to be able to apply these matrix units to the vectors on which the approximations are made. We introduce a technical device that permits us to do this.

A set $\mathcal{S}$ of vectors in the unit ball $(\mathcal{H})_1$ of a Hilbert space $\mathcal{H}$ will be said to be *closed* with respect to a partial isometry V on $\mathcal{H}$ whose initial projection E is orthogonal to its final projection F when Ex, Fx, Vx, V^*x, $(I-E)x$, $(I-F)x$, and $(I-E-F)x$ are in $\mathcal{S}$ for each x in $\mathcal{S}$. For each x in $\mathcal{H}$, the seven vectors, just listed, together with 0 form a closed set with respect to V as does this same set with x adjoined.

If y in $\mathcal{H}$ is such that $Ey = Fy = 0$, then $\{0, y\}$ is closed with respect to V. The union and intersection of sets closed with respect to V are closed with respect to V. If $\{E_{jk}\}$ is a system of $n \times n$ matrix units then, for each x in $\mathcal{H}$, $\{0, x, E_{jk}x, (I-E)x : j, k = 1, \ldots, n;\ E = E_{j_1j_1} + \cdots + E_{j_mj_m},\ 1 \le j_1 < j_2 < \cdots < j_m \le n\}$ is the minimal set containing x and closed with respect to all E_{jk}, $j \ne k$. Thus each finite set of vectors in $(\mathcal{H})_1$ is contained in a finite set closed with respect to all E_{jk}, $j \ne k$.

In loose terms, the programme that follows is aimed at showing that each unital subfactor $\mathcal{N}_0$ of type I_n of a matricial von Neumann algebra $\mathcal{R}$ is contained in another unital subfactor $\mathcal{N}$ of type I_{nr}, where r is a preassigned positive integer, and that there are operators in the unit ball of $\mathcal{N}$ that approximate a given finite set of operators in the unit ball of $\mathcal{R}$ on a given finite set of vectors to a preassigned degree of accuracy. These approximations performed successively, with greater and greater accuracy, assure us that a generating nest for a uniformly matricial C^*-algebra $\mathfrak{A}$ of $\mathcal{R}$ can be constructed that has strong-operator closure containing a (countable) generating family for $\mathcal{R}$; so that $\mathfrak{A}^- = \mathcal{R}$. The ability to introduce a given integer r as a divisor of the type at each stage, allows us to construct $\mathfrak{A}$, $*$ isomorphic to a given uniformly matricial C^*-algebra. In this form, the programme is most easily carried out when $\mathcal{R}$ is properly infinite. When $\mathcal{R}$ is finite, it will suffice, from Corollaries 2.4 and 2.5, to construct some one given $\mathfrak{A}$ as a dense subalgebra. In the finite case, constructing $\mathcal{N}$ to have type I_{nr} (and with the other properties described above) for a specified integer r is a more strenuous process; but it is relatively easy to make some (unspecified) multiple of r a divisor of the type of $\mathcal{N}$. Clearly, then, our goal in the finite case should be to construct $\mathfrak{A}_\infty$ (described in Section 1) as a dense subalgebra. It is in this respect that our argument differs strategically from the Murray–von Neumann argument—and simplification occurs. We use the freedom afforded by basic representation results applied to the special situation of uniformly matricial C^*-algebras to indulge the 'excesses' permitted by constructing $\mathfrak{A}_\infty$ rather than attempting the rigidly restrictive construction of a generating nest of unital, finite, type I factors each of whose types is specified. This last is what the Murray–von Neumann proof does (and is required for the properly-infinite case, where additional 'space' is available to allow this construction in simplified form).

In the first lemma, we show that if a 2×2 matrix unit system is approximable (in the strong $*$-operator topology) by operators in the unit ball of a finite type I subfactor then an approximation can be made by a 2×2 matrix unit system in the subfactor. Using this, we show the same, in the next lemma, for unital $n\times n$ matrix systems (with the resulting approximation dependent on n). The next three lemmas show that nearby matrix unit systems can be 'rotated' onto one another by a unitary operator in $\mathcal{R}$ near I. (The first of the three deals with the finite and the next two with the properly-infinite cases.) The two lemmas that follow these construct $\mathcal{N}$ (as described) in the properly-infinite and finite cases, respectively.

Lemma 4.1. *If V is a partial isometry in the von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$, $V^*V = E$, $VV^* = F$, $EF = 0$, $\mathcal{N}$ is a subfactor of $\mathcal{R}$ of type I_n, $\mathcal{S}$ is a closed set of vectors with respect to V, and*

A, B, T are operators in the unit ball of $\mathcal{N}$ such that $\|(A-E)x\| < b(<10^{-11})$, $\|(A^*-E)x\| < b$, $\|(B-F)x\| < b$, $\|(B^*-F)x\| < b$, $\|(T-V)x\| < b$, $\|(T^*-V^*)x\| < b$ for each x in $\mathcal{S}$, then there are orthogonal projections M and N and a partial isometry W in $\mathcal{N}$ such that

$$W^*W = M, \qquad WW^* = N, \qquad \|(M-E)x\| < 4b^{1/32},$$

$$\|(N-F)x\| < 4b^{1/32}, \qquad \|(W-V)x\| < 2b^{1/32}, \qquad \|(W^*-V^*)x\| < 2b^{1/32}$$

for all x in $\mathcal{S}$. If A is a projection, M can be chosen so that $M \le A$.

Proof. Replacing A by $\frac{1}{2}(A+A^*)$ and B by $\frac{1}{2}(B+B^*)$, we may assume that A and B are self-adjoint. With x in $\mathcal{S}$, we have

$$\|(A-A^2)x\| \le \|A(E-A)x\| + \|(A-E)(I-E)x\| < 2b.$$

Since A is a self-adjoint operator in the unit ball of $\mathcal{N}$; $A = \sum_{j=1}^m \lambda_j G_j$, where $|\lambda_j| \le 1$ and $\{G_j\}$ is an orthogonal family of projections in $\mathcal{N}$. If

$$d = b^{1/4}, \qquad X_0 = \{j : \lambda_j \notin [-d, d] \cup [1-d, d]\}$$

and $j \in X_0$; then $d^2 < |\lambda_j| \cdot |1-\lambda_j|$. Thus

$$d^4 \sum_{j\in X_0} \|G_j x\|^2 \le \sum_{j\in X_0} (\lambda_j - \lambda_j^2)^2 \|G_j x\|^2 = \sum_{j\in X_0} \|(\lambda_j - \lambda_j^2) G_j x\|^2$$
$$\le \sum_{j=1}^m \|(\lambda_j - \lambda_j^2) G_j x\|^2 = \|(A-A^2)x\|^2 < 4b^2.$$

If $-d \le \lambda_j \le d$ then $|\lambda_j| \le d = b^{1/4} < 10^{-1}$ so that $9|\lambda_j|/10 \le |\lambda_j| - |\lambda_j|^2 \le |\lambda_j - \lambda_j^2|$. Similarly, if $1-d \le \lambda_j \le 1$ then $|1-\lambda_j| = 1-\lambda_j \le d < 10^{-1}$ so that $9|1-\lambda_j|/10 \le 1-\lambda_j-(1-\lambda_j)^2 = |\lambda_j - \lambda_j^2|$. Let M_0 be the sum of those G_j's for which $1-d \le \lambda_j$. Then

$$\|(A-M_0)x\|^2 \le \left(\frac{10}{9}\right)^2 \sum_{j\notin X_0} \|(\lambda_j-\lambda_j^2)G_j x\|^2 + \sum_{j\in X_0} \|G_j x\|^2$$
$$\le \left(\frac{10}{9}\right)^2 \|(A-A^2)x\|^2 + 4b < 5b^2 + 4b$$

and (since $b < 10^{-1}$)

$$\|(M_0-E)x\| \le (5b^2+4b)^{1/2} + b < 3b^{1/2}.$$

If A is a projection, the preceding construction yields A as M_0. Now $\|EBx\| = \|E(B-F)x\| < b$; and $\|BEx\| = \|(B-F)Ex\| < b$; so that

$$\|BM_0x\| \le \|B(M_0-E)x\| + \|BEx\| < 3b^{1/2} + b$$

and

$$\begin{aligned}\|M_0Bx\| &\le \|(M_0-E)Bx\|+\|EBx\|\\ &\le \|(M_0-E)Fx\|+\|(M_0-E)(B-F)x\|+\|EBx\|\\ &< 3b^{1/2}+3b.\end{aligned}$$

Thus (since $b<10^{-2}$)

$$\|[B-(I-M_0)B(I-M_0)]x\| \le \|M_0Bx\|+2\,\|BM_0x\| < 9b^{1/2}+5b$$

and

$$\|[(I-M_0)B(I-M_0)-F]x\| < 10b^{1/2}(<10^{-4}).$$

Applying the previous argument with $(I-M_0)B(I-M_0)$ and F in place of A and E, we can construct a projection N_0 in $(I-M_0)\mathcal{N}(I-M_0)$ such that

$$\|(N_0-F)x\| < [5(10b^{1/2})^2+4(10b^{1/2})]^{1/2}+10b^{1/2} < 7b^{1/4}$$

(for $10b^{1/2}<\frac{1}{2}b^{1/4}$ and $[5(10b^{1/2})^2+4(10b^{1/2})]^{1/2}<\frac{13}{2}b^{1/4}$, since $b<10^{-6}$).

If $N_0TM_0=S=W(S^*S)^{1/2}$, where W is a partial isometry in $\mathcal{N}$ such that $W^*W=M\le M_0$ and $WW^*=N\le N_0$, then

$$\begin{aligned}\|(S-V)x\| = \|(N_0TM_0-FVE)x\| &\le \|N_0(TM_0-VE)x\|+\|(N_0-F)VEx\|\\ &\le \|T(M_0-E)x\|+\|(T-V)Ex\|+\|(N_0-F)Vx\|\\ &\le 3b^{1/2}+b+7b^{1/4}<8b^{1/4};\end{aligned}$$

and $\|(S^*-V^*)x\|<8b^{1/4}$. Thus

$$\begin{aligned}\|(S^*S-V^*V)x\| &= \|(S^*S-E)x\|\\ &\le \|S^*(S-V)x\|+\|(S^*-V^*)Vx\|<16b^{1/4};\end{aligned}$$

and $\|(SS^*-F)x\|<16b^{1/4}$. Hence

$$\|[S^*S-(S^*S)^2]x\|<32b^{1/4} \quad \text{and} \quad \|[SS^*-(SS^*)^2]x\|<32b^{1/4}.$$

Arguing, now, as at the beginning of this proof, with S^*S in place of A, let Y_0 be $\{j:\mu_j\notin[0,d_0]\cup[1-d_0,1]\}$, where $S^*S=\sum_{j=1}^{h}\mu_jG'_j$, $\{G'_j\}$ is an orthogonal family of projections in $\mathcal{N}$, and $d_0=(16b^{1/4})^{1/4}$. If $j\in Y_0$, then $d_0^2<|\mu_j|\cdot|1-\mu_j|$; so that $\sum_{j\in Y_0}\|G'_jx\|^2<64b^{1/4}$. Thus $\|[S^*S)^{1/2}-S^*S]x\|^2=\sum_{j\notin Y_0}(\mu_j^{1/2}-\mu_j)^2\cdot\|G'_jx\|^2+\sum_{j\in Y_0}(\mu_j^{1/2}-\mu_j)^2\,\|G'_jx\|^2\le d_0\|x\|^2+64b^{1/4}<3b^{1/16}$ (since $b<10^{-11}$); and $\|[(SS^*)^{1/2}-SS^*]x\|^2<3b^{1/16}$. Hence

$$\begin{aligned}\|(W-V)x\| &\le \|(W-WM_0)x\|+\|W(M_0-E)x\|+\|W(E-(S^*S)^{1/2})x\|\\ &\quad+\|(S-V)x\|<3b^{1/2}+16b^{1/4}+3^{1/2}b^{1/32}+8b^{1/4}<2b^{1/32}\end{aligned}$$

(since $b<10^{-7}$); and $\|(W^*-V^*)x\|<31b^{1/4}+3^{1/2}b^{1/32}<2b^{1/32}$ (since $b<10^{-7}$). It follows that

$$\|(W^*W-V^*V)x\|=\|(M-E)x\|<4b^{1/32},$$

and $\|(N-F)x\|<4b^{1/32}$. □

***Lemma* 4.2.** *For each positive integer n there is a positive integer m_n and a positive function $m \to f(n, m)$ on the positive integers such that $f(n, m) \to 0$ as $m \to \infty$ and such that if $\mathcal{R}$ is a von Neumann algebra, $\{E_{jk}\}$ is a unital system of $n \times n$ matrix units in $\mathcal{R}$, $\mathcal{N}$ is a unital subfactor of $\mathcal{R}$ of type I_n, $\mathcal{S}$ is a closed set of vectors with respect to all E_{jk}, $j \neq k$, and $\{A_{jk}\}$ is a family of operators in the unit ball of $\mathcal{N}$ such that $A_{jk}^* = A_{kj}$, $\|(E_{jk} - A_{jk})x\| < m^{-1}$ for each x in $\mathcal{S}$, then there is a unital system of $n \times n$ matrix units $\{F_{jk}\}$ in $\mathcal{N}$ such that $\|(E_{jk} - F_{jk})x\| < f(n, m)$ when $m_n < m$, for each x in $\mathcal{S}$.*

Proof. From Lemma 4.1, there are mutually orthogonal projections M_{11}^1, M_{22}^1, and a partial isometry M_{21}^1 in $\mathcal{N}$ such that $M_{21}^1M_{12}^1=M_{22}^1$, $M_{12}^1M_{21}^1=M_{11}^1$, where $M_{12}^1=M_{21}^*$, $\|(M_{11}^1-E_{11})x\|<4m^{-1/32}$, $\|(M_{22}^1-E_{22})x\|<4m^{-1/32}$, $\|(M_{21}^1-E_{21})x\|<2m^{-1/32}$, $\|(M_{12}^1-E_{12})x\|<2m^{-1/32}$, for each x in $\mathcal{S}$, provided $m^{-1}<10^{-11}$. For j, k in $\{2, 3\}$,

$$\begin{aligned}&\|[E_{jk}-(I-M_{11}^1)A_{jk}(I-M_{11}^1)]x\|\\&\quad=\|[(I-E_{11})E_{jk}(I-E_{11})-(I-M_{11}^1)A_{jk}(I-M_{11}^1)]x\|<8m^{-1/32}+m^{-1}.\end{aligned}$$

We can apply Lemma 4.1, again, to the algebra $(I-M_{11}^1)\mathcal{N}(I-M_{11}^1)$ with E_{22}, E_{33}, E_{32} in place of E, F, V of that lemma, M_{22}^1, $(I-M_{11}^1)A_{33}(I-M_{11}^1)$, $(I-M_{11}^1)A_{32}(I-M_{11}^1)$ in place of A, B, T and $8m^{-1/32}+m^{-1}$ in place of b. This application yields orthogonal projections M_{22}^2, M_{33}^2 and a partial isometry M_{32}^2 (all in $(I-M_{11}^1)\mathcal{N}(I-M_{11}^1)$) such that $M_{32}^2M_{23}^2=M_{33}^2$ (where $M_{23}^2=M_{32}^{2*}$), $M_{23}^2M_{32}^2=M_{22}^2\le M_{22}^1$ (since A is the projection M_{22}^1, in the present case) and such that

$$\|(E_{jh}-M_{jh}^2)x\|<4(8m^{-1/32}+m^{-1})^{1/32}\qquad (j,\ h=2,3)$$

for x in $\mathcal{S}$. Let M_{12}^2 be $M_{12}^1M_{22}^2$, M_{21}^2 be $M_{12}^{2*}(=M_{22}^2M_{21}^1)$, and M_{11}^2 be $M_{12}^2M_{21}^2$. Then, since

$$\begin{aligned}\|(M_{22}^1-M_{22}^2)x\|&\le\|(M_{22}^1-E_{22})x\|+\|(E_{22}-M_{22}^2)x\|\\&<4(8m^{-1/32}+m^{-1})^{1/32}+4m^{-1/32},\end{aligned}$$

we have

$$\begin{aligned}\|(M_{12}^2-E_{12})x\|&\le\|(M_{12}^1M_{22}^2-M_{12}^1M_{22}^1)x\|+\|(M_{12}^1M_{22}^1-M_{12}^1E_{22})x\|\\&\quad+\|(M_{12}^1E_{22}-E_{12}E_{22})x\|<5(8m^{-1/32}+m^{-1})^{1/32},\\ \|(M_{21}^2-E_{21})x\|&\le\|(M_{22}^2M_{21}^1-M_{22}^2E_{21})x\|+\|(M_{22}^2E_{21}-M_{22}^1E_{21})x\|\\&\quad+\|(M_{22}^1E_{21}-E_{22}E_{21})x\|<5(8m^{-1/32}+m^{-1})^{1/32}\end{aligned}$$

(provided m_n is suitably large). Thus

$$\|(M_{11}^2 - E_{11})x\| = \|(M_{12}^2 M_{21}^2 - E_{12}E_{21})x\| < 10(8m^{-1/32} + m^{-1})^{1/32};$$

so that, for each x in $\mathcal{S}$,

$$\|(M_{jh}^2 - E_{jh})x\| < 10(8m^{-1/32} + m^{-1})^{1/32} \qquad (j, h = 1, 2, 3, |j-h| \le 1).$$

Continuing in this way (at the next stage, we construct M_{33}^3, M_{44}^3 and M_{43}^3 such that $M_{43}^3 M_{34}^3 = M_{44}^3$ and $M_{34}^3 M_{43}^3 = M_{33}^3 \le M_{33}^2$; and then we replace M_{23}^2 by $M_{23}^2 M_{33}^3$ labelled as M_{23}^3, then M_{22}^2 by $M_{23}^3 M_{32}^3$, and so forth), we construct mutually orthogonal projections $M_{11}, \ldots, M_{nn}$ and partial isometries M_{j+1j} in $\mathcal{N}$ such that $M_{j+1j}M_{jj+1} = M_{j+1j+1}$, $M_{jj+1}M_{j+1j} = M_{jj}$, where $M_{jj+1} = M_{j+1j}^*$, and $\|(E_{jj} - M_{jj})x\| < f_0(n, m)$, $\|(E_{j+1j} - M_{j+1j})x\| < f_0(n, m)$, where, for each n, $f_0(n, m)$ tends to 0 as m tends to infinity. To construct M_{33}^3, M_{44}^3 and M_{43}^3, m^{-1} must have been chosen so small, at the outset, that $41(8m^{-1/32} + m^{-1})^{1/32} < 10^{-11}$ (so that Lemma 4.1 applies). If $1 \le k < j \le n$, define M_{jk} to be $M_{jj-1} \cdots M_{k+1k}$ and M_{kj} to be M_{jk}^*. Then $\{M_{jk}\}$ is a self-adjoint system of $n \times n$ matrix units in $\mathcal{N}$, and, for each x in $\mathcal{S}$,

$$\begin{aligned}
&\|(E_{jk} - M_{jk})x\| \\
&= \|(E_{jj-1} \cdots E_{k+1k} - M_{jj-1} \cdots M_{k+1k})x\| \\
&\le \|(E_{jj-1} \cdots E_{k+1k} - M_{jj-1}E_{j-1j-2} \cdots E_{k+1k})x\| \\
&\quad + \cdots + \|(M_{jj-1} \cdots M_{k+2k+1}E_{k+1k} - M_{jj-1} \cdots M_{k+1k})x\| \\
&< nf_0(n, m).
\end{aligned}$$

Now

$$\left\|\left(I - \sum_{j=1}^{n} M_{jj}\right)x\right\| \le \sum_{j=1}^{n} \|(E_{jj} - M_{jj})x\| < nf_0(n, m).$$

Since $M_{11}, \ldots, M_{nn}$ are orthogonal equivalent projections in $\mathcal{N}$ and $\mathcal{N}$ is of type I_{nr}; $I - \sum_{j=1}^{n} M_{jj}$ is the sum of n orthogonal equivalent projections $N_{11}, \ldots, N_{nn}$ in $\mathcal{N}$. Let $\{N_{rs}\}$ be a self-adjoint system of $n \times n$ matrix units in $\mathcal{N}$ (formed on $\{N_{jj}\}$); and let F_{jk} be $M_{jk} + N_{jk}$. Then $\{F_{jk}\}$ is a unital system of $n \times n$ matrix units in $\mathcal{N}$. Since

$$\|N_{jk}x\| = \left\|N_{jk}\left(I - \sum_{h=1}^{n} M_{hh}\right)x\right\| \le \left\|\left(I - \sum_{h=1}^{n} M_{hh}\right)x\right\| < nf_0(n, m);$$

we have, for each x in $\mathcal{S}$,

$$\|(E_{jk} - F_{jk})x\| \le \|(E_{jk} - M_{jk})x\| + \|N_{jk}x\| < 2nf_0(n, m).$$

Let $f(n, m)$ be $2nf_0(n, m)$; and let m_n be chosen so large that if $m^{-1} \le m_n^{-1}$ at the outset, then, at the last stage of the construction, Lemma 4.1 applies to allow us to find M_{nn} and M_{nn-1}. □

Lemma 4.3. *If $\{E_{jk}\}$ and $\{F_{jk}\}$ are unital systems of $n \times n$ matrix units* *in the finite von Neumann algebra $\mathscr{R}$ and $\mathscr{S}$ is a closed set of vectors with respect to all E_{jk}, $j \neq k$, such that $\|(E_{jk} - F_{jk})x\| < a$ for each x in $\mathscr{S}$ then there is a unitary operator U in $\mathscr{R}$ such that $UE_{jk}U^* = F_{jk}$ for j, k in $\{1, \ldots, n\}$ and $\|(U - I)x\| < (10n - 1)a$ for each x in $\mathscr{S}$.*

Proof. Let $V_{11}(E_{11}F_{11}E_{11})^{1/2}$ be the polar decomposition of $F_{11}E_{11}$; so that V_{11} is a partial isometry in $\mathscr{R}$ with initial projection E, the range projection of $E_{11}F_{11}$, and final projection F, the range projection of $F_{11}E_{11}$. We have,

$$\begin{aligned}[(E_{11}F_{11}E_{11})^{1/2} - E_{11}]^2 &= [(E_{11}F_{11}E_{11})^{1/2} - E_{11}]^2 E_{11} \\ &\leq [(E_{11}F_{11}E_{11})^{1/2} - E_{11}]^2[(E_{11}F_{11}E_{11})^{1/2} + E_{11}]^2 \\ &= [E_{11}F_{11}E_{11} - E_{11}]^2 = [E_{11}(F_{11} - E_{11})E_{11}]^2;\end{aligned}$$

and, with x in $\mathscr{S}$,

$$\begin{aligned}\|[(E_{11}F_{11}E_{11})^{1/2} - E_{11}]x\| &\leq \|E_{11}(F_{11} - E_{11})E_{11}x\| \\ &\leq \|(F_{11} - E_{11})E_{11}x\| < a,\end{aligned}$$

since $E_{11}x \in \mathscr{S}$. Similarly,

$$\begin{aligned}\|[(F_{11}E_{11}F_{11})^{1/2} - F_{11}]x\| \leq \|(E_{11} - F_{11})F_{11}x\| &\leq \|(E_{11}F_{11} - E_{11})x\| \\ &\quad + \|(E_{11} - F_{11})x\| < 2a.\end{aligned}$$

Thus

$$\begin{aligned}\|(V_{11} - E_{11})x\| = \|(V_{11}E_{11} - E_{11})x\| &\leq \|[V_{11}E_{11} - V_{11}(E_{11}F_{11}E_{11})^{1/2}]x\| \\ &\quad + \|(F_{11}E_{11} - E_{11})x\| < 2a\end{aligned}$$

and

$$\begin{aligned}\|(V_{11}^* - F_{11})x\| &\leq \|[V_{11}^*F_{11} - V_{11}^*(F_{11}E_{11}F_{11})^{1/2}]x\| \\ &\quad + \|(E_{11}F_{11} - F_{11})x\| < 4a.\end{aligned}$$

It follows that,

$$\begin{aligned}\|(E_{11} - E)x\| &\leq \|(E_{11} - V_{11}^*E_{11})x\| + \|(V_{11}^*E_{11} - V_{11}^*V_{11})x\| \\ &\leq \|(E_{11} - V_{11}^*)E_{11}x\| + \|(E_{11} - V_{11})x\| \\ &< \|(E_{11} - F_{11})E_{11}x\| + \|(F_{11} - V_{11}^*)E_{11}x\| + 2a < 7a.\end{aligned}$$

Since $E \sim F$ and $E_{11} \sim F_{11}$ in $\mathscr{R}$, we have $E_{11} - E \sim F_{11} - F$. (At this point, we use our present assumption that $\mathscr{R}$ is finite.) Let W_{11} be a partial isometry in $\mathscr{R}$ with initial projection $E_{11} - E$ and final projection $F_{11} - F$. If $U_1 = W_{11} + V_{11}$, then U_1 is a partial isometry in $\mathscr{R}$ with initial projection E_{11} and final projection F_{11}. Moreover, with x in $\mathscr{S}$,

$$\begin{aligned}\|(U_1 - E_{11})x\| &\leq \|W_{11}x\| + \|(V_{11} - E_{11})x\| \\ &= \|W_{11}(E_{11} - E)x\| + \|(V_{11} - E_{11})x\| < 9a.\end{aligned}$$

Let U_j be $F_{j1}U_1E_{1j}$. Then $U_j^*U_j = E_{jj}$, $U_jU_j^* = F_{jj}$; and, with j in $\{2, 3, \ldots, n\}$ and x in $\mathscr{S}$, we have

$$\|(U_j - E_{jj})x\| = \|(F_{j1}U_1E_{1j} - E_{j1}E_{11}E_{1j})x\| < 10a.$$

If $U = \sum_{j=1}^n U_j$ then U is a unitary operator in $\mathfrak{R}$ fulfilling the conditions in the statement of this lemma: $\|(U - I)x\| \le (10n - 1)a$ and $UE_{jk}U^* = F_{j1}U_1E_{1j}E_{jk}E_{k1}U_1^*F_{1k} = F_{jk}$. □

Lemma 4.4. If E_0 and F_0 are equivalent, properly-infinite projections in the von Neumann algebra $\mathfrak{R}$ and $x_1, \ldots, x_m$ are vectors in $\mathscr{H}$, the Hilbert space on which $\mathfrak{R}$ acts, such that $\|(E_0 - F_0)x_j\| < a_j$, for each x_j, then there is a partial isometry W in $\mathfrak{R}$ such that $W^*W = E_0$, $WW^* = F_0$, and $\|(W - E_0)x_j\| < 25a_j$.

Proof. Using the Halving Lemma, we can find orthogonal families $\{E_n\}$ and $\{F_n\}$ with sums E_0 and F_0, respectively, such that each E_n is equivalent to E_0 and each F_n is equivalent to F_0. Let ε_j be $(a_j - a')/50$ where $\|(E_0 - F_0)x_j\| < a' < a_j$. Since $\sum_{n=1}^\infty \|E_nx_j\|^2 \le \|x_j\|^2$ and $\sum_{n=1}^\infty \|F_nx_j\|^2 \le \|x_j\|^2$, we can find E' in $\{E_n\}$ and F' in $\{F_n\}$ such that $\|E'x_j\| < \varepsilon$ ($\varepsilon = \min \varepsilon_j$) and $\|F'x_j\| < \varepsilon$. If $E = E_0 - E'$ and $F = F_0 - F'$ then $\|(E - F)x_j\| < a_j + 2\varepsilon (= a)$.

Let $V(EFE)^{1/2}$ be the polar decomposition of FE. Since

$$\begin{aligned}[(EFE)^{1/2} - E]^2 &= [(EFE)^{1/2} - E]^2E \le [(EFE)^{1/2} - E]^2[(EFE)^{1/2} + E]^2\\ &= [EFE - E]^2 = [E(F - E)E]^2,\end{aligned}$$

we have

$$\begin{aligned}\|[(EFE)^{1/2} - E]x_j\| &\le \|E(F - E)Ex_j\| \le \|(F - E)Ex_j\|\\ &= \|(F - F^2 + FE - E)x_j\| \le 2\,\|(F - E)x_j\| < 2a.\end{aligned}$$

Thus

$$\|(V - E)x_j\| \le \|[VE - V(EFE)^{1/2}]x_j\| + \|(FE - E)x_j\| < 4a,$$

and $\|(V - F)x_j\| < 5a$.

Similarly $\|[(FEF)^{1/2} - F]x_j\| < 2a$. Since $V^*(FEF)^{1/2}$ is the polar decomposition of EF, $\|(V^* - F)x_j\| < 4a$ and $\|(V^* - E)x_j\| < 5a$. Thus, if $V^*V = E''$ and $VV^* = F''$, then $E'' \sim F''$ and

$$\begin{aligned}\|(E'' - E)x_j\| &= \|(V^*V - E)x_j\| = \|(V^*V - V^*F + V^* - E)x_j\|\\ &\le \|(V - F)x_j\| + \|(V^* - E)x_j\| < 10a.\end{aligned}$$

Since $E_0 \sim E' \le E_0 - E''$ and $F_0 \sim F' \le F_0 - F''$; we have $E_0 - E'' \sim E_0 \sim F_0 \sim F_0 - F''$. Let V_0 be a partial isometry in $\mathfrak{R}$ with initial projection

$E_0 - E''$ and final projection $F_0 - F''$. If $W = V_0 + V$ then $W^*W = E_0$, $WW^* = F_0$ and

$$\begin{aligned}\|(W - E_0)x_j\| &\le \|(V_0 - E')(E_0 - E'')x_j\| + \|(V - E)x_j\| \\ &< 2(\|E'x_j\| + \|(E - E'')x_j\|) + 4a < 2\varepsilon + 24a < 25a_j. \quad \square\end{aligned}$$

Lemma 4.5. *If $\{E_{jk}\}$ and $\{F_{jk}\}$ are unital systems of $n \times n$ matrix units in a properly-infinite von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ and $\mathcal{S}$ is a finite set of vectors in $\mathcal{H}$ such that, for each x in $\mathcal{S}$, $\|(E_{jk} - F_{jk})x\| < a$, then there is a unitary operator U in $\mathcal{R}$ such that $UE_{jk}U^* = F_{jk}$ for all j and k in $\{1, \ldots, n\}$ and $\|(U - I)x\| < 52na$.*

Proof. Note that, for each x in $\mathcal{S}$,

$$\begin{aligned}\|(E_{jk} - F_{jk})E_{kh}x\| &= \|(E_{jh} - F_{jk}F_{kh} + F_{jk}F_{kh} - F_{jk}E_{kh})x\| \\ &\le \|(E_{jh} - F_{jh})x\| + \|(F_{kh} - E_{kh})x\| < 2a.\end{aligned}$$

Since $\|(E_{11} - F_{11})E_{1j}x\| < 2a$ and $\|(E_{11} - F_{11})x\| < a$, Lemma 4.4. provides us with a partial isometry U_1 in $\mathcal{R}$ with initial projection E_{11} and final projection F_{11} such that $\|(U_1 - E_{11})x\| < 25a$ and $\|(U_1 - E_{11})E_{1j}x\| < 50a$ for j in $\{1, \ldots, n\}$ and x in $\mathcal{S}$. Let U_j be $F_{j1}U_1E_{1j}$, for j in $\{2, \ldots, n\}$. Then $U_j^*U_j = E_{jj}$, $U_jU_j^* = F_{jj}$, and, for x in $\mathcal{S}$,

$$\begin{aligned}\|(U_j - E_{jj})x\| &= \|(F_{j1}U_1E_{1j} - E_{j1}E_{11}E_{1j})x\| \\ &\le \|(F_{j1}U_1E_{1j} - F_{j1}E_{11}E_{1j})x\| + \|(F_{j1}E_{11}E_{1j} - E_{j1}E_{11}E_{1j})x\| \\ &\le \|(U_1 - E_{11})E_{1j}x\| + \|(F_{j1} - E_{j1})E_{1j}x\| < 52a.\end{aligned}$$

Let U be $\sum_{j=1}^n U_j$. Then U is a unitary operator in $\mathcal{R}$, $UE_{jk}U^* = F_{jk}$ for all j and k in $\{1, \ldots, n\}$, and, for x in $\mathcal{S}$, $\|(U - I)x\| = \|\sum_{j=1}^n (U_j - E_{jj})x\| \le 52na$. $\square$

Lemma 4.6. *If $\mathcal{R}$ is a properly-infinite, matricial von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{N}_0$ is a unital subfactor of $\mathcal{R}$ of type I_n, $\mathcal{S}_0$ is a finite set of vectors in $(\mathcal{H})_1$, $\{A_1, \ldots, A_h\}$ is a finite set of operators in the unit ball of $\mathcal{R}$, $\{r_j\}$ is a strictly increasing sequence of positive integers, and ε is a positive number, then there is a subfactor $\mathcal{N}$ of $\mathcal{R}$ of type I_{nr_j}, for some j, containing $\mathcal{N}_0$ and containing operators $\{B_1, \ldots, B_h\}$ in its unit ball such that $\|(A_j - B_j)x\| < \varepsilon$ for all x in $\mathcal{S}_0$ and j in $\{1, \ldots, h\}$.*

Proof. Let $\{E_{jk}\}$ be a unital system of $n \times n$ matrix units for $\mathcal{N}_0$. As described at the beginning of this section, we can enlarge $\mathcal{S}_0$ to a (finite) set $\mathcal{S}$ containing $A_1x, \ldots, A_hx$ for each x in $\mathcal{S}_0$ and closed with respect to each E_{jk}, $j \ne k$. Let r be a positive integer chosen sufficiently large that $r^{-1} < \varepsilon/3$ and that $f(n, r) \le \varepsilon/3(52n)$, where $r \to f(n, r)$ is the function introduced in Lemma 4.2. Since $\mathfrak{R}$ is matricial, there is a unital subfactor $\mathcal{N}_1$ of $\mathfrak{R}$ of type I_m containing operators $\{A_{jk}\}$ and $\{T_1, \ldots, T_h\}$ in its unit ball such that $A_{jk}^* = A_{kj}$,

$$\|(A_{jk} - E_{jk})x\| < \tfrac{1}{2}r^{-1} \quad \text{and} \quad \|(T_i - A_i)x\| < \tfrac{1}{2}r^{-1} \tag{$*$}$$

for all j and k in $\{1, \ldots, n\}$, i in $\{1, \ldots, h\}$, and x in $\mathcal{S}$. Let $\{M_{jk}\}$ be a unital system of $m \times m$ matrix units for $\mathcal{N}_1$. Since $M_{11}, \ldots, M_{mm}$ are equivalent, finite in number, have sum I, and I is properly infinite; each M_{jj} is properly infinite in $\mathfrak{R}$ and each M_{jj} is equivalent to I.

Repeated use of the Halving Lemma permits us to express M_{11} as the sum of a countable, orthogonal family $\{M_k\}$ of projections equivalent to M_{11}. As $\sum_{k=1}^{\infty} \|M_k x\|^2 \le \|x\|^2$ and $\sum_{k=1}^{\infty} \|M_k M_{1j} x\|^2 \le \|x\|^2$, for j in $\{1, \ldots, m\}$; there are orthogonal projections M and N in $\{M_k\}$ such that $\|MM_{1j}x\| < r^{-1}/14m+2$, $\|Mx\| < r^{-1}/14m+2$, $\|NM_{1j}x\| < r^{-1}/14m+2$, and $\|Nx\| < r^{-1}/14m+2$, for j in $\{1, \ldots, m\}$ and x in $\mathcal{S}$. With V a partial isometry in $\mathfrak{R}$ such that $V^*V = M+N$ and $VV^* = N$, we have that $M_{11} - M - N + V(=W)$ is a partial isometry in $\mathfrak{R}$ such that $W^*W = M_{11}$ and $WW^* = M_{11} - M$. Moreover, if j is in $\{1, \ldots, m\}$ and $x \in \mathcal{S}$, then

$$\|(W - M_{11})M_{1j}x\| \le \|MM_{1j}x\| + \|NM_{1j}x\| + \|V(M+N)M_{1j}x\| \le \frac{4r^{-1}}{14m+2}$$

and

$$\|(W^* - M_{11})x\| \le \|Mx\| + \|Nx\| + \|V^*Nx\| \le \frac{3r^{-1}}{14m+2}.$$

Let N_{11} be $M_{11} - M$, N_{12} be WM_{12}, N_{jk} be M_{jk} when $j \ne 1 \ne k$, N_{1j} be $N_{12}N_{2j}(= WM_{1j})$ when $j \ne 1$, and N_{j1} be $N_{1j}^*(= M_{j1}W^*)$. Then $\{N_{jk}\}$ is a system of $m \times m$ matrix units and $\sum_{j=1}^{m} N_{jj} = I - M$.

If A is in the unit ball of $\mathcal{N}_1$, then there are scalars a_{jk} such that $A = \sum_{j,k=1}^{m} a_{jk}M_{jk}$ and $|a_{jk}| \le 1$. Let B be $\sum_{j,k=1}^{m} a_{jk}N_{jk}$. Then $1 \ge \|A\| = \|B\|$ and, with x in $\mathcal{S}$,

$$\begin{aligned}
\|(A-B)x\| &= \left\| \sum_{j=2}^{m} [a_{j1}(M_{j1} - N_{j1}) + a_{1j}(M_{1j} - N_{1j})]x + a_{11}(M_{11} - N_{11})x \right\| \\
&\le \sum_{j=2}^{m} (\|(M_{j1} - N_{j1})x\| + \|(M_{1j} - N_{1j})x\|) + \|Mx\| \\
&= \sum_{j=2}^{m} (\|M_{j1}(M_{11} - W^*)x\| + \|(M_{1j} - N_{1j})x\|) + \|Mx\| \\
&\le m\frac{3r^{-1}}{14m+2} + m\frac{4r^{-1}}{14m+2} + \frac{r^{-1}}{14m+2} = \frac{1}{2}r^{-1}.
\end{aligned}$$

Thus there are operators B_{jk} and S_j, in the unit ball of $\mathfrak{R}$, that are linear combinations of $\{N_{jk}\}$ such that $B_{jk}^* = B_{kj}$,

$$\|(B_{jk} - A_{jk})x\| < \tfrac{1}{2}r^{-1} \quad \text{and} \quad \|(S_j - T_j)x\| < \tfrac{1}{2}r^{-1}. \tag{$**$}$$

Combining $(*)$ and $(**)$, we have

$$\|(B_{jk} - E_{jk})x\| < r^{-1} \quad \text{and} \quad \|(S_j - A_j)x\| < r^{-1} < \frac{\varepsilon}{3} \tag{$***$}$$

for all x in $\mathcal{S}$. Choosing r_j so that $nr_j - m$ is positive, use the Halving Lemma to find $nr_j - m$ mutually orthogonal projections equivalent to M with sum M; and adjoin these projections, together with suitable partial isometries between them and the projections N_{jj}, to $\{N_{jk}\}$ to form a unital system of $nr_j \times nr_j$ matrix units and its generated unital subfactor $\mathcal{N}_2$ of $\mathfrak{R}$ of type I_{nr_j}. Since $\mathcal{N}_2$ contains the operators B_{jk} and S_j, and the type of $\mathcal{N}_2$ is divisible by n; Lemma 4.2 applies (see $(***)$, above, and recall the choice of r), and there is a unital system $\{F_{jk}\}$ of $n \times n$ matrix units in $\mathcal{N}_2$ such that, for each x in $\mathcal{S}$, $\|(E_{jk} - F_{jk})x\| < \varepsilon/3(52n)$. From Lemma 4.5, there is a unitary operator U in $\mathfrak{R}$ such that $U^* F_{jk} U = E_{jk}$ and, for each x in $\mathcal{S}$

$$\|(U - I)x\| < 52n \frac{\varepsilon}{3(52n)} = \frac{\varepsilon}{3}.$$

In particular $\|(U - I)A_j x\| < \varepsilon/3$ for each j in $\{1, \ldots, h\}$ and each x in $\mathcal{S}_0$, since $A_j x \in \mathcal{S}$ for each such x. It follows that $U^* \mathcal{N}_2 U$ is a unital subfactor, $\mathcal{N}$, of $\mathfrak{R}$ of type I_{nr_j} containing $\mathcal{N}_0$ and operators $U^* S_j U (= B_j)$ such that

$$\begin{aligned}\|(A_j - B_j)x\| &= \|(A_j - U^* S_j U)x\| = \|(UA_j - S_j U)x\| \\ &\le \|(U - I)A_j x\| + \|(A_j - S_j)x\| + \|S_j(U - I)x\| < \varepsilon\end{aligned}$$

for all x in $\mathcal{S}_0$ and j in $\{1, \ldots, h\}$. □

The lemma that follows is a simpler consequence of Lemmas 4.2, 4.3 and 4.5, than the preceding lemma; but is applicable to *finite*, matricial von Neumann algebras as well. It suffices, when combined with the easy observations on finite representations of uniformly, matricial C^*-algebras (contained in Section 2), for the proof of the Murray–von Neumann theorem on the uniqueness of the finite, matricial von Neumann algebra (Theorem 4.9).

***Lemma* 4.7.** *If $\mathfrak{R}$ is a matricial von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{N}_0$ is a unital subfactor of $\mathfrak{R}$ of type I_n, $\{A_1, \ldots, A_h\}$ is a finite set of operators in the unit ball of $\mathfrak{R}$, $\mathcal{S}_0$ is a finite set of vectors in*

$(\mathscr{H})_1$, ε is a positive number, and r_0 is a positive integer, there is a unital subfactor $\mathscr{N}$ of $\mathscr{R}$ of type I_{mnr_0} containing $\mathscr{N}_0$ and containing operators $\{B_1, \ldots, B_h\}$ in its unit ball such that $\|(A_j - B_j)x\| < \varepsilon$ for all j in $\{1, \ldots, h\}$ and x in $\mathscr{S}_0$.

Proof. As in the argument of Lemma 4.6, we choose a unital system $\{E_{jk}\}$ of $n \times n$ matrix units for $\mathscr{N}_0$ and enlarge $\mathscr{S}_0$ to a finite set $\mathscr{S}$ of vectors in $\mathscr{H}$ containing $A_1x, \ldots, A_hx$, for each x in $\mathscr{S}_0$, and closed with respect to each E_{jk}, $j \neq k$. Again, choose a positive integer, r, so large that $f(n, r) \leq \varepsilon/3(52n)(<\varepsilon/3(10n-1))$ and $r^{-1} < \varepsilon/3$; and choose a unital subfactor $\mathscr{N}_1$ of $\mathscr{R}$ of type I_m containing operators $\{A_{jk}\}$ and $\{T_1, \ldots, T_h\}$ in its unit ball such that $A_{jk}^* = A_{kj}$,

$$\|(A_{jk} - E_{jk})x\| < r^{-1} \quad \text{and} \quad \|(T_i - A_i)x\| < r^{-1} \tag{$*$}$$

for all j and k in $\{1, \ldots, n\}$, i in $\{1, \ldots, h\}$, and x in $\mathscr{S}$.

From Lemma 2.6, there is a unital subfactor $\mathscr{N}_2$ of $\mathscr{R}$ of type I_{mnr_0} containing $\mathscr{N}_1$. Since $\{A_{jk}\}$ are in $\mathscr{N}_2$ and the type of $\mathscr{N}_2$ is divisible by n, we have, from $(*)$ and the choice of r, by application of Lemma 4.2, that $\mathscr{N}_2$ contains a unital system $\{F_{jk}\}$ of $n \times n$ matrix units such that for each x in $\mathscr{S}$, $\|(E_{jk} - F_{jk})x\| < f(n, r) \leq \varepsilon/3(52n)$.

Now $\mathscr{R}$ is a direct sum of its finite and properly-infinite central summands. Applying Lemmas 4.3 and 4.5 to the appropriate summand and to the suitably restricted matrix unit systems $\{E_{jk}\}$ and $\{F_{jk}\}$, we find a unitary operator U in $\mathscr{R}$ such that $U^*F_{jk}U = E_{jk}$ and, for each x in $\mathscr{S}$, $\|(U-I)x\| < \varepsilon/3$. In particular, $\|(U-I)A_jx\| < \varepsilon/3$ for each j in $\{1, \ldots, h\}$ and each x in $\mathscr{S}_0$. It follows that $U^*\mathscr{N}_2U$ is a unital subfactor, $\mathscr{N}$, of $\mathscr{R}$ of type I_{mnr_0} containing $\mathscr{N}_0$ and operators $U^*T_jU(=B_j)$ such that

$$\begin{aligned}\|(A_j - B_j)x\| &= \|(A_j - U^*T_jU)x\| \\ &= \|(UA_j - T_jU)x\| \leq \|(U-I)A_jx\| + \|(A_j - T_j)x\| + \|T_j(U-I)x\| < \varepsilon,\end{aligned}$$

for all x in $\mathscr{S}_0$ and j in $\{1, \ldots, h\}$. □

Lemma 4.8. *If $\mathscr{R}$ is a matricial von Neumann algebra with a countably decomposable centre, there is a representation of the uniformly matricial C^*-algebra $\mathfrak{A}_\infty$ as an ultraweakly dense subalgebra of $\mathscr{R}$.*

Proof. From Lemma 2.7, we may assume that $\mathscr{R}$ acts on a separable Hilbert space $\mathscr{H}$. Let $\{A_1, \ldots, A_h\}$ be a countable generating family of operators for $\mathscr{R}$ (as a von Neumann algebra) in the unit ball of $\mathscr{R}$; and let $\{x_1, x_2, \ldots\}$ be a denumerable dense subset of $(\mathscr{H})_1$. From Lemma 4.7, we can find an ascending sequence $\{\mathscr{N}_h\}$ of unital subfactors of $\mathscr{R}$ such that $\mathscr{N}_h$ contains operators $\{B_1, \ldots, B_h\}$ in its unit ball for which

$\|(A_j - B_j)x_k\| < h^{-1}$ for all j and k in $\{1, \ldots, h\}$ and such that $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \ldots$, are of types I_{2m_1}, $I_{2^2 3^2 m_2}$, $I_{2^3 3^3 5^3 m_3}, \ldots$, respectively. It follows that $\bigcup_{h=1}^{\infty} \mathcal{N}_h$ has norm closure isomorphic to $\mathfrak{A}_\infty$ and ultraweakly dense in $\mathcal{R}$. □

***Theorem* 4.9.** *All finite matricial von Neumann algebras with countably-decomposable centres are factors and are * isomorphic.*

Proof. Our assertion follows from Lemma 4.8, Proposition 2.1, and Corollary 2.4. □

***Theorem* 4.10.** *If $\mathcal{R}$ is a matricial von Neumann algebra with a countably-decomposable centre $\mathcal{C}$ then each uniformly matricial C^*-algebra has a (faithful) representation as an ultraweakly-dense subalgebra of $\mathcal{R}$.*

Proof. Let P be the central projection in $\mathcal{R}$ such that $\mathcal{R}P$ is finite and $\mathcal{R}(I-P)$ is properly infinite. Then $\mathcal{R}P$ and $\mathcal{R}(I-P)$ have countably-decomposable centres and are matricial, provided $0 \neq P \neq I$. It follows from Corollary 2.5 that each uniformly matricial C^*-algebra $\mathfrak{A}$ has a (faithful) representation as an ultraweakly-dense subalgebra of a finite matricial von Neumann algebra. From Theorem 4.9, all such matricial von Neumann algebras are * isomorphic. Thus there is a (faithful) representation φ_1 of $\mathfrak{A}$ as an ultraweakly-dense subalgebra of $\mathcal{R}P$, provided $P \neq 0$. If we can prove that there is such a representation φ_2 of $\mathfrak{A}$ in $\mathcal{R}(I-P)$ (provided $P \neq I$) then, with $\psi(A)$ defined as $\varphi_1(A) + \varphi_2(A)$ for each A in $\mathfrak{A}$, ψ is a (faithful) representation of $\mathfrak{A}$ in $\mathcal{R}$. Now ψ is (unitarily equivalent to) $\varphi_1 \oplus \varphi_2$; and φ_1 and φ_2 are disjoint, since each subrepresentation of φ_1 is finite while no subrepresentation of φ_2 is finite. From [8; Corollary 10.3.4, Theorem 10.3.5], $\psi(\mathfrak{A})^- = \varphi_1(\mathfrak{A})^- \oplus \varphi_2(\mathfrak{A})^- = \mathcal{R}P \oplus \mathcal{R}(I-P) = \mathcal{R}$.

It suffices, therefore, to deal with the case where $\mathcal{R}$ is properly infinite; and, applying Lemma 2.7, we may assume, in addition, that $\mathcal{R}$ acts on a separable Hilbert space $\mathcal{H}$. Let $\{A_j\}$ be a countable set of operators in the unit ball of $\mathcal{R}$ that generates $\mathcal{R}$ (as a von Neumann algebra); and let $\{x_j\}$ be a countable, dense subset of $(\mathcal{H})_1$. If $2^{n_1}3^{n_2}\cdots$ is a supernatural number, let $\{r_j\}$ (of Lemma 4.6) be $\{2^{n_1}3^{n_2}\cdots p_j^{n_j}\}$ and apply Lemma 4.6 to find a unital subfactor $\mathcal{N}_1$ of $\mathcal{R}$ of type I_{r_1} and an operator B_{11} in the unit ball of $\mathcal{N}_1$ such that $\|(A_1 - B_{11})x_1\| < 1$. Replace $\{r_j\}$, now, by $\{p_{j+1}^{n_{j+1}} \cdots p_{j+k}^{n_{j+k}}\} (= \{r_k'\})$ and apply Lemma 4.6, again, to find a unital subfactor $\mathcal{N}_2$ of $\mathcal{R}$ of type $I_{r_j r_k'}$, for some k, containing $\mathcal{N}_1$ and operators B_{12}, B_{22} in its unit ball such that $\|(A_j - B_{j2})x_k\| < \frac{1}{2}$ for each j and k in $\{1, 2\}$. Continuing in this way, we construct a generating nest for an ultraweakly-dense, uniformly matricial C^*-subalgebra in $\mathcal{R}$ with $2^{n_1}3^{n_2}\cdots$ as its supernatural number. □

Corollary **4.11.** *All uniformly matricial C^*-algebras are factor- and cyclic-isoreductive.*

References

1. Araki, H. and Woods, E. J. A classification of factors. *Publ. RIMS Kyoto Univ.*, **4** (1968), 51–130.
2. Connes, A. Une classification des facteurs de type III, *Ann. Sci. École Normale Sup. 4 ème Série*, **6** fasc. 2 (1973), 133–252.
3. Connes, A. Classification of injective factors. *Ann. Math.*, **104** (1976), 73–115.
4. Elliott, G. On approximately finite-dimensional von Neumann algebras. *Math. Scand.*, **39** (1976), 91–101.
5. Elliott, G. and Woods, E. J. The equivalence of various definitions for a properly infinite von Neumann algebra to be approximately finite dimensional. *Proc. Amer. Math. Soc.*, **60** (1976), 175–178.
6. Glimm, J. On a certain class of operator algebras. *Trans. Amer. Math. Soc.*, **95** (1960), 318–340.
7. Glimm, J. Type I C^*-algebras. *Ann. Math.*, **73** (1961), 572–612.
8. Kadison, R. and Ringrose, J. *Fundamentals of the Theory of Operator Algebras*. Academic Press, New York, 1983.
9. Maréchal, O. Une remarque sur un théorèm de Glimm. *Bull. Sci. Math.*, **99** (1975), 41–44.
10. Murray, F. and Neumann, J. von. On rings of operators. IV. *Ann. Math.*, **44** (1943), 716–804.
11. Pedersen, G. K. Isomorphisms of UHF algebras. *J. Functional Anal.*, **30** (1978), 1–16.
12. Pedersen, G. K. *C^*-algebras and their Automorphism Groups*. Academic Press, London, 1979.
13. Powers, R. T. Representations of uniformly hyperfinite algebras and their associated von Neumann rings. *Ann. Math.*, **86** (1967), 138–171.
14. Rieffel, M. Morita equivalence for C^*-algebras and W^*-algebras. *J. Pure Appl. Alg.*, **5** (1974), 51–96.
15. Takesaki, M. *Tomita's Theory of Modular Hilbert Algebras and its Applications*. Springer-Verlag, Heidelberg, 1970.

R. V. Kadison
Department of Mathematics, E1
University of Pennsylvania
Philadelphia PA 19104
USA

Received June, 1981