DIAGONALIZING MATRICES

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Dedicated to Marshall H. Stone on the occasion of his eightieth birthday

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1. Introduction. One of the primitive forms of the spectral theorem tells us that a self-adjoint (hermitian) matrix A over the complex numbers $\mathbf{C}$ can be "diagonalized"—there is a unitary matrix U such that UAU^{-1} has all its non-zero entries on the diagonal. It is important that we are dealing with matrices over $\mathbf{C}$ (rather than over $\mathbf{R}$), for the process of diagonalizing A involves solving polynomial equations (either explicitly or implicitly). Despite this, matrices over rings (not fields) of one sort or another appear in many mathematical situations; the analysis of the forms to which such matrices, satisfying special conditions, can be reduced often plays a crucial role in dealing with particular problems. In the theory of operator algebras in general and von Neumann algebras especially, the process of forming matrices over such algebras is a construction of great importance. It occurs in a critical way in the proof of the Double Commutant theorem, the main result of the first paper [4] in that subject.

There is a natural structure on the set $\mathfrak{M}_n(\mathfrak{A})$ of $n \times n$ matrices over the operator algebra $\mathfrak{A}$ that makes it a C^*-algebra when $\mathfrak{A}$ is a C^*-algebra and a von Neumann algebra when $\mathfrak{A}$ is a von Neumann algebra. The algebraic structure on $\mathfrak{M}_n(\mathfrak{A})$ corresponds to the usual addition and multiplication of matrices (employing the addition and multiplication in $\mathfrak{A}$). The adjoint of a matrix $[A_{jk}]$ (with A_{jk} in $\mathfrak{A}$ as its j, kth entry) is the matrix whose k, jth entry is A_{jk}^*. The norm on $\mathfrak{M}_n(\mathfrak{A})$ is most easily described in an "extrinsic" form by supposing (as we may) that $\mathfrak{A}$ is represented faithfully on a Hilbert space $\mathcal{H}$ whereupon $\mathfrak{M}_n(\mathfrak{A})$ is represented faithfully on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ $(=\tilde{\mathcal{H}})$, the n-fold direct sum of $\mathcal{H}$ with itself, through the usual matrix action on "column vectors." The norm on $\mathfrak{M}_n(\mathfrak{A})$ is the one it inherits from $\mathcal{B}(\tilde{\mathcal{H}})$, the algebra of all bounded linear operators on $\tilde{\mathcal{H}}$. (It is a basic result that this norm is independent of the representation of $\mathfrak{A}$ on $\mathcal{H}$ [3, Proposition 11.1.2].)

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A self-adjoint element of $\mathfrak{M}_n(\mathfrak{A})$ in its representation on $\tilde{\mathcal{H}}$ is simply a self-adjoint operator on $\tilde{\mathcal{H}}$. In matrix terms, it is an element $[A_{jk}]$ such that $A_{jk} = A_{kj}^*$ for all j, k in $\{1, \ldots, n\}$. Similar comments apply to normal and unitary elements in $\mathfrak{M}_n(\mathfrak{A})$. The fundamental problem concerning diagonalization is whether or not each normal element of $\mathfrak{M}_n(\mathfrak{A})$ can be diagonalized. In the next section, we give an example, based primarily on higher homotopy groups of spheres, of a C^*-algebra $\mathfrak{A}$ and a unitary element of $\mathfrak{M}_2(\mathfrak{A})$ that *cannot* be diagonalized. In the third section, we prove the principal result of this article, namely that when $\mathfrak{A}$ is a von Neumann algebra each normal element *can* be diagonalized. We show (Theorem 3.19), in fact, that each abelian self-adjoint subset of $\mathfrak{M}_n(\mathfrak{A})$ is simultaneously diagonalizable (that is, one unitary element in $\mathfrak{M}_n(\mathfrak{A})$ transforms all elements of the subset to diagonal form). This is proved as the last of a series of twenty results the main thrust of which is to construct a comparison theory for projections in a maximal abelian subalgebra of a von Neumann algebra relative to that von Neumann algebra. We consider only countably decomposable algebras to avoid complicated but peripheral cardinality problems.

Many questions and topics for further investigation stem from these results. We list some of these with a few related comments.

1.1 *Carry out the "relative" comparison theory for other than the maximal abelian subalgebras.* Of course this program takes clearer form after studying the results of the third section; but without that further study, general principles might indicate that the case of abelian subalgebras is the easiest and the case of comparison in a factor (relative to a factor) is the most difficult. This might have been so if the "absolute" comparison theory for a factor were not complete; but with the "absolute" theory available there is little to do in the "relative" factor case. For example, two projections in a subfactor of a II_1 factor are equivalent in the larger factor if and only if they are equivalent in the smaller factor. (The normalized trace on the larger factor restricts to the (unique) normalized trace on the smaller factor.) All non-zero projections in a II_1 subfactor of an infinite factor (acting on a separable space) are infinite hence, equivalent in the larger factor. But the problem for other von Neumann subalgebras of a von Neumann algebra has interest, and the problem for C^*-subalgebras has special interest with relation to the rapidly developing K-theory of C^*-algebras.

1.2. The problem of diagonalization can be refined in various ways. Although we note that some normal elements of 2×2 matrix algebras over certain C^*-algebras cannot be diagonalized, some normal elements can be. *Which are they—in a general context?—in special contexts?*

1.3. The example we give to illustrate the failure of diagonalization involves the C^*-algebra of continuous mappings of a compact Hausdorff space X into the algebra of $n \times n$ complex matrices (equivalently, the algebra of $n \times n$ matrices over $C(X)$, the algebra of continuous complex-valued functions on X). More specifically, the X of our example is the 4-sphere and n is 2. The construction of a non-diagonalizable normal (unitary) element is based on homotopy results. We may ask: what topological properties of X guarantee that it is *diagonalizable* (that is, that normal $n \times n$ matrices over $C(X)$ can be brought to diagonal form)?

1.4. What topological properties of X will guarantee that normal 2×2 matrices over $C(X)$ can be diagonalized (2-*diagonalizability*)? What is the relation of n-diagonalizability to m-diagonalizability?

2. An example. From [2], $\pi_4(S^3)$ is the additive group of integers modulo 2. Let f_0 be an essential mapping of S^4 into S^3. We represent S^3 as the surface of the unit ball in $\mathbf{C}^2$. If $U \in SU(2)$ and $U(1, 0) = (1, 0)$, then $U(0, 1) = \lambda(0, 1)$ for some complex number λ of modulus 1. Since $\det(U) = 1 = 1 \cdot \lambda$, we have that $U(0, 1) = (0, 1)$ and $U = I$. Thus the mapping, $g_0 : U \to U(1, 0)$ of $SU(2)$ into S^3 is one-to-one. If $\{f_1, f_2\}$ is an orthonormal basis for $\mathbf{C}^2$ and V_θ is the unitary that maps $(1, 0)$ onto f_1 and $(0, 1)$ onto θf_2 where $|\theta| = 1$, then $\det(V_\theta) = \theta \det(V_1)$. Thus $V_\theta \in SU(2)$ if $\theta = \overline{\det(V_1)}$. Hence g_0 is a homeomorphism of $SU(2)$ onto S^3 and $g_0^{-1} f_0$ is an essential mapping h_0 of S^4 into $SU(2)$. For topological purposes, we can identify S^3 and $SU(2)$.

Let $\mathfrak{A}$ be the C^*-algebra $C(S^4, \mathcal{B}(\mathbf{C}^2))$ of continuous mappings of S^4 into $\mathcal{B}(\mathbf{C}^2)$ (equivalently, the algebra of 2×2 matrices over $C(S^4)$). Then h_0 is a unitary element U_0 in $\mathfrak{A}$. Suppose there is a unitary element U in $\mathfrak{A}$ such that UU_0U^{-1} is diagonal (in the representation of $\mathfrak{A}$ as 2×2 matrices over $C(S^4)$). Then $U(p)U_0(p)U(p)^{-1}$ is a diagonal 2×2 matrix over $\mathbf{C}$ for each p in S^4. Let $V(p)$ be $U_1(p)U(p)$, where

$$U_1(p) = \begin{bmatrix} \overline{\det U(p)} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $V(p) \in SU(2)$, V is a unitary element in $\mathfrak{A}$, and $V(p)U_0(p)V(p)^{-1} = U(p)U_0(p)U(p)^{-1}$.

Again, V and V^{-1} are mappings of S^4 into $SU(2)$ and correspond to mappings of S^4 into S^3. Since $V(p)U_0(p)V(p)^{-1}$ is diagonal and in $SU(2)$, it has the form $\begin{bmatrix} \lambda(p) & 0 \\ 0 & \overline{\lambda(p)} \end{bmatrix}$, where $|\lambda(p)| = 1$. Thus $p \to V(p)U_0(p)V(p)^{-1}$ is a mapping of S^4 into $SU(2)$ whose image lies in a subset homeomorphic to S^1. Since $\pi_4(S^1) = \{0\}$, this mapping is not essential. Let $\{f\}$ denote the homotopy class of a mapping f of S^4 (with base point p_0) into $SU(2)$ with base point I. Since $SU(2)$ is a topological group, the class $\{V \cdot U_0 \cdot V^{-1}\}$ of $V \cdot U_0 \cdot V^{-1}$ is the product of the homotopy classes $\{V\}$, $\{U_0\}$, and $\{V^{-1}\}$. (See [1] Satz II where this is proved for "Γ_e-spaces"—we include a proof below.) As $\pi_4(SU(2))$ is abelian, $\{V\} \cdot \{U_0\} \cdot \{V^{-1}\} = \{U_0\} \cdot \{V\} \cdot \{V^{-1}\} = \{U_0\} \cdot \{V \cdot V^{-1}\} = \{U_0\}$. But $\{VU_0V^{-1}\}$ is 0 if VU_0V^{-1} is diagonal, as noted; while $\{U_0\}$ is not 0 by choice. It follows that U_0 cannot be diagonalized.

We show that two mappings f and g of S^n into a topological group $\mathcal{G}$ have group product $f \cdot g$ in the homotopy class $\{f\} \cdot \{g\}$. For this, choose (standard) mappings σ_+ and σ_- of S^n into S^n (with base point p_0) each homotopic to ι, the identity transform of S^n onto S^n, with the property that $\sigma_+(p) = p_0$ for each p in S^n_- (the "Southern Hemisphere") and $\sigma_-(p) = p_0$ for each p in S^n_+ (the "Northern Hemisphere"). Let H_+ and H_- be homotopies of σ_+ and ι and σ_- and ι, respectively. (Thus $H_+(0, p) = p = H_-(0, p)$ for each p in S^n; $H_+(1, p) = \sigma_+(p)$ and $H_-(1, p) = \sigma_-(p)$ for each p in S^n; $H_+(t, p_0) = p_0 = H_-(t, p_0)$ for t in $[0, 1]$; H_+ and H_- are continuous mappings of $[0, 1] \times S^n$ into S^n.) Then $\{f\} \cdot \{g\}$ is, by definition, the homotopy class of h, where $h(p) = f(\sigma_+(p))$ if $p \in S^n_+$ and $h(p) = g(\sigma_-(p))$ if $p \in S^n_-$. Define $F(t, p)$ to be $f(H_+(t, p)) \cdot g(H_-(t, p))$ (this is the product in $\mathcal{G}$) and note that, for all t in $[0, 1]$, $F(t, p_0) = f(p_0)g(p_0) = e$ (since $f(p_0) = g(p_0) = e$ by assumption), $F(0, p) = f(p) \cdot g(p)$ for each p in S^n, and $F(1, p) = f(H_+(1, p)) \cdot g(H_-(1, p)) = f(\sigma_+(p))g(\sigma_-(p)) = h(p)$ for each p in S^n, for if $p \in S^n_+$, $h(p) = f(\sigma_+(p))$ and $g(\sigma_-(p)) = g(p_0) = e$, while if $p \in S^n_-$, $h(p) = g(\sigma_-(p))$ and $f(\sigma_+(p)) = f(p_0) = e$. Since H_+, H_-, f, g, and multiplication in $\mathcal{G}$ are continuous, F is continuous. Hence F is a homotopy of h and $f \cdot g$.

Another example is obtained by replacing S^4 by S^3 and f_0 by the identity mapping. The argument is unchanged.

Added in proof Jan. 5, 1984: Compare this example with Section 4 of "Diagonalizing Matrices Over $C(X)$" (to appear in J. Functional Analysis) by K. Grove and G. K. Pedersen. This paper provides a splendid analysis and complete answer to 1.3.

3. The main results. Let $\mathcal{R}_0$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and let $\mathcal{R}$ be the von Neumann algebra $\mathfrak{M}_n(\mathcal{R}_0)$ acting on $\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0 (= \mathcal{H})$ as described in the introduction. We shall show, as one of the last (Theorem 3.19) of the series of results that follow, that each commutative self-adjoint subset $\mathcal{S}$ of $\mathcal{R}$ is simultaneously diagonalizable—that is, there is a unitary (matrix) U in $\mathcal{R}$ such that UAU^{-1} has all off-diagonal entries 0 for every A in $\mathcal{S}$. In particular, if A is normal, there is a unitary U such that UAU^{-1} is normal. Before beginning what is essentially the proof of this result, we sketch the structure of the argument. If we find n orthogonal equivalent projections $F_1, \ldots, F_n$ in $\mathcal{R}$ with sum I, then each F_j is equivalent to E_j, where E_j is the projection in $\mathcal{R}$ whose matrix has entry I at the j, j position and all other entries 0. If we can arrange, moreover, that each F_j commutes with every A in $\mathcal{S}$, then each such A is diagonal with respect to every matrix unit system for $\mathcal{R}$ in which $F_1, \ldots, F_n$ are the principal units. There is a unitary operator U in $\mathcal{R}$ such that $UF_jU^{-1} = E_j$ for each j in $\{1, \ldots, n\}$. It follows that UAU^{-1} commutes with each E_j, and is therefore diagonal, for each A in $\mathcal{S}$. Our problem then is to find n projections $F_1, \ldots, F_n$ with the properties described. The larger the "relative commutant" of $\mathcal{S}$ in $\mathcal{R}$ (the von Neumann subalgebra consisting of those operators in $\mathcal{R}$ that commute with all A in $\mathcal{S}$) the easier it should be to find these projections; but we have very little "general" control over this relative commutant. At the very least, $\mathcal{S}$ is contained in a *maximum abelian self-adjoint* (henceforth, simply *maximal abelian*) subalgebra of $\mathcal{R}$ and each such is contained in the relative commutant of $\mathcal{S}$. Now $\mathcal{S}$ may very well generate (or *be*) a maximal abelian subalgebra of $\mathcal{R}$, in which case, $\mathcal{S}$ is its own relative commutant. We see therefore that we must be prepared to (and it will suffice to) find n (orthogonal) equivalent projections with sum I in each maximal abelian subalgebra of $\mathcal{R}$. We prove this for $\mathcal{R}$ in the eighteen results that follow.

3.1 Lemma. *If $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$, and E is a projection in $\mathcal{A}$ minimal among the projections in $\mathcal{A}$ that have the same central support as E (relative to $\mathcal{R}$), then E is an abelian projection in $\mathcal{R}$.*

Proof. From [3; Proposition 5.5.6], the center of $E\mathcal{R}E$ (acting on $E(\mathcal{H})$) is $\mathcal{C}E$, where $\mathcal{C}$ is the center of $\mathcal{R}$, and $\mathcal{A}E$ is a maximal abelian subalgebra of $E\mathcal{R}E$. If F is a non-zero projection in $\mathcal{A}E$, then $F \in \mathcal{A}$ and $F \le E$. Thus $F = C_F F \le C_F E$. If $F < C_F E$, then $(I - C_F)E + F$ is a projection in $\mathfrak{A}$ (note that $\mathcal{C} \subseteq \mathfrak{A}$) with central support C_E. Moreover $(I - C_F)E + F < E$. But this contradicts the minimality of E. Hence $F = C_F E$. It follows that each projection in $\mathfrak{A}E$ is in $\mathcal{C}E$, so that $\mathfrak{A}E = \mathcal{C}E$. Thus the center $\mathcal{C}E$ of $E\mathfrak{R}E$ is a maximal abelian subalgebra $\mathfrak{A}E$ of $E\mathfrak{R}E$. Hence $E\mathfrak{R}E = \mathcal{C}E$, $E\mathfrak{R}E$ is abelian, and E is an abelian projection in $\mathfrak{R}$. ■

3.2 Lemma. *If $\mathfrak{R}$ is a von Neumann algebra and $\mathfrak{A}$ is a maximal abelian subalgebra of $\mathfrak{R}$, then either $\mathfrak{A} = \mathfrak{R}$ or $\mathfrak{A}$ contains two non-zero orthogonal projections E and F such that $C_E = C_F$ and $E \precsim F$.*

Proof. If G is a projection in $\mathfrak{A}$ such that $C_G C_{I-G} = 0$, then $G = C_G$, for

$$G \leq C_G \leq I - C_{I-G} \leq I - (I - G) = G.$$

It follows that either each projection in $\mathfrak{A}$ is central in $\mathfrak{R}$, in which case the center of $\mathfrak{R}$ coincides with $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{R}$ (since $\mathfrak{A}$ is maximal abelian in $\mathfrak{R}$), or $(P =)C_G C_{I-G} \neq 0$ for some projection G in $\mathfrak{A}$. In the latter case, PG and $P(I - G)$ have the same central carrier P. From the Comparison Theorem [3; Theorem 6.2.7], there is a non-zero central projection Q such that $Q \leq P$ and either $QG \precsim Q(I - G)$ or $Q(I - G) \precsim QG$. In any event, one of QG, $Q(I - G)$ serves as E and the other serves as F, when $\mathfrak{R}$ is not abelian. ■

3.3 Lemma. *If $\mathfrak{R}$ is a von Neumann algebra that has no abelian central summands and $\mathfrak{A}$ is a maximal abelian subalgebra of $\mathfrak{R}$, then $\mathfrak{A}$ contains a projection E such that $C_E = C_{I-E} = I$ and $E \precsim I - E$.*

Proof. Let $\{E_a\}$ be a family of non-zero projections in $\mathfrak{A}$ maximal with respect to the properties that $\{C_{E_a}\}$ is an orthogonal family and $E_a \precsim I - E_a$ for each a. From Lemma 3.2, $\mathfrak{A}$ contains non-zero orthogonal projections E_0 and F_0 such that $E_0 \precsim F_0$ $(\leq I - E_0)$. Thus the family $\{E_a\}$ is non-null. Let E be $\Sigma_a E_a$. Then $C_E = \Sigma_a C_{E_a}$ $(=P)$. If $P \neq I$, then $\mathfrak{R}(I - P)$ is a non-abelian von Neumann algebra (since $\mathfrak{R}$ is assumed to have no abelian central summands) and $\mathfrak{A}(I - P)$ is a maximal abelian subalgebra of $\mathfrak{R}(I - P)$. Again from Lemma 3.2, there is a non-zero projection E_1 in $\mathfrak{A}(I - P)$ such that $E_1 \precsim (I - P) - E_1$. If we adjoint E_1 to the family $\{E_a\}$, we have a properly larger family than $\{E_a\}$, contradicting the maximality of $\{E_a\}$. Thus $P = I$. Since

$$E_a = C_{E_a} E_a \precsim C_{E_a}(I - E_a) = C_{E_a} - E_a$$

for each a, we have that

$$E = \sum_a E_a \precsim \sum_a C_{E_a} - E_a = I - E,$$

and $C_E = P = I$. ∎

3.4 Lemma. *If $\mathcal{R}$ is a countably decomposable von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$ that contains no non-zero finite projections, then for each positive integer n, $\mathcal{A}$ contains n orthogonal projections with sum I equivalent in $\mathcal{R}$.*

Proof. Note, first, that under the hypotheses on $\mathcal{A}$, $\mathcal{R}$ has no abelian central summand, for if P is a non-zero central projection such that $\mathcal{R}P$ is abelian, then P is in $\mathcal{A}$ and P is an abelian, hence finite, projection in $\mathcal{R}$. From Lemma 3.3, there is a projection E in $\mathcal{A}$ such that $C_E = C_{I-E} = I$. From [3; Corollary 6.3.5], two countably decomposable properly infinite projections in $\mathcal{R}$ with the same central support are equivalent. Thus E, $I - E$, and I, are equivalent in $\mathcal{R}$. Again $E\mathcal{R}E$ acting on $E(\mathcal{H})$ is a countably decomposable von Neumann algebra in which $\mathcal{A}E$ is a maximal abelian subalgebra that contains no non-zero finite projections. Applying Lemma 3.3 once again, we find a projection F in $\mathcal{A}E$ equivalent to $E - F$ and E (in $E\mathcal{R}E$, and hence in $\mathcal{R}$). We now have three orthogonal equivalent projections, F, $E - F$, and $I - E$, in $\mathcal{A}$ (with sum I). Continuing in this way, we construct n orthogonal equivalent projections with sum I in $\mathcal{A}$.

3.5 Lemma. *If $\mathcal{R}$ is a von Neumann algebra of type I with no infinite central summand, then each maximal abelian subalgebra of $\mathcal{R}$ contains an abelian projection with central support I.*

Proof. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{R}$. By assumption on $\mathcal{R}$ and the type decomposition theorem [3; Theorem 6.5.2], $\mathcal{R}$ has a central summand $\mathcal{R}P_n$ of type I_n, with n a positive integer. If $n = 1$, then P is a non-zero abelian projection in the center of $\mathcal{R}$ and hence in $\mathcal{A}$. If $n > 1$, then $\mathcal{R}P_n$ acting on $P_n(\mathcal{H})$ is a von Neumann algebra without abelian central summands and $\mathcal{A}P_n$ is a maximal abelian subalgebra of it. From Lemma 3.3, $\mathcal{A}P_n$ contains a projection E_1 such that $C_{E_1} = P_n$ and $E_1 \precsim P_n - E_1$. Now $E_1\mathcal{R}E_1$ acting on $E_1(\mathcal{H})$ is a type I von Neumann algebra with no infinite central summand. (See [3; Corollary 6.5.5].) Again, either $\mathcal{A}E_1$ has a non-zero abelian projection F, in which case $F\mathcal{R}F = FE_1\mathcal{R}E_1F$ is abelian and F is an abelian projection in $\mathcal{A}$, or there is a non-zero projection E_2 in $\mathcal{A}E_1$ such that $E_2 \precsim E_1 - E_2$. Continuing in this way (we consider $E_2\mathcal{R}E_2$, next), we produce either a non-zero abelian projection in $\mathcal{A}$ or a set of n non-zero projections $E_1, \ldots, E_n$ in $\mathcal{R}P_n$ such that $E_{j+1} \precsim E_j - E_{j+1}, E_1 \precsim P_n - E_1$, and $E_{j+1} < E_j$. If Q is the central support of E_n, then $E_n, Q(E_{n-1} - E_n), Q(E_{n-2} - E_{n-1}), \ldots, Q(E_1 - E_2), Q(P_n - E_1)$ are $n + 1$ orthogonal projections in $\mathcal{R}P_n$ with the same non-zero central support, which contradicts the fact that $\mathcal{R}P_n$ is of type I_n. Thus the process must end with a non-zero abelian projection for $\mathcal{R}$ in $\mathcal{A}$ before we construct E_n.

Let $\{E_a\}$ be a family of non-zero projections in $\mathcal{A}$ abelian for $\mathcal{R}$ and maximal with respect to the property that $\{C_{E_a}\}$ is an orthogonal family. Let P be $\Sigma_a C_{E_a}$. If $P \neq I$, then $\mathcal{R}(I - P)$ is a von Neumann algebra of type I with no infinite central summand. From what we have just proved $\mathcal{A}(I - P)$, a maximal abelian subalgebra of $\mathcal{R}(I - P)$, contains a non-zero abelian projection E_0 for $\mathcal{R}(I - P)$. But then, adjoining E_0 to $\{E_a\}$ produces a family properly larger than $\{E_a\}$ of non-zero abelian projections for $\mathcal{R}$ in $\mathcal{A}$ with mutually orthogonal central supports, which contradicts the maximal property of $\{E_a\}$. Thus $P = I$. From [3, Proposition 6.4.5], $\Sigma_a E_a$ is an abelian projection for $\mathcal{R}$, has central support I, and lies in $\mathcal{A}$. ∎

3.6 LEMMA. *If $\mathcal{R}$ is a von Neumann algebra of type I_{mn}, with m and n positive integers, then each maximal abelian subalgebra of $\mathcal{R}$ contains n orthogonal projections with sum I equivalent in $\mathcal{R}$.*

Proof. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{R}$. Since $\mathcal{R}$ is of type I with no infinite central summands, Lemma 3.5 applies and $\mathcal{A}$ contains an abelian projection E_1 with central support I. If $E_1 = I$, then $m = n = 1$ and the proof is complete. If $E_1 \neq I$, then $(I - E_1)\mathcal{R}(I - E_1)$ is of type I_{mn-1} and $\mathcal{A}(I - E_1)$ is a maximal abelian subalgebra of $(I - E_1)\mathcal{R}(I - E_1)$. Again $\mathcal{A}(I - E_1)$ contains an abelian projection E_2 for $(I - E_1)\mathcal{R}(I - E_1)$ with central support $I - E_1$ relative to $(I - E_1)\mathcal{R}(I - E_1)$. Thus E_2 is abelian for $\mathcal{R}$ and has central support I relative to $\mathcal{R}$. To see this last, note that if P is a central projection in $\mathcal{R}$ such that $E_2 \leq P$, then $E_2 \leq P(I - E_1)$ and $P(I - E_1)$ is central in $(I - E_1)\mathcal{R}(I - E_1)$. Hence $P(I - E_1) = I - E_1$ and $I - E_1 \leq P$. But $I - E_1$ has central support I in $\mathcal{R}$ by choice of E_1 and our assumption that $\mathcal{R}$ is of type I_{mn} with mn not 1. Thus $P = I$. If $I \neq E_1 + E_2$, we can apply the same procedure to $(I - E_1 - E_2)\mathcal{R}(I - E_1 - E_2)$ to produce E_3. Continuing in this way, we find mn orthogonal projections in $\mathcal{A}$ with central support I relative to $\mathcal{R}$. Adding m

of these projections at a time, we construct n orthogonal projections in $\mathcal{A}$ with sum I equivalent in $\mathcal{R}$. ■

3.7 Lemma. *If $\mathcal{R}$ is a von Neumann algebra of type I_n (n finite) acting on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$, then there is a set of n (orthogonal, equivalent) projections in $\mathcal{A}$ with sum I each abelian with central support I in $\mathcal{R}$ and $\mathcal{A}$ is generated algebraically by these projections and the center $\mathcal{C}$ of $\mathcal{R}$.*

Proof. We proceed by induction on n. When $n = 1$, $\mathcal{R}$ is abelian, $\mathcal{R} = \mathcal{A}$, and I is a projection in $\mathcal{A}$ abelian in $\mathcal{R}$ with central support I. Moreover, $\mathcal{A}(=\mathcal{R})$ is the center of $\mathcal{R}$. Suppose $n > 1$ and we have established our assertion when $\mathcal{R}$ is of type I_k with k less than n. Then $\mathcal{R}$ has no infinite central summands. Lemma 3.5 applies and there is a projection E_1 in $\mathcal{A}$ such that E_1 is abelian and $C_{E_1} = I$. It follows now that $(I - E_1)\mathcal{R}(I - E_1)$ acting on $E_1(\mathcal{H})$ is a von Neumann algebra of type I_{n-1} and $\mathcal{A}(I - E_1)$ is a maximal abelian subalgebra of it. The inductive hypothesis applies and $I - E_1$ is the sum of $n - 1$ projections $E_2, \ldots, E_n$ in $\mathcal{A}(I - E_1)$ abelian in $(I - E_1)\mathcal{R}(I - E_1)$ (and hence, in $\mathcal{R}$) with central support $I - E_1$ in $(I - E_1)\mathcal{R}(I - E_1)$. As in the proof of Lemma 3.6, it follows that $I = C_{E_1} = C_{E_2} = \cdots = C_{E_n}$.

It remains to show that $E_1, \ldots, E_n$ and $\mathcal{C}$ generate $\mathcal{A}$ algebraically. Since $E_j\mathcal{R}E_j$ acting on $E_j(\mathcal{H})$ is an abelian von Neumann algebra with center $\mathcal{C}E_j$ and in which $\mathcal{A}E_j$ is a maximal abelian subalgebra, we have that $E_j\mathcal{R}E_j = \mathcal{C}E_j = \mathcal{A}E_j$ for each j in $\{1, \ldots, n\}$. If $A \in \mathcal{A}$, then

$$A = A\sum_{j=1}^{n} E_j = \sum_{j=1}^{n} AE_j = \sum_{j=1}^{n} C_jE_j,$$

where $C_j \in \mathcal{C}$. ■

3.8 Corollary. *If $\mathcal{R}$ is a von Neumann algebra of type I with no infinite central summand and $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$, then for each positive integer n for which $\mathcal{R}$ has a central summand $\mathcal{R}P_n$ of type I_n, $\mathcal{A}$ contains a set of n (orthogonal, equivalent) projections abelian with central supports P_n and sum P_n in $\mathcal{R}$, and $\mathcal{A}P_n$ is generated algebraically by these n abelian projections and the center of $\mathcal{R}P_n$.*

Proof. If $\mathcal{R}$ has the central summand $\mathcal{R}P_n$, then $\mathcal{R}P_n$ is a von Neumann algebra of type I_n and $\mathcal{A}P_n$ is a maximal abelian subalgebra of it. Our assertion follows by applying Lemma 3.7 to $\mathcal{R}P_n$ and $\mathcal{A}P_n$. ■

3.9 LEMMA. *If $\mathcal{R}$ is a countably decomposable von Neumann algebra of type I_∞ and $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$ with the property that I is the union of projections in $\mathcal{A}$ finite in $\mathcal{R}$, then there is an infinite orthogonal family of projections in $\mathcal{A}$ abelian with central support I and sum I. For each positive integer n, $\mathcal{A}$ contains n orthogonal projections with sum I equivalent in $\mathcal{R}$.*

Proof. Let $\{F_b\}$ be a family of projections in $\mathcal{A}$ finite in $\mathcal{R}$ and maximal with the property that $\{C_{F_b}\}$ is an orthogonal family. If $P = \Sigma_b C_{F_b}$ and $P \neq I$, then $I - P$ is a non-zero projection in $\mathcal{A}$. If $I - P$ is orthogonal to all finite projections of $\mathcal{R}$ in $\mathcal{A}$, the union of these finite projections is not I, contrary to assumption. Thus there is a projection F_0 in $\mathcal{A}$ finite in $\mathcal{R}$ such that $F_0(I - P) \neq 0$. But then $\{F_0(I - P), F_b\}$ is a family of finite projections in $\mathcal{A}$, properly larger than $\{F_b\}$, whose central supports form an orthogonal family. This contradicts the maximal property of $\{F_b\}$. Thus $P = I$. From [3; Lemma 6.3.6], $\Sigma_b F_b$ is a projection F in $\mathcal{A}$ finite with central support I in $\mathcal{R}$. The von Neumann algebra $F\mathcal{R}F$ is of type I with no infinite central summand and $\mathcal{A}F$ is a maximal abelian subalgebra of it. From Lemma 3.5, $\mathcal{A}F$ contains a projection E_0 abelian in $F\mathcal{R}F$ (and hence, in $\mathcal{R}$) with central support F in $F\mathcal{R}F$. Since F has central support I in $\mathcal{R}$ so has E_0. Thus E_0 is an abelian projection in $\mathcal{R}$ with central support I and E_0 lies in $\mathcal{A}$.

Let $\{E_a\}$ be a maximal orthogonal family of projections in $\mathcal{A}$ abelian with central support I in $\mathcal{R}$, and let E be $\Sigma_a E_a$. If $E \neq I$, then $(I - E)\mathcal{R}(I - E)$ is a von Neumann algebra of type I in which $\mathcal{A}(I - E)$ is a maximal abelian subalgebra. Moreover, $I - E$ is the union of projections in $\mathcal{A}(I - E)$ finite in $(I - E)\mathcal{R}(I - E)$. From what we have proved to this point, $\mathcal{A}(I - E)$ contains a projection E_1 abelian with central support $I - E$ in $(I - E)\mathcal{R}(I - E)$. It follows that E_1 is abelian with central support C_{I-E} in $\mathcal{R}$. If $C_{I-E} = I$, we can adjoin E_1 to $\{E_a\}$ contradicting the maximal property of $\{E_a\}$. Thus $(Q=)I - C_{I-E} \neq 0$. Now $Q(I - E) = 0$ so that $\{QE_a\}$ is a family of projections in $\mathcal{A}$ abelian with central supports Q and sum Q in $\mathcal{R}$.

We have just established that if $\mathcal{R}$ is a von Neumann algebra of type I and $\mathcal{A}$ is a maximal abelian subalgebra of it with the property that I is the union of projections in $\mathcal{A}$ finite in $\mathcal{R}$, then there is a non-zero central projection Q in $\mathcal{R}$ that is the sum of projections in $\mathcal{A}$ abelian and equivalent in $\mathcal{R}$ with central supports Q. Let $\{Q_c\}$ be a maximal orthogonal family of such central projections. If $0 \neq I - \Sigma_c Q_c$ $(=Q_0)$, then $\mathcal{R}Q_0$ is a von Neumann algebra of type I and $\mathcal{A}Q_0$ is a maximal abelian subalgebra of it with the property that Q_0 is the union of projections in $\mathcal{A}Q_0$ finite in $\mathcal{R}Q_0$. Thus there is a non-zero central projection Q_1 in $\mathcal{R}Q_0$ that is the sum of projections in $\mathcal{A}Q_0$ abelian with central supports Q_1 in $\mathcal{R}Q_0$ (hence, in $\mathcal{R}$). Adjoining Q_1 to $\{Q_c\}$ produces a family that contradicts the maximal property of $\{Q_c\}$. Hence $\Sigma_c Q_c = I$.

Since $\mathcal{R}$ is countably decomposable of type I_∞, the same is true of $\mathcal{R}Q_c$. We can index the set of projections in $\mathcal{A}Q_c$ with sum Q_c (abelian in $\mathcal{R}$ with central supports Q_c) as $E_{1c}, E_{2c}, \ldots$. From [3; Proposition 6.4.5], $\Sigma_c E_{nc}$ $(=E_n)$ is an abelian projection with central support I $(=\Sigma_c Q_c)$ in $\mathcal{R}$ for each positive integer n. Moreover, each $E_n \in \mathcal{A}$ and $\Sigma_{n=1}^{\infty} E_n = \Sigma_c Q_c = I$.

If $F_k = \Sigma_{j=0}^{\infty} E_{k+jn}$ for k in $\{1, \ldots, n\}$, then $\{F_1, \ldots, F_n\}$ are n orthogonal projections in $\mathcal{A}$, equivalent in $\mathcal{R}$ with sum I. ∎

3.10 *Remark.* If $\mathcal{R}$ is not assumed to have a "uniform decomposability character" (that is, if $\mathcal{R}$ has central summands corresponding to different infinite cardinal types), the first assertion of Lemma 3.9 is not valid for $\mathcal{R}$ but the second remains true. In any event, we can partition the family of abelian projections (in $\mathcal{A}$) with sum Q_c into n subfamilies with the same cardinality (as each other). Summing each of these subfamilies, we arrive at n equivalent projections $F_{1c}, \ldots, F_{nc}$ in $\mathcal{A}$ with sum Q_c. If we let F_k be $\Sigma_c F_{kc}$, then $F_1, \ldots, F_n$ is a set of projections in $\mathcal{A}$ equivalent in $\mathcal{R}$ with sum I. ∎

3.11 LEMMA. *If $\mathcal{R}$ is a countably decomposable von Neumann algebra of type I_∞ acting on a Hilbert space $\mathcal{H}$ and n is a positive integer, then each maximal abelian subalgebra of $\mathcal{R}$ contains n orthogonal projections with* sum I *equivalent in $\mathcal{R}$.*

Proof. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{R}$, and let E_0 be the union of all the projections in $\mathcal{A}$ finite in $\mathcal{R}$. If $E_0 = 0$, then $\mathcal{A}$ has no finite non-zero projections and an application of Lemma 3.4 completes the argument. If $E_0 = I$, then Lemma 3.9 completes the argument. We may suppose that $0 < E_0 < I$. In this case, $(I - E_0)\mathcal{R}(I - E_0)$ acting on $(I - E_0)(\mathcal{H})$ is a von Neumann algebra of type I and $\mathcal{A}(I - E_0)$ is a maximal abelian subalgebra of it that contains no finite non-zero projections. From Lemma 3.4, there are n orthogonal projections $E_1, \ldots, E_n$ in $\mathcal{A}(I - E_0)$ with sum $I - E_0$ equivalent in $(I - E_0)\mathcal{R}(I - E_0)$ (and hence, in $\mathcal{R}$). From [3; Proposition 6.3.7] there is a central projection P_0 in $\mathcal{R}$ such that P_0E_0 is finite and either $I - P_0$ is 0 or $(I - P_0)E_0$ is properly infinite.

Suppose $P_0 \neq 0$. Then P_0 is properly infinite, since $\mathcal{R}$ is of type I_∞, and $P_0 = P_0E_0 + P_0(I - E_0)$ with P_0E_0 finite. Hence $P_0(I - E_0)$ is properly infinite from [3; Theorem 6.3.8]. Now $P_0(I - E_0) = P_0(\Sigma_{j=1}^{n} E_j)$ and $P_0E_1, \ldots, P_0E_n$ are equivalent, by choice of $E_1, \ldots, E_n$. Thus $P_0E_1, \ldots, P_0E_n$ are equivalent (countably decomposable) properly infinite projections in $\mathcal{A}P_0$, and $P_0(E_1 + E_0), P_0E_2, \ldots, P_0E_n$ are n equivalent (countably decomposable, properly infinite) projections with sum P_0 in $\mathcal{A}P_0$. (Use [3; Corollary 6.3.5] for this.)

It will suffice to locate n orthogonal equivalent projections in $\mathcal{A}(I - P_0)$ with sum $I - P_0$. In effect, we may assume that E_0 is properly infinite with central support I (that is, that $P_0 = 0$). With this assumption, $E_0\mathcal{R}E_0$ acting on $E_0(\mathcal{H})$ is a countably decomposable von Neumann algebra of type I_∞ and $\mathcal{A}E_0$ is a maximal abelian subalgebra with the property that E_0 is the union of projections in $\mathcal{A}E_0$ finite in $E_0\mathcal{R}E_0$. From Lemma 3.9, there are n projections $F_1, \ldots, F_n$ in $\mathcal{A}E_0$ equivalent in $E_0\mathcal{R}E_0$ (hence in $\mathcal{R}$) with sum E_0. The n projections $E_1 + F_1, \ldots, E_n + F_n$ are equivalent in $\mathcal{R}$, have sum I, and lie in $\mathcal{A}$. ∎

3.12 Lemma. *If $\mathcal{R}$ is a von Neumann algebra of type II_1 acting on a Hilbert space $\mathcal{H}$, $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$, E is a projection in $\mathcal{A}$, P is a non-zero central subprojection of C_E, τ is the (normalized) center-valued trace on $\mathcal{R}$, and ϵ is a positive real number, then there is a non-zero subprojection F of E in $\mathcal{A}$ such that $\tau(F) \leq \epsilon P$.*

Proof. Replacing $\mathcal{R}$, $\mathcal{H}$, $\mathcal{A}$, and τ, by $\mathcal{R}C_E$, $C_E(\mathcal{H})$, $\mathcal{A}C_E$, and the (normalized) center-valued trace on $\mathcal{R}C_E$, respectively, we may suppose, without loss of generality that $C_E = I$. Since $E\mathcal{R}E$ acting on $E(\mathcal{H})$ is a von Neumann algebra of type II_1 and $\mathcal{A}E$ is a maximal abelian subalgebra of it, from Lemma 3.3 there is a projection E_1 in $\mathcal{A}E$ such that $C'_{E_1} = E$ and $E_1 \precsim E - E_1$, where C'_{E_1} is the central support of E_1 relative to $E\mathcal{R}E$. If Q is a central projection in $\mathcal{R}$ containing E_1, then QE is a central projection in $E\mathcal{R}E$ containing E_1. Since $C'_{E_1} = E, E \leq QE$. Hence $E = QE$ and $E \leq Q$. Under our present assumption, $C_E = I$, so that $Q = I$. It follows that $C_{E_1} = I$.

In the same way, we can find a subprojection E_2 of E_1 in $\mathcal{A}$ such that $C_{E_2} = I$ and $E_2 \precsim E_1 - E_2$. Continuing in this way, we find a sequence $\{E_n\}$ of projections in $\mathcal{A}$ (where $E_0 = E$) such that $C_{E_n} = I, E_{n+1} < E_n$, and $E_{n+1} \precsim E_n - E_{n+1}$. By construction there is a subprojection E_j' of $E_{j-1} - E_j$ equivalent to E_n for each j in $\{1, \ldots, n\}$. (Note that E_j' is not assumed to lie in $\mathcal{A}$.) Hence $\tau(E_n) \leq (n + 1)^{-1}I$ and $\tau(PE_n) = P\tau(E_n) \leq$

$(n+1)^{-1}P$. For large n, $(n+1)^{-1} \leq \epsilon$ and $\tau(PE_n) \leq \epsilon P$. We complete the proof by choosing F to be PE_n and noting that $E \geq F \in \mathfrak{A}$. ∎

3.13 PROPOSITION. *Let $\mathfrak{R}$ be a von Neumann algebra of type II_1, $\mathfrak{A}$ be a maximal abelian subalgebra of $\mathfrak{R}$, and τ be the (normalized) center-valued trace on $\mathfrak{R}$. If H is an element in the center of $\mathfrak{R}$ such that $0 \leq H \leq I$, then there is a projection E in $\mathfrak{A}$ such that $\tau(E) = H$.*

Proof. From the spectral theorem, there is a central projection P and a positive ϵ such that $0 < \epsilon P < H$ if $0 < H$. From Lemma 3.12, there is a non-zero projection E in $\mathfrak{A}$ such that $\tau(E) < \epsilon P$. Thus there is at least one non-null orthogonal family of non-zero projections in $\mathfrak{A}$ whose traces have sum dominated by H. Let $\mathfrak{F}$ be the set of such families partially ordered by inclusion. The union of each chain in $\mathfrak{F}$ is a member of $\mathfrak{F}$ that serves as an upper bound of that chain (for the sum of the traces of projections in an element of $\mathfrak{F}$ is the least upper bound in the operator ordering of the set of finite subsums and each finite subsum is dominated by H). Thus $\mathfrak{F}$ has a maximal element $\{E_a\}$. Since τ is normal (see [3; Theorem 8.2.8(vi)]), $\tau(E) = \Sigma_a \tau(E_a)$ $(\leq H)$, where $E = \Sigma_a E_a$ $(\in \mathfrak{A})$. If $\tau(E) < H(\leq I)$, then there is a non-zero central projection Q and a positive ϵ' such that

$$\epsilon' Q \leq Q(H - \tau(E)) \leq Q(I - \tau(E)) = \tau(Q(I - E)) \leq C_{Q(I-E)} = QC_{I-E}.$$

(See [3; Proposition 5.5.3].) Thus $Q = QC_{I-E}$, and $Q \leq C_{I-E}$. From Lemma 3.12, there is a non-zero subprojection E_0 of $I - E$ in $\mathfrak{A}$ such that $\tau(E_0) \leq \epsilon' Q$. Now

$$\tau(E + E_0) = \tau(E) + \tau(E_0) \leq \tau(E) + \epsilon' Q \leq \tau(E) + Q(H - \tau(E)) \leq H$$

and $\{E_0, E_a\}$ is an orthogonal family of non-zero projections in $\mathfrak{A}$ properly larger than $\{E_a\}$. This contradicts the maximal property of $\{E_a\}$. Thus $\tau(E) = H$. ∎

3.14 COROLLARY. *Let $\mathfrak{R}$ be a von Neumann algebra of type II_1 acting on a Hilbert space $\mathcal{H}$, $\mathfrak{A}$ be a maximal abelian subalgebra of $\mathfrak{R}$, E be a projection in $\mathfrak{A}$, and τ be the (normalized) center-valued trace on $\mathfrak{R}$. If H is an element of the center $\mathfrak{C}$ of $\mathfrak{R}$ such that $0 \leq H \leq \tau(E)$, then there is a projection F in $\mathfrak{A}$ such that $\tau(F) = H$ and $F \leq E$.*

Proof. Proposition 3.13 applies to the von Neumann algebra $E\mathfrak{R}E$

of type II_1 acting on $E(\mathcal{H})$ and the maximal abelian subalgebra $\mathcal{A}E$ of it to yield a projection F in $\mathcal{A}E$ such that $\tau(F)E = HE$. (Note for this that $HE \in \mathcal{C}E$, the center of $E\mathcal{R}E$, and that $0 \leq HE \leq E$, since $0 \leq H \leq \tau(E) \leq I$.) As $\tau(F) - H \in \mathcal{C}$ and $(\tau(F) - H)E = 0$, we have that $(\tau(F) - H)C_E = 0$. Now $F \leq E \leq C_E$ so that $\tau(F) = \tau(FC_E) = \tau(F)C_E$ and $\tau(E) = \tau(E)C_E$. Moreover, $0 \leq H \leq \tau(E) = \tau(E)C_E \leq C_E$, so that $HC_E = H$. Thus $\tau(F) = H$. ■

3.15 COROLLARY. *If $\mathcal{R}$ is a von Neumann algebra of type II_1 and n is a positive integer, then each maximal abelian subalgebra of $\mathcal{R}$ contains n orthogonal equivalent projections with sum I.*

Proof. Suppose $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$. From Corollary 3.14, there are projections $E_1, \ldots, E_n$ in $\mathcal{A}$ such that $\tau(E_j) = n^{-1}I$ and $E_{j+1} \leq I - (E_1 + \cdots + E_j)$ for j in $\{1, \ldots, n-1\}$. Since $\tau(E_j) = \tau(E_k)$, $E_1, \ldots, E_n$ are equivalent and by construction they are mutually orthogonal. Moreover, $\tau(\Sigma_j E_j) = \Sigma_j \tau(E_j) = I$. Since τ is faithful and $\Sigma_j E_j \leq I$, $E_1, \ldots, E_n$ have sum I. ■

3.16 LEMMA. *If $\mathcal{R}$ is a countably decomposable von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{R}$ has no central summand of type I, $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$, and n is a positive integer, then each non-zero projection in $\mathcal{A}$ contains n non-zero orthogonal projections in $\mathcal{A}$ equivalent in $\mathcal{R}$.*

Proof. Let E be a non-zero projection in $\mathcal{A}$. Then $E\mathcal{R}E$ acting on $E(\mathcal{H})$ is a countably decomposable von Neumann algebra with no central summand of type I and $\mathcal{A}E$ is a maximal abelian subalgebra of $E\mathcal{R}E$. If we show that $\mathcal{A}E$ contains n non-zero orthogonal projections equivalent in $E\mathcal{R}E$, then these n projections are orthogonal subprojections of E in $\mathcal{A}$ and are equivalent in $\mathcal{R}$. It suffices to show that $\mathcal{A}$ contains n non-zero orthogonal projections equivalent in $\mathcal{R}$.

If $\mathcal{A}$ contains a non-zero finite projection F, then $F\mathcal{R}F$ acting on $F(\mathcal{H})$ is a von Neumann algebra of type II_1 and $\mathcal{A}F$ is a maximal abelian subalgebra of it. From Corollary 3.15, $\mathcal{A}F$ contains n orthogonal subprojections equivalent in $F\mathcal{R}F$ with sum F.

We may suppose now that $\mathcal{A}$ has no non-zero finite projections. In this case, Lemma 3.4 applies and $\mathcal{A}$ has n orthogonal projections with sum I equivalent in $\mathcal{R}$. ■

3.17 LEMMA. *If $\mathcal{R}$ is a countably decomposable von Neumann algebra with no central summand of type I and n is a positive integer, then each* *maximal abelian subalgebra of $\mathcal{R}$ contains n orthogonal projections with sum I equivalent in $\mathcal{R}$.*

Proof. From Lemma 3.16, a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{R}$ contains n non-zero orthogonal projections equivalent in $\mathcal{R}$. Thus the set $\mathcal{S}$ of sets $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ of n elements, where each $\mathcal{F}_j$ is an orthogonal family $\{E_{ja}\}_{a\in\mathbf{A}}$ of non-zero projections in $\mathcal{A}$, each $\mathcal{F}_j$ is indexed by $\mathbf{A}$, $E_{1a}, \ldots, E_{na}$ are equivalent for each a in $\mathbf{A}$, and $\cup_{j=1}^{n} \mathcal{F}_j$ is an orthogonal family, is non-null. We partially order $\mathcal{S}$ in such a way that $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\} \leq \{\mathcal{F}_1', \ldots, \mathcal{F}_n'\}$ precisely when the indexing of the families of the second set extends the indexing of those of the first set (which entails, in particular, that $\mathcal{F}_j \subseteq \mathcal{F}_j'$ for j in $\{1, \ldots, n\}$). Let $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ be a maximal element of $\mathcal{S}$ relative to this ordering; and let E_j be the union of the projections in $\mathcal{F}_j$. Then $\{E_1, \ldots, E_n\}$ is an orthogonal family of non-zero projections in $\mathcal{A}$ equivalent in $\mathcal{R}$ (see [3; Proposition 6.2.2]). It remains to show that $\Sigma_{j=1}^{n} E_j = I$. Suppose the contrary. Then $I - \Sigma_{j=1}^{n} E_j$ contains n non-zero orthogonal projections $F_1, \ldots, F_n$ in $\mathcal{A}$ equivalent in $\mathcal{R}$ (from Lemma 3.16). By adjoining F_j to $\mathcal{F}_j$, we construct a set in $\mathcal{S}$ properly larger than $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ (relative to the given partial ordering on $\mathcal{S}$). This contradicts the maximal property of $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$. Hence $\Sigma_{j=1}^{n} E_j = 1$. ■

3.18 THEOREM. *If $\mathcal{R}_0$ is a countably decomposable von Neumann algebra and $\mathcal{R}$ is the von Neumann algebra of $n \times n$ matrices over $\mathcal{R}_0$, then each maximal abelian subalgebra of $\mathcal{R}$ contains n (orthogonal) equivalent projections with sum I.*

Proof. Suppose $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{R}$ and $\mathcal{C}$ is the center of $\mathcal{R}$. Then $\mathcal{C} \subseteq \mathcal{A}$ so that each of the central projections corresponding to the central type decomposition of $\mathcal{R}$ lies in $\mathcal{A}$. If P is a central projection in $\mathcal{R}$, then $\mathcal{A}P$ is a maximal abelian subalgebra of $\mathcal{R}P$. From [3; Theorem 6.5.2], there are central projections $P_\infty, P_1, P_2, \ldots,$ and P_c with sum I such that either $P_c = 0$ or $\mathcal{R}P_c$ is a von Neumann algebra with no central summand of type I; and either $P_m = 0$ or $\mathcal{R}P_m$ is a von Neumann algebra of type I_m for all m in $\{\infty, 1, 2, \ldots\}$. We shall note that $\mathcal{R}$ is countably decomposable so that $\mathcal{R}P$ is countably decomposable for each central projection P in $\mathcal{R}$. We shall also see that P_m is 0 unless m is divisible by n. Thus $\mathcal{A}P_c$ contains n equivalent projections $E_{1c}, \ldots, E_{nc}$ with sum P_c from Lemma 3.17 and $\mathcal{A}P_\infty$ contains n equivalent projections $E_{1\infty}, \ldots, E_{n\infty}$ with sum P_∞ from Lemma 3.11. From Lemma 3.7, $\mathcal{A}P_m$ contains m equivalent projections with sum P_m. If $P_m = 0$, then $\mathcal{A}P_m$ contains n equivalent projections with sum P_m (each projection equal to 0). If $P_m \neq 0$, then m is divisible by n and, again, $\mathfrak{A}P_m$ contains n equivalent projections $E_{1m}, \ldots, E_{nm}$ with sum P_m for each m. Let E_j be $E_{jc} + E_{j\infty} + \Sigma_{m=1}^{\infty} E_{jm}$ for each j in $\{1, \ldots, n\}$. Then $\{E_1, \ldots, E_n\}$ is a set of n equivalent projections in $\mathfrak{A}$ with sum I.

It remains to show that $\mathfrak{R}$ is countably decomposable and that m is divisible by n if $P_m \neq 0$. Let F_j be the projection in $\mathfrak{R}$ whose matrix has I at the j, j entry and 0 at all others. If $\{G_a : a \in \mathbf{A}\}$ is an orthogonal family of non-zero subprojections of F_j in $\mathfrak{R}$, then each G_a has matrix whose only non-zero entry is a (non-zero) projection E_a in $\mathfrak{R}_0$, and $\{E_a : a \in \mathbf{A}\}$ is an orthogonal family of projections in $\mathfrak{R}_0$. Since $\mathfrak{R}_0$ is countably decomposable, $\mathbf{A}$ is countable; and F_j is a countably decomposable projection in $\mathfrak{R}$. From [3; Proposition 5.5.9], each F_j is the union of an orthogonal family of cyclic projections in $\mathfrak{R}$. Since each F_j is countably decomposable, each orthogonal family of non-zero cyclic projections is countable. It follows that $\Sigma_{j=1}^{n} F_j$, the identity in $\mathfrak{R}$, is the union of a countable family of cyclic projections in $\mathfrak{R}$, and from [3; Proposition 5.5.19], the identity in $\mathfrak{R}$ is countably decomposable.

The matrix of P_m has a central projection Q_0 of $\mathfrak{R}_0$ at each diagonal position and 0 at all others. Suppose $P_m \neq 0$ so that $\mathfrak{R}P_m$ is of type I_m. From [3; Corollary 6.5.5] the projection M in $\mathfrak{R}P_m$ whose matrix has Q_0 at the 1, 1 entry and 0 at all others is a sum of projections abelian in $\mathfrak{R}P_m$. Using [3; Proposition 6.4.5], we can find a subprojection G of M abelian in $\mathfrak{R}P_m$ with the same central carrier P_m as M. The matrix of G has some projection G_0 in $\mathfrak{R}_0 Q_0$ at the 1, 1 entry and 0 at all others. Since G is abelian in $\mathfrak{R}P_m$ with central support P_m, G_0 is abelian in $\mathfrak{R}_0 Q_0$ with central support Q_0. Thus $\mathfrak{R}_0 Q_0$ is of type I. Suppose $M_1, \ldots, M_k$ are orthogonal abelian projections in $\mathfrak{R}_0 Q_0$ each with non-zero central support Q. By placing each M_j at any one of the diagonal positions and 0 at all other positions, we form nk orthogonal abelian projections in $\mathfrak{R}$ each with central support Q_1, the diagonal matrix with Q at each diagonal entry. Since $\mathfrak{R}P_m$ is of type I_m; $nk \leq m$. It follows that each central summand of $\mathfrak{R}_0 Q_0$ is of type I_j, where $j \leq m/n$. Now $\mathfrak{R}_0 Q_0$ has a central summand of type I_k for some finite k. If we assume, in the preceding argument, that $M_1 + \cdots + M_k = Q$, then the nk abelian projections formed from $M_1, \ldots, M_k$ have sum Q_1. Thus $\mathfrak{R}Q_1$ is of type I_{nk} as well as of type I_m. By uniqueness of type decomposition (see Theorem 6.5.2 of [3] and the comments preceding it), $m = nk$. Hence m is divisible by n, $k = m/n$, and $\mathfrak{R}_0 Q_0$ is of type I_k. ∎

3.19 THEOREM. *If $\mathcal{R}_0$ is a countably decomposable von Neumann algebra, $\mathcal{R}$ is the von Neumann algebra of $n \times n$ matrices with entries in $\mathcal{R}_0$, and $\mathcal{S}$ is an abelian self-adjoint subset of $\mathcal{R}$, then there is a unitary element (matrix) U in $\mathcal{R}$ such that UAU^{-1} has all its non-zero entries on the diagonal for each A in $\mathcal{S}$.*

Proof. We follow the pattern described in the introductory comments to this section and make use of the notation established there. By Zorn's lemma, $\mathcal{S}$ is contained in a maximal abelian self-adjoint subfamily $\mathcal{A}$ of $\mathcal{R}$. From maximality, $\mathcal{A}$ is a maximal abelian (self-adjoint) subalgebra of $\mathcal{R}$. From Theorem 3.18, there are n orthogonal equivalent projections $F_1, \ldots, F_n$ in $\mathcal{A}$ with sum I. The matrix with entry I in the k, j position is a partial isometry in $\mathcal{R}$ with initial projection E_j and final projection E_k. Thus $E_1, \ldots, E_n$ are n orthogonal equivalent projections in $\mathcal{R}$ with sum I. We prove that E_j and F_k are equivalent in $\mathcal{R}$ for all j and k in $\{1, \ldots, n\}$. Of course, it suffices to show that E_1 and F_1 are equivalent in $\mathcal{R}$. Suppose the contrary. From the comparison theorem [3; Theorem 6.2.7], there is a non-zero central projection P in $\mathcal{R}$ such that either $QE_1 \prec QF_1$ for each non-zero central subprojection Q of P in $\mathcal{R}$ or $QF_1 \prec QE_1$ for each such Q. We may suppose that $PE_1 \prec PF_1$ without loss of generality. Replacing $E_1, \ldots, E_n, F_1, \ldots, F_n$, and $\mathcal{R}$, by $PE_1, \ldots, PE_n, PF_1, \ldots, PF_n$, and $\mathcal{R}P$, respectively, we may suppose, without loss of generality, that $QE_1 \prec QF_1$ for each non-zero central projection Q in $\mathcal{R}$. If QE_1 is finite for some non-zero central projection Q, then so are $QE_2, \ldots, QE_n$, and $Q\,(= QE_1 + \cdots + QE_n)$ from Proposition 6.3.2 and Theorem 6.3.8 of [3]. But $QE_j \sim G_j < QF_j$ so that $\Sigma_{j=1}^n QE_j = Q \sim \Sigma_{j=1}^n G_j < \Sigma_{j=1}^n QF_j = Q$. Thus E_1 and all E_j are properly infinite projections with central carrier I. Since $E_1 \prec F_1$, F_1 and all F_j are properly infinite with central carrier I (see [3; Proposition 6.3.7]). From the proof of Theorem 3.18, $\mathcal{R}$ is countably decomposable. Thus $E_1 \sim F_1$ from [3; Corollary 6.3.5], contradicting our assumption that $E_1 \prec F_1$. Thus $E_j \sim F_k$ for all j and k in $\{1, \ldots, n\}$. (Note that this last conclusion is valid even when $\mathcal{R}$ is not countably decomposable; but using the countable decomposability of $\mathcal{R}$, which is available to us under the present hypotheses, it is possible to give this brief proof).

Let V_j be a partial isometry in $\mathcal{R}$ with initial projection F_j and final projection E_j for j in $\{1, \ldots, n\}$ and let U be $\Sigma_{j=1}^n V_j$. Then U is a unitary element in $\mathcal{R}$ and $UF_jU^{-1} = E_j$. Since F_j commutes with every element in

$\mathcal{A}$ and hence in $\mathcal{S}$, E_j commutes with UAU^{-1} for all A in $\mathcal{S}$. Hence UAU^{-1} has all its non-zero entries on the diagonal when $A \in \mathcal{S}$. ∎

3.20 COROLLARY. *With the notation and assumptions of the preceding theorem, if A is a normal element of $\mathcal{R}$, then there is a unitary element U in $\mathcal{R}$ such that UAU^{-1} has all the non-zero entries of its matrix on the diagonal.*

Proof. Since A is normal, $\{A, A^*\}$ is an abelian self-adjoint family in $\mathcal{R}$ and Theorem 3.19 applies. ∎

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