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Limits of States

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Abstract. Estimates for vector representations of states are used to prove that $\{C_nC_0\}$ is strong-operator convergent to C_0, where C_n is the universal central support of ϱ_n and $\{\varrho_n\}$ is a sequence of states of a C^*-algebra $\mathfrak{A}$ converging in norm to ϱ_0. States of $\mathfrak{A}$ of a given type are shown to form a norm-closed convex subset of the (norm) dual of $\mathfrak{A}$. The pure states of $\mathfrak{A}$ form a norm-closed subset of the dual.

1. Introduction and Preliminaries

Let $\mathfrak{A}$ be a C^*-algebra (with unit element I) and ϱ be a (normalized) state of $\mathfrak{A}$. We denote by $(\pi_\varrho, \mathscr{H}_\varrho, x_\varrho)$ the representation π_ϱ of $\mathfrak{A}$ on the Hilbert space $\mathscr{H}_\varrho$ resulting from the GNS construction applied to $\mathfrak{A}$ and ϱ, where x_ϱ is a (unit) cyclic vector in $\mathscr{H}_\varrho$ for $\pi_\varrho(\mathfrak{A})$ such that $\varrho(A) = \langle \pi_\varrho(A)x_\varrho, x_\varrho \rangle$ for each A in $\mathfrak{A}$. We say that the state ϱ is of type I_n, II_1, II_∞, III when $\pi_\varrho(\mathfrak{A})^-$, the strong-operator closure of $\pi_\varrho(\mathfrak{A})$ is of the corresponding type. We show (Theorem 4.1) that the set of states of a given type is a norm-closed convex subset of the space of states of $\mathfrak{A}$.

The direct sum of all representations π_ϱ of $\mathfrak{A}$ arising from states ϱ of $\mathfrak{A}$ is the universal representation ψ of $\mathfrak{A}$ (on $\mathscr{H}_u$). The "universal" property of ψ is expressed in the fact that each cyclic representation is (unitarily) equivalent to a *subrepresentation* $\psi_{E'}$ of ψ [that is, the representation $A \to \psi(A)E'$ of $\mathfrak{A}$ on $E'(\mathscr{H}_u)$] with E' a cyclic projection for $\psi(\mathfrak{A})$ in $\psi(\mathfrak{A})'$, the commutant of $\psi(\mathfrak{A})$. Recall that a normal state ω of a von Neumann algebra $\mathfrak{R}$ acting on $\mathscr{H}$ has as *support* the largest projection E (in $\mathfrak{R}$) orthogonal to all projections annihilated by ω and as *central support* the largest central projection P in $\mathfrak{R}$ orthogonal to all central projections annihilated by ω. Thus P is C_E, the central support of the support projection E. If ω is the vector state $\omega_x|\mathfrak{R}$ of $\mathfrak{R}$ then E has range $[\mathfrak{R}'x]$ and C_E has range $[\mathfrak{R}\mathfrak{R}'x]$ (where $[\mathscr{V}]$ denotes the norm closure of the linear span of a set $\mathscr{V}$ of vectors).

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By construction $\varrho\circ\psi^{-1}$ is a vector state, $\omega_{x_\varrho}|\psi(\mathfrak{A})$, of $\psi(\mathfrak{A})$ and, so, has a unique normal extension, $\omega_{x_\varrho}|\psi(\mathfrak{A})^-$ to $\psi(\mathfrak{A})^-$. We denote the central support of this extension by C_ϱ and refer to it as the *universal central support* of ϱ. Recall that two representations π_1 and π_2 of $\mathfrak{A}$ are defined to be *quasi-equivalent* when there is an isomorphism φ of $\pi_1(\mathfrak{A})^-$ onto $\pi_2(\mathfrak{A})^-$ such that $\varphi\circ\pi_1=\pi_2$. We say that two states ϱ and η are quasi-equivalent when π_ϱ and π_η are quasi-equivalent. We use the notation $\sim_q$ to indicate quasi-equivalence (both for states and representations). We say that π_1 is *quasi-subequivalent* to π_2 (and write $\pi_1 \precsim_q \pi_2$) when π_1 is quasi-equivalent to a subrepresentation of π_2. (We use the corresponding terminology and notation for states.) In Proposition 4.3 we note that $\varrho \precsim_q \eta$ if and only if $C_\varrho \leqq C_\eta$; so that $\precsim_q$ induces a (lattice) partial ordering of the (quasi-equivalence classes of) states of $\mathfrak{A}$, and the minimal (classes of) states relative to this partial ordering are precisely the (classes of) factor states. (This structure in a more general setting and with a slightly different terminology is discussed in [14, Sect. 1].) We prove, in Theorem 3.3, that if $\{\varrho_n\}$ is a sequence of states of $\mathfrak{A}$ converging in norm to ϱ_0 then $\{C_{\varrho_n}C_{\varrho_0}\}$ is strong-operator convergent to C_{ϱ_0}. We deduce, as a corollary, that the set of factor states of $\mathfrak{A}$ is norm closed. In Proposition 4.7 we note that the norm limit of a sequence of equivalent states is a state subequivalent to them and, as a corollary, that the set of pure states of a C^*-algebra is norm closed. (The assertion for factor states is a result of Combes [7, Corollaire 2.3].)

To prove Theorem 3.3, we establish a convergence result (Lemma 3.1) for the support and central support of a vector state of a von Neumann algebra. We also need an estimate on $\|x-y\|$ in terms of $\|\omega-\omega_y|\mathfrak{R}\|$, where ω is a vector state of $\mathfrak{R}$ and x is to be chosen so that $\omega_x|\mathfrak{R}=\omega$. This topic is treated in the next section (estimates are obtained in Proposition 2.2 and Theorems 2.3 and 2.4).

The results in this article were found during an investigation stemming from a question posed to the author by G. Price, B. Blackadar, and J. Rosenberg: Is a norm limit of type I states of an AF algebra a type I state? Price encountered this question in his thesis work under the direction of R.T. Powers. Powers felt that the answer was affirmative (and that an argument with finite rank density matrices would prove it). He suggested that Price check with others to see if the answer were known or if a simpler argument were available. The combination of norm estimates on vector state representatives ("Bures distance") and the "magic" of universal representation techniques produces a complete analysis of this sort of question. There are further (and deeper) avenues of investigation possible along the lines of "Bures distance" through the more sophisticated techniques of the "V_α cones" of Araki, Connes, and Haagerup [5, 8, 13] and some of these have been explored in recent papers. The simpler techniques used here suffice for the results on "stability of type" under norm limits.

2. Differences of Vector States

In this section, we study the extent to which "nearby" normal states of a concretely represented C^*-algebra can be realized as vector states corresponding to "nearby" vectors. Bures introduces [6, Definitions 1.2 and 1.3] as a distance between normal states ω and ω' of a von Neumann algebra $\mathfrak{R}$, the infimum $d(\omega,\omega')$ of $\|x-x'\|$ over all x, x' and faithful normal representations φ of $\mathfrak{R}$, where $\omega=\omega_x\circ\varphi$ and $\omega'=\omega_{x'}\circ\varphi$. He studies this distance function for the purpose of extending Kakutani's theorem on the existence of infinite product measures to the noncommutative case. Bures proves, among other things, that $d(\omega,\omega')$ is attained in some representation of $\mathfrak{R}$ and that $d(\omega,\omega')\leqq\|\omega-\omega'\|^{1/2}$. This inequality is established with the aid of the Sakai-Radon-Nikodým theorem (cf. [17, Theorem 7.3.6]). That the Sakai-Radon-Nikodým theorem is very much a part of this study is emphasized by Araki's results [1, 2]. Evidently inspired by the work of Bures, Araki subjects the function d to a penetrating analysis – emerging with an important extension of the Sakai-Radon-Nikodým theorem (to the case where no order restriction is imposed on the positive, normal functionals). From the current viewpoint, these estimates are most efficiently obtained by using the (unique) representing vectors in the "natural" (self-dual) cones of $\mathfrak{R}$ (at least when $\mathfrak{R}$ has a separating and generating vector) [4, 5, 8, 13, 18]. (Our thanks are due to Richard Herman for recalling to us the relevance of the Bures Distance Function for these problems.)

For our use (see Theorem 4.1), we will want the representation of $\mathfrak{R}$ and the vector representing one of the normal states given. This introduces special problems of "fitting" that are dealt with by the detailed comparison theory of projections in Theorem 2.3. Araki treats this problem [1, Theorem 3]. Our Proposition 2.2, used in the proof of Theorem 2.3 is similar to but not as strong as [1, Theorem 4 and Remark] (our constant, $\sqrt{2}$, is replaced by 1 and it suffices to assume that ω and ω' are vector states of $\mathfrak{R}$ in [1]). Our proof is correspondingly simpler (we use the Radon-Nikodým derivative in $\mathfrak{R}'$ in the situation where $\omega_0\leqq\omega$).

The first question, of course, is when a normal state of a von Neumann algebra is a vector state. There are circumstances under which each normal state of $\mathfrak{R}$ is a vector state. This is the case if $\mathfrak{R}$ has a separating vector (cf. [17, Theorem 7.2.3]). It is also the case for $\mathfrak{R}$ represented in its "universal normal representation" (cf. [17, Lemma 7.1.6]). In general a normal state of $\mathfrak{R}$ is a vector state if and only if its support projection is cyclic in $\mathfrak{R}$ (cf. [17, Proposition 7.2.7]). The proposition that follows records a necessary and sufficient condition for each normal state of $\mathfrak{R}$ to be a vector state.

2.1. Proposition. *Each normal state of a von Neumann algebra $\mathfrak{R}$ acting on $\mathscr{H}$ is a vector state if and only if each countably-decomposable projection in $\mathfrak{R}$ is cyclic.*

Proof. The support of each normal state of $\mathfrak{R}$ is countably decomposable. Conversely, if E is countably decomposable in $\mathfrak{R}$, it is the sum of a countable family $\{E_1,E_2,\dots\}$ of cyclic projections E_n in $\mathfrak{R}$. With x_n a generating vector of norm $2^{-n/2}$ for E_n, $\sum_{n=1}^{\infty}\omega_{x_n}|\mathfrak{R}$ is a normal state ω of $\mathfrak{R}$ whose support is E. In fact, $\omega(E)=\sum_{n=1}^{\infty}2^{-n}=1$; and if $\omega(E_0)=0$ for some subprojection E_0 of E, then $E_0x_n=0$, so that $E_0E_n=0$, for all n. Hence $0=E_0\left(\sum_{n=1}^{\infty}E_n\right)=E_0E=E_0$.

We have identified the set of countably-decomposable projections in $\mathfrak{R}$ with the set of support projections of normal states of $\mathfrak{R}$. Our result follows, now, from [17, Proposition 7.2.7]. □

In case $\mathfrak{R}$ acts on a separable Hilbert space $\mathscr{H}$, each projection in $\mathfrak{R}$ is countably decomposable, and the requirement that each such projection be cyclic reduces to the assumption that I be cyclic in $\mathfrak{R}$-equivalently, that $\mathfrak{R}$ have a separating vector. In any event, the family of vector states of $\mathfrak{R}$ is a norm-closed subset of the (continuous) dual of $\mathfrak{R}$ [15, Theorem D] (see, also, [17, Theorem 7.3.11]).

In [12] it is proved that if ϱ and τ are pure states of a C^*-algebra $\mathfrak{A}$ and $\|\varrho-\tau\|<2$ then π_ϱ and π_τ are unitarily equivalent representations of $\mathfrak{A}$ and ϱ and τ correspond to unit vectors in the representation space of some irreducible representation of $\mathfrak{A}$. If $\mathfrak{A}$ acts irreducibly on $\mathscr{H}$, it is a consequence of [11, Lemma 3.2] that, with x and y unit vectors in $\mathscr{H}$,

$$\begin{aligned}1-|\langle x,y\rangle|^2 &\leqq \|\omega_x|\mathfrak{A}-\omega_y|\mathfrak{A}\| \leqq \inf\{2\|ax-y\| : |a|=1\}\\ &\leqq 2^{3/2}(1-|\langle x,y\rangle|^2)^{1/2}.\end{aligned}$$

(These are proved simply and are entirely sufficient for Glimm's purposes.) In [19, Lemma 2.4], Powers and Størmer develop the precise formula:

$$\|\omega_x|\mathfrak{A}-\omega_y|\mathfrak{A}\| = 2[1-|\langle x,y\rangle|^2]^{1/2}, \tag{1}$$

from which, for some θ of modulus 1,

$$\|x-\theta y\| = \frac{\|\omega_x-\omega_y\|}{[2(1+|\langle x,y\rangle|)]^{1/2}} \leqq 2^{-1/2}\|\omega_x|\mathfrak{A}-\omega_y|\mathfrak{A}\|.$$

Since $\mathfrak{A}$ acts irreducibly on $\mathscr{H}$, $\mathfrak{A}^- = \mathfrak{B}(\mathscr{H})$; and, from the Kaplansky density theorem, $\|\omega_x|\mathfrak{A}-\omega_y|\mathfrak{A}\| = \|\omega_x-\omega_y\|$. If $\mathscr{H}_0$ is the space spanned by x and y, $\|\omega_x-\omega_y\| = \|(\omega_x-\omega_y)(\mathfrak{B}(\mathscr{H}_0))\|$. Another proof of (1) proceeds as follows. With E and F the one-dimensional projections of $\mathscr{H}_0$ on $[x]$ and $[y]$, $(\omega_x-\omega_y)(A) = \mathrm{tr}((E-F)A)$ for each A in $\mathfrak{B}(\mathscr{H}_0)$, where "tr" denotes the (non-normalized) trace on $\mathfrak{B}(\mathscr{H}_0)$. Clearly, then, $\|\omega_x-\omega_y\| = \mathrm{tr}(|E-F|)$. Since $(E-F)^2$ commutes with both E and F, x and y are eigenvectors for $(E-F)^2$. We may assume that $\langle x,y\rangle \neq 0$ [for (1) is immediate when $\langle x,y\rangle = 0$], in which case, x and y must correspond to the same eigenvalue for $(E-F)^2$ (a self-adjoint operator). Thus $(E-F)^2$ is a scalar. But $\langle x,(E-F)^2x\rangle = \langle x, x-EFx\rangle = 1-|\langle x,y\rangle|^2$; from which (1) follows.

2.2. Proposition. *If ω and ω' are normal positive linear functionals on a von Neumann algebra $\mathfrak{R}$ acting on a Hilbert space $\mathscr{H}$ and the union of the support projections of ω and ω' is a cyclic projection E in $\mathfrak{R}$ then there are vectors x and x' in $\mathscr{H}$ such that $\omega_x|\mathfrak{R}=\omega$, $\omega_{x'}|\mathfrak{R}=\omega'$, and*

$$\|x-x'\| \leqq [2\|\omega-\omega'\|]^{1/2}.$$

Proof. From [17, Theorem 7.4.7], $\omega-\omega'=\omega_1-\omega_2$, where ω_1 and ω_2 are normal positive linear functionals on $\mathfrak{R}$ with orthogonal supports such that $\|\omega_1\|+\|\omega_2\| = \omega_1(I)+\omega_2(I) = \|\omega-\omega'\|$. The supports of ω_1 and ω_2 are contained in E as is that of $\omega+\omega_2$ $(=\omega'+\omega_1)$; so that all these supports are cyclic, by assumption. From

[17, Proposition 7.2.7], there is a vector z in $\mathscr{H}$ such that $\omega_z|\mathfrak{R}=\omega+\omega_2=\omega'+\omega_1$. Since $\omega\leqq\omega_z|\mathfrak{R}$, there is an operator H' in $\mathfrak{R}'$ such that $0\leqq H'\leqq I$ and $\omega=\omega_{H'z}|\mathfrak{R}$. It follows that $(I-H')^2\leqq I-H'^2$ so that

$$\begin{aligned}\|z-H'z\|^2&=\langle(I-H')^2z,z\rangle\leqq\langle(I-H'^2)z,z\rangle\\&=\langle z,z\rangle-\langle H'z,H'z\rangle=(\omega_z-\omega_{H'z})(I)=\|\omega_2\|\,.\end{aligned}$$

Similarly there is a K' in $\mathfrak{R}'$ such that, $0\leqq K'\leqq I$, $\omega'=\omega_{K'z}|\mathfrak{R}$, and $\|K'z-z\|^2\leqq\|\omega_1\|$. Hence

$$\|K'z-z\|^2+\|z-H'z\|^2\leqq\|\omega_1\|+\|\omega_2\|=\|\omega-\omega'\|$$

and

$$\|K'z-H'z\|\leqq\|K'z-z\|+\|z-H'z\|\leqq[2\|\omega-\omega'\|]^{1/2}\,.$$

The proof is completed by choosing $H'z$ as x and $K'z$ as x' in our assertion. □

2.3. Theorem. *If ω is a normal positive linear functional on a von Neumann algebra $\mathfrak{R}$, v is a vector in $\mathscr{H}$, the Hilbert space on which $\mathfrak{R}$ acts, and $E\vee M$, the union of the support projection E of $\omega_v|\mathfrak{R}$ and the support projection M of ω, is cyclic, then there is a vector u in $\mathscr{H}$ such that $\omega=\omega_u|\mathfrak{R}$ and*

$$\|u-v\|\leqq 2\|\omega-\omega_v|\mathfrak{R}\|^{1/2}\,.$$

Proof. If we use $\omega_v|\mathfrak{R}$ in place of ω', the hypotheses of Proposition 2.2 are fulfilled; and there are vectors x and x' in $\mathscr{H}$ such that $\omega_x|\mathfrak{R}=\omega$, $\omega'_x|\mathfrak{R}=\omega_v|\mathfrak{R}$ and

$$\|x-x'\|\leqq[2\|\omega-\omega_v|\mathfrak{R}\|]^{1/2}\,.$$

When $\omega=\omega_v|\mathfrak{R}$, our theorem is proved by choosing v for u. We may assume that $\omega\neq\omega_v|\mathfrak{R}$ so that $[2\|\omega-\omega_v|\mathfrak{R}\|]^{1/2}<2\|\omega-\omega_v|\mathfrak{R}\|^{1/2}$. Since $\omega_{x'}|\mathfrak{R}=\omega_v|\mathfrak{R}$, the mapping $Ax'\to Av$ $(A\in\mathfrak{R})$ extends to a partial isometry V' in $\mathfrak{R}'$ with initial space $[\mathfrak{R}x']$ and final space $[\mathfrak{R}v]$. If V' can be extended to an isometry W' (in $\mathfrak{R}'$), then $W'x'=v$, $\omega_{w'x}|\mathfrak{R}=\omega_x|\mathfrak{R}=\omega$, and $\|W'x-v\|=\|W'x-W'x'\|=\|x-x'\|\leqq[2\|\omega-\omega_v|\mathfrak{R}\|]^{1/2}$. Thus $W'x$ will serve as u.

Now $[\mathfrak{R}x']$ and $[\mathfrak{R}v]$ are equivalent subspaces relative to $\mathfrak{R}'$. If E' is the projection (in $\mathfrak{R}'$) with range $[\mathfrak{R}v]$, from [17, Proposition 6.3.7], there is a (unique) central projection P in $\mathfrak{R}'$ such that, $P\leqq C_{E'}$, PE' is either properly infinite or 0 (in which case, P must be 0) and $(I-P)E'$ is finite. If either PE' is 0 or both PE' and $P(I-E')$ are properly infinite there is an isometric extension W' of V' with the properties employed in the preceding paragraph. It remains to deal with the case where PE' is properly infinite and $P(I-E')$ is not.

In any event, with PE' properly infinite, there is a countable (orthogonal) family $\{PE'_n\}$ of projections equivalent to PE' with sum PE'. (See the proof of [17, Theorem 6.3.4] for the construction of $\{PE'_n\}$.) Given a positive ε, we can choose n' $(\neq 1)$ such that $\|PE'_{n'}v\|<\varepsilon$ whence $\|v'-v\|<\varepsilon$, where $v'=(I-P)v+\left(\sum_{n\neq n'}PE'_n\right)v$. The range of $(I-P)E'+(PE'-PE'_{n'})$ $(=F')$ is $[\mathfrak{R}v']$ and $C_{F'}=C_{E'}$. Moreover, P is, again, the (unique) central subprojection of $C_{F'}$ such that (in this case) PF' is properly infinite and $(I-P)F'$ is finite. In addition $P(I-F')$ $(\geqq PE'_{n'})$ is properly infinite. Thus we are in the situation in which we can assert the existence of a vector u such that $\omega_u|\mathfrak{R}=\omega$ and

$$\begin{aligned} &\|u-v'\| \leqq [2\|\omega-\omega_{v'}|\mathfrak{R}\|]^{1/2}, \\ &\|u-v\| \leqq \|u-v'\| + \|v'-v\| \leqq \|u-v'\| + \varepsilon \leqq [2\|\omega-\omega_{v'}|\mathfrak{R}\|]^{1/2} + \varepsilon, \end{aligned} \tag{2}$$

once we note that $M \vee F$ is cyclic in $\mathfrak{R}$, where F is the support of $\omega_{v'}|\mathfrak{R}$. To see this last, note that $E(\mathscr{H})=[\mathfrak{R}'v]$ and $F(\mathscr{H})=[\mathfrak{R}'v']$ (from [17, Remark 7.2.6]); so that $(I-P)E=(I-P)F$. Since $(I-P)(F \vee M)=(I-P)F \vee (I-P)M=(I-P)(E \vee M)$ and $E \vee M$ is cyclic in $\mathfrak{R}$, by hypothesis; $(I-P)(F \vee M)$ is cyclic. At the same time, $P(F \vee M)$ has PF and, hence, PE_1 as subprojections, where $PE_1(\mathscr{H})=[\mathfrak{R}'PE_1'v]$. Now PE_1 is properly infinite with central carrier P relative to $\mathfrak{R}$ since the same is true of PE_1' relative to $\mathfrak{R}'$ and $PE_1'(\mathscr{H})=[\mathfrak{R}PE_1'v]$ (cf. [17, Proposition 9.1.2]). As F and M are countably decomposable projections in $\mathfrak{R}$, $F \vee M$ and $P(F \vee M)$ are countably decomposable projections in $\mathfrak{R}$. Thus, from [17, Theorem 6.3.4], $P(F \vee M) \sim PE_1$; and $P(F \vee M)$ is cyclic. It follows that $M \vee F$ is cyclic. Using

$$\|\omega_v|\mathfrak{R}-\omega_{v'}|\mathfrak{R}\| \leqq (\|v\| + \|v'\|)\|v-v'\|$$

and (2), we have

$$\|u-v\| \leqq [2\|\omega-\omega_v|\mathfrak{R}\|]^{1/2} + \varepsilon' < 2\|\omega-\omega_v|\mathfrak{R}\|^{1/2}$$

when ε is chosen suitably small. □

If we use the stronger Theorem 4 (and "Remark" following it) of [1] in place of our Proposition 2.2, we obtain the following stronger version of Theorem 2.3.

2.4. Theorem. *If ω is a normal positive linear functional on a von Neumann algebra $\mathfrak{R}$ with cyclic support projection and $a>1$, there is a vector u in $\mathscr{H}$ such that $\omega=\omega_u|\mathfrak{R}$ and*

$$\|u-v\| \leqq a\|\omega-\omega_v|\mathfrak{R}\|^{1/2}.$$

The proof of this strengthened version of Theorem 2.3 replaces the first sentence of that proof by a direct application of [1, Theorem 4 and Remark] (the inequality there replaces the factor "2" by "1"). When $\omega \neq \omega_v|\mathfrak{R}$, we have $\|\omega-\omega_v|\mathfrak{R}\|^{1/2} < a\|\omega-\omega_v|\mathfrak{R}\|^{1/2}$. If V' extends to an isometry $\|W'x'-v\| \leqq \|\omega-\omega_v|\mathfrak{R}\|^{1/2}$. In (2), we replace the factor "2" by "1", again, and apply [1, Theorem 4 and Remark] so that there is no further need to prove that $M \vee F$ is cyclic in $\mathfrak{R}$. We may now proceed to the last sentence of that proof (and replace the factors "2" of the last inequality by "1" and "a", respectively).

3. Supports of States

3.1. Lemma. *If $\mathfrak{R}$ is a von Neumann algebra acting on the Hilbert space $\mathscr{H}$ and $\{x_n\}$ is a sequence of unit vectors in $\mathscr{H}$ tending in norm to x_0 then $\{E_nE_0\}$ and $\{P_nP_0\}$ tend in the strong-operator topology to E_0 and P_0, respectively, where E_n and P_n are the support and central support projections of $\omega_{x_n}|\mathfrak{R}$, respectively.*

Proof. The set consisting of vectors of the form $A'x_0$ with A' in $\mathfrak{R}'$ and vectors y orthogonal to the range of E_0 is a total set. Of course $E_nE_0y=0\to 0=E_0y$. Now $E_0A'x_0=A'x_0$ and $E_nA'x_n=A'x_n$, so that

$$\|E_nE_0A'x_0-E_0A'x_0\|=\|E_nA'x_0-A'x_0\|=\|(I-E_n)A'x_0\|$$
$$=\|(I-E_n)(A'x_n-A'x_0)\|\leqq\|A'\|\ \|x_n-x_0\|\to 0\,.$$

Similarly, with A in $\mathfrak{R}$, $P_0AA'x_0=AA'x_0$ and $P_nAA'x_n=AA'x_n$; so that

$$\|P_nP_0AA'x_0-P_0AA'x_0\|=\|(P_nAA'-AA')x_0\|$$
$$=\|(I-P_n)(AA'x_n-AA'x_0)\|\leqq\|AA'\|\ \|x_n-x_0\|\to 0\,.$$

Thus $\{E_nE_0\}$ and $\{P_nP_0\}$ are strong-operator convergent to E_0 and P_0, respectively. □

As an immediate application of Theorem 2.4 and the preceding lemma, we have:

3.2. Theorem. *If $\mathfrak{R}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and $\{\omega_n\}$ is a sequence of vector states (not normalized) of $\mathfrak{R}$ converging in norm to ω_0 then $\{E_nE_0\}$ and $\{P_nP_0\}$ are strong-operator convergent to E_0 and P_0, respectively, where E_n is the support and P_n is the central support of ω_n.*

Proof. From Theorem 2.4, we can choose vectors x_n in $\mathscr{H}$ such that $\omega_{x_n}|\mathfrak{R}=\omega_{x_n}$ and $\|x_n-x_0\|\to 0$ as $n\to\infty$. Our theorem follows, now, from Lemma 3.1. □

For the proof of the theorem that follows, we can apply the theorem above; however the simpler Theorem 2.3 is applicable in this case.

3.3. Theorem. *If $\{\varrho_n\}$ is a sequence of states of a C^*-algebra $\mathfrak{A}$ tending in norm to the state ϱ_0 then $\{P_nP_0\}$ is strong-operator convergent to P_0, where P_n is the universal central support of ϱ_n, $n=0,1,2,\dots$.*

Proof. Let ψ be the universal representation of $\mathfrak{A}$ on $\mathscr{H}_u$. From [17, Proposition 10.1.1], $\varrho_n\circ\psi^{-1}$ has a (unique) normal extension $\bar{\varrho}_n$ from $\psi(\mathfrak{A})$ to $\psi(\mathfrak{A})^-$ and $\|\bar{\varrho}_n-\bar{\varrho}_0\|=\|\varrho_n\circ\psi^{-1}-\varrho_0\circ\psi^{-1}\|=\|\varrho_n-\varrho_0\|\to 0$. Choose x_0 in $\mathscr{H}_u$ so that $\bar{\varrho}_0=\omega_{x_0}|\psi(\mathfrak{A})^-$. Since each normal state of $\psi(\mathfrak{A})^-$ is a vector state, it follows from Proposition 2.1 that each countably-decomposable projection in $\psi(\mathfrak{A})^-$ is cyclic; and Theorem 2.3 applies. For each n, there is a (unit) vector x_n in $\mathscr{H}_u$ such that $\bar{\varrho}_n=\omega_{x_n}|\psi(\mathfrak{A})^-$ and

$$\|x_n-x_0\|\leqq 2\|\bar{\varrho}_n-\bar{\varrho}_0\|^{1/2}\to 0\,.$$

The central support of $\bar{\varrho}_n$ is P_n. From Lemma 3.1, $\{P_nP_0\}$ is strong-operator convergent to P_0. □

In [10, Theorem 1] Dell'Antonio shows that each weak* convergent *sequence* of states of a type I von Neumann algebra with totally-atomic center (that is, a von Neumann algebra generated by its minimal projections) is norm convergent. By combining this result with Theorem 3.3, we have:

3.4. Corollary. *If $\{\omega_n\}$ is a sequence of normal states of a type I von Neumann algebra $\mathfrak{R}$ with totally-atomic center that is weak* convergent to ω_0 then $\{E_nE_0\}$ and $\{P_nP_0\}$ are strong-operator convergent to E_0 and P_0, respectively, where E_n is the support and P_n is the central support of ω.*

In [10, Theorems 2 and 3] Dell'Antonio shows that for von Neumann algebras with a "continuous" center and for factors of type II there is a weak* convergent sequence of states that is not norm convergent. He conjectures the same for factors of type III, and this was proved recently in [9, Corollary 9].

4. Properties of Limit States

4.1. Theorem. *If ψ is the universal representation of a C^*-algebra $\mathfrak{A}$ and P_n, P_{c_1}, P_{c_∞}, P_∞ are the central projections corresponding to the summands of $\psi(\mathfrak{A})^-$ of types I_n, II_1, II_∞, and III, respectively, then a state ϱ of $\mathfrak{A}$ is of type I_n, II_1, II_∞, or III, if and only if $\varrho\circ\psi^{-1}$ has a (unique) ultraweakly-continuous (normal) extension $\bar{\varrho}$ to $\psi(\mathfrak{A})^-$ such that, respectively, $\bar{\varrho}(P_n)$, $\bar{\varrho}(P_{c_1})$, $\bar{\varrho}(P_{c_\infty})$, or $\bar{\varrho}(P_\infty)$ is 1. The set of states of $\mathfrak{A}$ of a given type is a norm-closed convex subset of $\mathfrak{A}^\#$.*

Proof. From the definition of ψ, there is a projection E' in $\psi(\mathfrak{A})'$ such that $\psi_{E'}: A\to\psi(A)E'$ $(A\in\mathfrak{A})$ is unitarily equivalent to π_ϱ. Now $\psi(A)E'\to\psi(A)C_{E'}$ extends to a * isomorphism of $\psi(\mathfrak{A})^-E'$ onto $\psi(\mathfrak{A})^-C_{E'}$ (cf. [17, Proposition 5.5.5]). It follows that ϱ is of a given type if and only if $C_{E'}$ is a subprojection of the central projection corresponding to the summand of $\psi(\mathfrak{A})^-$ of that type. There is a unit vector y_ϱ in the range of E' (corresponding to x_ϱ through the unitary equivalence of π_ϱ and $\psi_{E'}$) such that $\varrho\circ\psi^{-1}=\omega_{y_\varrho}|\psi(\mathfrak{A})$. Hence $\bar{\varrho}=\omega_{y_\varrho}|\psi(\mathfrak{A})^-$ and $1=\bar{\varrho}(E')\leqq\bar{\varrho}(C_{E'})\leqq\bar{\varrho}(P_b)$, where b is one of n, c_1, c_∞, or ∞. Conversely, if $\bar{\varrho}(P_b)=1$, for b one of n, c_1, c_∞, or ∞, then $P_by_\varrho=y_\varrho$ and $[\psi(\mathfrak{R})'\psi(\mathfrak{R})^-y_\varrho]$, the range of $C_{E'}$ (see [17, Proposition 5.5.2]), is contained in the range of P_b. Consequently the type of ϱ corresponds to that of $\psi(\mathfrak{A})^-P_b$.

The mapping $\eta\to\bar{\eta}$, where η is a continuous linear functional on $\mathfrak{A}$ and $\bar{\eta}$ is the (unique) ultraweakly-continuous extension of $\eta\circ\psi^{-1}$ from $\psi(\mathfrak{A})$ to $\psi(\mathfrak{A})^-$, is a linear isometry of the (continuous) dual of $\mathfrak{A}$ onto the space of ultraweakly-continuous linear functionals on $\psi(\mathfrak{A})^-$ (see [17, Proposition 10.1.1]), the predual of $\psi(\mathfrak{A})^-$. Since the mapping $\eta\to\bar{\eta}$ carries states onto states and the set of normal states of $\psi(\mathfrak{A})^-$ taking the value 1 at P_b is convex and norm closed, the set of states of $\mathfrak{A}$ of a given type (I_n, II_1, II_∞, or III) is a norm-closed convex subset of $\mathscr{S}(\mathfrak{A})$. □

4.2. Remark. The situation described in the preceding theorem is markedly different from that which obtains in the case where the weak* topology is used. If $\mathfrak{A}$ is a norm-separable C^*-algebra the weak* topology on the unit ball of the norm dual of $\mathfrak{A}$ is metrizable so that weak* convergence of states is equivalent to weak* *sequential* convergence. In [16, Definition (2.1)] a *full family* of states of $\mathfrak{A}$ is defined as a convex set S_0 of states such that $A\geqq 0$ if $\varrho(A)\geqq 0$ for each ϱ in S_0, when $A\in\mathfrak{A}$. In [16, Theorem (2.2)] it is proved that a convex set of states is full if and only if it is weak* dense in the set of all states; and it is noted that the weak-operator continuous states of $\mathfrak{A}$ in each faithful representation of $\mathfrak{A}$ is full. If P is the central projection corresponding to the type I summand of $\psi(\mathfrak{A})^-$, then the universal central support of each pure state of $\mathfrak{A}$ is contained in P. Since the pure states separate $\mathfrak{A}$, ψ_P is a faithful representation of $\mathfrak{A}$. Thus the normal states of $\psi_P(\mathfrak{A})$ form a full family. Each such normal state is a vector state arising from a vector in $P(\mathscr{H}_u)$; and is, therefore, a type I state of $\mathfrak{A}$. It follows that the type I states of $\mathfrak{A}$ form a full family – and (with $\mathfrak{A}$ norm separable) *each* state is a weak* limit of a sequence of type I states. In particular the (unique) tracial state of the CAR algebra, a type II_1 state, is such a limit. □

4.3. Proposition. *If ϱ_1 and ϱ_2 are states of a C^*-algebra $\mathfrak{A}$, ψ is the universal representation of $\mathfrak{A}$ and E'_1, E'_2 are projections in $\psi(\mathfrak{A})'$ such that π_{ϱ_1} and π_{ϱ_2} are quasi-equivalent to $\psi_{E'_1}$ and $\psi_{E'_2}$, respectively, then $\varrho_1 \precsim_q \varrho_2$ if and only if $C_{E'_1} \leqq C_{E'_2}$. The set $Q(\mathfrak{A})$ of quasi-equivalence classes of states of $\mathfrak{A}$ is partially ordered by the relation $\precsim_q$. The quasi-equivalence classes corresponding to the factor states are the minimal elements of $Q(\mathfrak{A})$.*

Proof. By definition, $\varrho_1 \precsim_q \varrho_2$ if and only if there is a subprojection E'_0 of E'_2 in $\psi(\mathfrak{A})'$ such that $\psi_{E'_1}$ is quasi-equivalent to $\psi_{E'_0}$; and, from [17, Theorem 10.3.3(ii)], this is the case if and only if $C_{E'_0} = C_{E'_1}$ for some such subprojection E'_0 of E'_2, that is, if and only if $C_{E'_1} \leqq C_{E'_2}$.

It follows from the above that $\varrho_1 \sim_q \varrho_2$ when $\varrho_1 \precsim_q \varrho_2$ and $\varrho_2 \precsim_q \varrho_1$. Hence $\precsim_q$ is a partial ordering of $Q(\mathfrak{A})$.

If ϱ is a state of $\mathfrak{A}$ and E' is a projection in $\psi(\mathfrak{A})'$ such that π_ϱ is quasi-equivalent to $\psi_{E'}$ then ϱ is a factor state of $\mathfrak{A}$ if and only if $\psi(\mathfrak{A})^- E'$ and, hence, $\psi(\mathfrak{A})^- C_{E'}$ are factors. The center of $\psi(\mathfrak{A})^- C_{E'}$ is $\mathscr{C} C_{E'}$, where $\mathscr{C}$ is the center of $\psi(\mathfrak{A})^-$. Thus ϱ is a factor state of $\mathfrak{A}$ if and only if $\mathscr{C} C_{E'} = \{\lambda C_{E'}\}$. From [17, Proposition 6.4.3] this last is the case if and only if $C_{E'}$ is a minimal projection in $\mathscr{C}$. The preceding identification of the partial ordering relation $\precsim_q$ with the (usual) ordering of central support projections makes it clear, now, that (the quasi-equivalence class of) ϱ is a minimal element of $Q(\mathfrak{A})$ if and only if $C_{E'}$ is a minimal projection in $\mathscr{C}$; so that the quasi-equivalence classes of the factor states of $\mathfrak{A}$ are precisely the minimal elements of $Q(\mathfrak{A})$ relative to $\precsim_q$. □

4.4. Corollary. *If $\{\varrho_n\}$ is a sequence of states of a C^*-algebra $\mathfrak{A}$, $\varrho_{n+1} \precsim_q \varrho_n$ for $n = 1, 2, \ldots$, and $\{\varrho_n\}$ tends in norm to ϱ_0, then $\varrho_0 \precsim_q \varrho_n$ for all n.*

Proof. From Theorem 3.3, $\{P_n P_0\}$ converges to P_0 in the strong-operator topology, where P_n is the universal central support of ϱ_n. By assumption and Proposition 4.3, $P_{n+1} \leqq P_n$; so that $\{P_n\}$ converges in the strong-operator topology to $\bigcap P_n$ $(=P)$. Thus $\{P_n P_0\}$ converges to PP_0 and $P_0 = PP_0$. It follows that $P_0 \leqq P \leqq P_n$ for all n. From Proposition 4.3, $\varrho_0 \precsim_q \varrho_n$. □

E. Størmer remarked to us that the preceding corollary can also be proved, without (indirect) reference to Theorem 2.3, as follows. As in the proof of Theorem 3.3, let $\bar{\varrho}_n$ be the (unique) normal extension of $\varrho_n \circ \psi^{-1}$ from $\psi(\mathfrak{A})$ to $\psi(\mathfrak{A})^-$ with $\|\bar{\varrho}_n - \bar{\varrho}_0\| = \|\varrho_n - \varrho_0\| \to 0$. As above $P_{n+1} \leqq P_n$; so that $\bar{\varrho}_n(P_m) \geqq \bar{\varrho}_n(P_n) = 1$, when $n \geqq m$, and $\bar{\varrho}_n(P_m) \to 1$ as $n \to \infty$. But $\bar{\varrho}_n(P_m) \to \bar{\varrho}_0(P_m)$. Thus $\bar{\varrho}_0(P_m) = 1$, $E'_0 \leqq P_m$, and $P_0 \leqq P_m$, for all m.

We noted in Proposition 4.3, that the factor states of $\mathfrak{A}$ correspond to the minimal elements in $Q(\mathfrak{A})$. If we assume, in the preceding corollary, that ϱ_n is a factor state when $n = 1, 2, \ldots$, the condition "$\varrho_{n+1} \precsim_q \varrho_n$" becomes "$\varrho_{n+1} \sim_q \varrho_n$" and the conclusion becomes "$\varrho_0 \sim_q \varrho_n$" for all n. In particular, ϱ_0 is a factor state. Of course each convex combination of quasi-equivalent factor states has as its central support [in $\psi(\mathfrak{A})^-$] the same (minimal) central projection as each of the states of which it is a combination – and is, therefore, a factor state quasi-equivalent to each of these states. Combining these observations, we have a result of Combes [7, Corollaire 2.3]: each minimal element of $Q(\mathfrak{A})$ is a norm-closed, convex subset of the (continuous) dual of $\mathfrak{A}$. Corollary 4.4 is, in reality, a (partial) generalization of the result just quoted. Note that it is not difficult to give a direct proof for [7, Corollaire 2.3]. It is apparent that a norm limit of normal states of $\psi(\mathfrak{A})^-$, each of which has a given minimal central projection as its central support has that minimal central projection as its central support; so that, from [17, Theorem 10.3.3(ii)], the limit is a factor state quasi-equivalent to each of the others. If we add the information just noted to one of the basic observations of [12] (see, also, [17, Corollary 10.3.6]): if π_ϱ and π_τ are disjoint, for states ϱ and τ of $\mathfrak{A}$, then $\|\varrho-\tau\|=2$; we can state the following stronger result.

4.5. Proposition. *If $\{\varrho_n\}$ is a sequence of factor states of the C^*-algebra $\mathfrak{A}$ converging in norm to the state ϱ, then all but a finite number of states of $\{\varrho_n\}$ are quasi-equivalent to ϱ.*

We examine a single example that illustrates several phenomena related to the preceding results:

There is a sequence $\{\varrho_n\}$ of normal states of a von Neumann algebra $\mathfrak{R}$ tending in norm to the (normal) state ϱ, and

(i) the sequence of support projections of $\{\varrho_n\}$ does not converge to the support projection of ϱ in the strong-operator topology; – cf. Lemma 3.1.

(ii) The sequence of central support projections of $\{\varrho_n\}$ does not converge to the central support projection of ϱ in the strong-operator topology; – cf. Lemma 3.1.

(iii) There is a strong-operator-dense C^*-subalgebra $\mathfrak{A}$ of $\mathfrak{R}$ such that the restrictions $\varrho_n|\mathfrak{A}$ are quasi-equivalent (indeed, they give rise to unitarily equivalent representations of $\mathfrak{A}$), but $\varrho|\mathfrak{A}$ is quasi-equivalent to none of them; – cf. Corollary 4.4.

4.6. Example. Let $\mathfrak{R}$ be the (maximal abelian) algebra of operators on the Hilbert space $\mathscr{H}$ whose matrix representation relative to a fixed orthonormal basis $\{e_n\}$ for $\mathscr{H}$ consists of bounded diagonal matrices. Let x_n be $\sum_{m=1}^{\infty} a_{mn}e_m$, where $a_{mn}=\sqrt{6}(\pi mn)^{-1}$ when $m=2,3,\dots$ and $a_{1n}=((n^2-1)\pi^2+6)^{1/2}(\pi n)^{-1}$. With these choices, $\{x_n\}$ is a sequence of unit vectors converging in norm to e_1 and each x_n is generating for $\mathfrak{R}$ (since $a_{mn}\neq 0$ for all m and n). The C^*-subalgebra $\mathfrak{A}$ of $\mathfrak{R}$ consisting of all A in $\mathfrak{R}$ for which $\{\langle Ae_n, e_n\rangle\}$ converges is a strong-operator-dense subalgebra of $\mathfrak{R}$. If ϱ_n is $\omega_{x_n}|\mathfrak{R}$, both the support and central support (since $\mathfrak{R}$ is abelian) projections of ϱ_n have range $[\mathfrak{R}x_n]$ $(=\mathscr{H})$. But $\{\varrho_n\}$ converges in norm to $\omega_{e_1}|\mathfrak{R}$ $(=\varrho_0)$; and the support and central support projections of ϱ_0 have range $[\mathfrak{R}e_1]$, which is $[e_1]$. Of course the sequence of projections each of which is I does not converge in the strong-operator topology to the projection E_1 with one-dimensional range $[e_1]$ – which illustrates (i) and (ii).

Each of the representations $\pi_{\varrho_1'}, \pi_{\varrho_2'}, \dots$ is unitarily equivalent to the given representation of $\mathfrak{A}$ on $\mathscr{H}$, where $\varrho_n'=\omega_{x_n}|\mathfrak{A}$, since each x_n is generating for $\mathfrak{A}$ (cf. [17, Proposition 4.5.3]); but $\pi_{\varrho_0'}$ is one-dimensional, where ϱ_0' is the multiplicative linear functional, $\omega_{e_1}|\mathfrak{A}$, and ϱ_0' is quasi-equivalent to none of $\{\varrho_n'\}$ – which illustrates (iii). □

4.7. Proposition. *The norm limit of a sequence of equivalent states of a C^*-algebra is subequivalent to these states.*

Proof. If $\mathfrak{A}$ is the C^*-algebra, $\{\varrho_n\}$ the sequence of equivalent states of $\mathfrak{A}$, and ϱ_0 is the norm limit of $\{\varrho_n\}$, then ϱ_0 annihilates the (common) kernel of π_{ϱ_n}. With π a representation of $\mathfrak{A}$ on $\mathscr{H}$ unitarily equivalent to each π_{ϱ_n}, there are unit vectors x_n in $\mathscr{H}$ such that $\varrho_n = \omega_{x_n} \circ \pi$, and there is a linear functional ϱ on $\pi(\mathfrak{A})$ such that $\varrho_0 = \varrho \circ \pi$. Since π maps the unit ball and the set of positive operators in $\mathfrak{A}$ onto the corresponding sets in $\pi(\mathfrak{A})$; ϱ is a state of $\pi(\mathfrak{A})$ and $\|\varrho_0 - \varrho_n\| = \|\varrho - \omega_{x_n}|\pi(\mathfrak{A})\|$. From [15, Theorem D], ϱ is a vector state, $\omega_x|\pi(\mathfrak{A})$, of $\pi(\mathfrak{A})$. If E' is the projection in $\pi(\mathfrak{A})'$ with range $[\pi(\mathfrak{A})x]$, then $\pi_{E'}$ is unitarily equivalent to the representation of $\mathfrak{A}$ arising from the GNS construction applied to $\omega_x \circ \pi$ $(=\varrho_0)$, that is, to π_{ϱ_0}. Thus ϱ_0 is subequivalent to each ϱ_n. □

4.8. Corollary. *A norm limit of a sequence of pure states of a C^*-algebra is a pure state unitarily equivalent to all but a finite number of them.*

Proof. If $\mathfrak{A}$ is the C^*-algebra and $\{\varrho_n\}$ is the sequence of pure states of $\mathfrak{A}$ converging in norm to ϱ_0, there is an n_0 such that $\|\varrho_n - \varrho_m\| < 2$ when $n_0 \leqq n$ and $n_0 \leqq m$. It follows from [12, Corollary 9] that all ϱ_n are equivalent when $n_0 \leqq n$. Proposition 4.7 applies, and ϱ_0 is subequivalent to each ϱ_n when $n_0 \leqq n$. Since each π_{ϱ_n} is irreducible, π_{ϱ_0} is equivalent to π_{ϱ_n} when $n_0 \leqq n$; so that π_{ϱ_0} is irreducible and ϱ_0 is pure. □

The formula (1) of Sect. 2 can be used to give a proof of the preceding corollary without appeal to Proposition 4.7 (and [15, Theorem D]).

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