Colloques Internationaux C.N.R.S.
N° 274 – ALGÈBRES D'OPÉRATEURS ET LEURS APPLICATIONS EN PHYSIQUE MATHÉMATIQUE

UNBOUNDED SIMILARITY

Richard V. KADISON *
Department of Mathematics
University of Pennsylvania, Philadelphia

ABSTRACT

The main theorem of Tomita-Takesaki theory is viewed as a consequence of the existence of an unbounded similarity between a dense self-adjoint subalgebra of a von Neumann algebra and a dense self-adjoint subalgebra of its commutant. This aspect of the theory is extended to unbounded similarities between arbitrary self-adjoint operator algebras.

RESUMÉ

Le théorème principal de la théorie de Tomita-Takesaki est vu comme conséquence de l'existence d'une similarité non bornée entre une sous * algèbre dense d'une algèbre de Von Neumann et une sous * algèore dense de son commutant. Cet aspect de la théorie est étendu aux similarités non bornées entre des * algèbres arbitraires d'opérateurs.

*With partial support of the National Science Foundation (USA).

I. Introduction

If A and B are self-adjoint (everywhere-defined) bounded operators acting on complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and T is a bounded linear transformation of $\mathcal{H}$ onto $\mathcal{K}$ with bounded inverse such that $TAT^{-1} = B$ (so that A and B are <u>similar</u>) then A and B are unitarily equivalent. The polar decomposition UH of T provides us with a positive operator $H(= (T^*T)^{\frac{1}{2}})$ and a unitary operator $U(= T(T^*T)^{-\frac{1}{2}})$ such that $T = UH$. Then $B = TAT^{-1} = UHAH^{-1}U^{-1}$ and $HAH^{-1} = U^{-1}BU = (U^{-1}BU)^* = H^{-1}AH$ so that $H^2A = AH^2$. Now $H(=(H^2)^{\frac{1}{2}})$ is a norm limit of polynomials in H^2 (each of which commutes with B); so that $HAH^{-1} = A$ and $B = UAU^{-1}$. With some cautious interpretation of the meaning of TAT^{-1}, we can allow T and T^{-1} to be closed, densely-defined operators in this discussion.

Replacing A and B by C*-algebras, in the preceding argument, we conclude that $H^2\mathfrak{A}H^{-2} = \mathfrak{A}$ (as sets). It is not quite as simple, now, to deduce that $H\mathfrak{A}H^{-1} = \mathfrak{A}$ (for the polynomial approximation to $(H^2)^{\frac{1}{2}}$ is not effective in this situation); but that conclusion is, nevertheless, valid. The Gardner Invariance Theorem [1; Theorem 3.5] supplies what is needed. (Basically, the Cauchy Integral Formula for the square-root function is applied to the mapping $A \to H^2AH^{-2}$ of $\mathfrak{A}$ into itself.) As before, it follows that $U\mathfrak{A}U^{-1} = \mathfrak{B}$ (and that $A \to H^{it}AH^{-it}$ is a *-automorphism of $\mathfrak{A}$). There is a clear relation between this bounded similarity result and the interesting results of T. Okayasu [4, 5, 7; p. 170] on polar decompositions of automorphisms and isomorphisms between C*-algebras.

We describe in this note, extensions of this (bounded) similarity result for C*-algebras to unbounded similarities of C*-algebras and self-adjoint operator algebras. In the first case, we will note that the unbounded similarity of the full C*-algebras may just as well have been effected by a bounded similarity.

The situation of unbounded similarities between self-adjoint operator algebras arises in connection with the Tomita-Takesaki theory [10, 11]. To describe this, let R be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and let x_0 be a separating and generating vector for R. As is familiar in the Tomita-Takesaki theory, define conjugate-linear operators S_0 and F_0 mapping Rx_0 and $R'x_0$ onto themselves by: $S_0Ax_0 = A^*x_0$, $F_0A'x_0 = A'^*x_0$. A simple computation shows that $R'x_0$ is in the domain of S_0^*, Rx_0 is in the domain of F_0^*, S_0^* extends F_0, and F_0^* extends S_0. It follows that S_0 and F_0 have closures S and F that are adjoints of one another. With $J\Delta^{\frac{1}{2}}$ the polar decomposition of S (where $\Delta = S^*S = FS$), Δ is a positive (in general, unbounded) operator on $\mathcal{H}$ and J is a conjugate-linear isometry of $\mathcal{H}$ onto $\mathcal{H}$. Since $S^2 = I$; $J^2 = I$ and J is an <u>involution</u>. The main result of Tomita-Takesaki theory is expressed in the:

THEOREM. $JRJ = R'$ <u>and</u> $\Delta^{it}R\,\Delta^{-it} = R$.

To see how this fits into the similarity pattern, note that:

$$SASBCx_0 = SAC^*B^*x_0 = BCA^*x_0 = BSASCx_0.$$

Loosely speaking, then, SAS commutes with each B in R and "lies in" R'. In effect, $S(= S^{-1})$ tends to set up a similarity between R and R'; and the conclusions noted in the theorem above are those that we derive from our similarity results. In order to apply these results in a rigorous manner, to recapture that theorem, we must account for the discrepancies associated with the conjugate-linearity of S and the fact that SAS is not bounded, in general (and certainly not in R', in that event). By introducing the conjugate Hilbert space to $\mathcal{H}$, we can convert consideration of conjugate-linear operators to that of linear operators. A less simple (but well-understood) matter is that of finding an appropriate self-adjoint subalgebra $\mathfrak{A}_0$ (dense in R) on which the similarity induced by S is "well-behaved" (in particular, on which the resulting operators are bounded). A convenient route to these "Tomita subalgebras" of R and R' can be constructed from portions of the elegant proofs of Haagerup [2] and Zsido [12] for the theorem stated above.

While the similarity view of the Tomita-Takesaki theory gives clear motivation to each step of the argument, it does not reduce, in a significant way, the amount of work required to prove the main theorem. There is still the need to locate the "well-behaved" subalgebras of R and R' and then the work involved in proving the unbounded similarity result. But the point to the similarity study is not to redo the Tomita-Takesaki theory. Considering the importance of this theory in the recent development of the subject, we have tried to isolate the basic features of the theory and free them from the restrictions of algebra and commutant and a distinguished state (or weight). In our view, these basic features are the structural relation between R and R' expressed in the relation $JRJ = R'$ (i.e. $A \to (JAJ)^*$ is a *-anti-isomorphism between R and R') and the one-parameter group of modular automorphisms of R (i.e. the automorphisms $A \to \Delta^{it} A \Delta^{-it}$ of R). As we see them, these structural features arise from the "coupling" of R and R' through the "similarity" effected by S (associated with the choice of separating and generating vector x_0 for R). The unbounded similarity theorem we describe extracts the same features from algebras $\mathfrak{A}$ and $\mathfrak{B}$ coupled by a similarity.

If the algebras $\mathfrak{A}$ and $\mathfrak{B}$ represent physical systems, the coupling effected by the similarity can be that resulting from the physical interaction of the systems. The one-parameter group of automorphisms on $\mathfrak{A}''$ resulting from the similarity expresses the dynamical effect that the interaction between $\mathfrak{A}$ and $\mathfrak{B}$ has on $\mathfrak{A}$. (A unitary coupling between $\mathfrak{A}$ and $\mathfrak{B}$ would correspond to trivial interaction.)

II. The similarity results

The first problem that immediately presents itself in the similarity study is the meaning of, $T\mathfrak{A}T^{-1} = \mathfrak{B}$, when T is unbounded. We restrict attention to the case where T is closed and densely defined. Roughly this should mean that TAT^{-1} has a bounded extension in $\mathfrak{B}$ for each A in $\mathfrak{A}$, and each B in $\mathfrak{B}$ is such an extension. But we have various options in our requirement for the domain of TAT^{-1}. Shall it vary with A; and can it be an arbitrary dense linear manifold in $D(T^{-1})$, the domain of T^{-1}? Since the operators involved are bounded (though not everywhere defined), it might seem that it does not matter which (reasonable) version of ' $T\mathfrak{A}T^{-1} = \mathfrak{B}$ ' we adopt - they will all suffice. This is not the case, as we illustrate in the example that follows.

EXAMPLE. Let $\mathcal{H}$ be a Hilbert space and (e_n) be an orthonormal basis for $\mathcal{H}$. Let T^{-1} be the operator that assigns $\sum_{n=1}^{\infty} n\lambda_n e_n$ to $\sum_{n=1}^{\infty} \lambda_n e_n$, with domain $\{\sum_{n=1}^{\infty} \lambda_n e_n : \sum_{n=1}^{\infty} n^2 |\lambda_n|^2 < \infty\}$. Then T^{-1} is self-adjoint. Let E_0 be the one-dimensional projection with range generated by $\sum_{n=1}^{\infty} n^{-1} e_n$. Let D_0 be the set of those vectors in $D(T^{-1})$ such that $\sum_{n=1}^{\infty} \lambda_n = 0$ (so that D_0 is a linear space). We prove that D_0 is dense by showing that we can approximate each e_n in norm as closely as we wish by an element of D_0. Note, for this, that $e_n - \sum_{j=1}^{m} m^{-1} e_{n+j}$ $(= x_m)$ lies in D_0 and that $\|e_n - x_m\|^2 = m^{-1}$. Since $(T^{-1}x, x_0) = 0$ for each x in D_0; $E_0 T^{-1}|D_0$ is 0. It follows that $(aE_0 + bI)T^{-1}|D_0 = bT^{-1}|D_0$; so that $T(aE_0 + bI)T^{-1}|D_0 = bI|D_0$ for all scalars a and b. If $\mathfrak{A}$ is the (two-dimensional) C*-algebra generated by E_0 and I and $\mathcal{B}$ is the algebra of scalar multiples of I, then $T\mathfrak{A}T^{-1} = \mathcal{B}$ (in the sense that there is a dense linear submanifold D_0 of $D(T^{-1})$ such that $TAT^{-1}|D_0$ has a bounded extension to $\mathcal{H}$ in $\mathcal{B}$ and each B in $\mathcal{B}$ is such an extension) - but $\mathfrak{A}$ and $\mathcal{B}$ are not unitarily equivalent (not even isomorphic).

In the preceding example, D_0 is not a <u>core</u> for T^{-1} (i.e. the restriction of T^{-1} to D_0 does not have closure T^{-1}). If we impose this restriction on D_0 the unbounded similarity theory yields the desired results. Thus when we write '$T\mathfrak{A}T^{-1} = \mathcal{B}$' with $\mathfrak{A}$ and $\mathcal{B}$ self-adjoint operator algebras; we shall mean that T is closed and densely-defined, that there is a core D_0 for T^{-1} such that $TAT^{-1}|D_0$ has a unique bounded extension in $\mathcal{B}$ for each A in $\mathfrak{A}$, and each B in $\mathcal{B}$ is such an extension. One of the main results is contained in the:

THEOREM. *If $\mathfrak{A}$ and $\mathcal{B}$ are C*-algebras and $T\mathfrak{A}T^{-1} = \mathcal{B}$ then $U\mathfrak{A}U^{-1} = \mathcal{B}$, H^zAH^{-z} has a (unique) bounded extension in $\mathfrak{A}$ for each complex z, and there is a positive H_0 in $\mathfrak{A}''$ such that $H_0AH_0^{-1}|D(H^{-1}) = HAH^{-1}$ for each A in $\mathfrak{A}$, where UH is the polar decomposition of T.*

In proving this theorem, crucial use is made of the fact that $\mathfrak{A}$ and $\mathcal{B}$ are C*-algebras, in particular, Banach spaces, so that the Uniform Boundedness Principle applies to the restrictions of the mappings $A \to TAT^{-1}$ and $A \to HAH^{-1}$ to finite-interval spectral projections for H. It follows, first, that the latter mapping φ is bounded and, then, that it is an automorphism of $\mathfrak{A}$. Employing the reduced atomic representation and some C*-algebra theory, the spectrum of φ can be shown to be positive. Applying the holomorphic function calculus to φ and noting various relations between functions of φ and those of H, the theorem is established.

The strong (and surprising) conclusion that the dynamical effect of the coupling of C*-algebras through an unbounded similarity corresponds to a bounded Hamiltonian and is weakly inner (H_0 is in $\mathfrak{A}''$) is another aspect of the phenomenon that certain mappings of operator algebras, when defined on the full C*- algebra, must be bounded [6,8] and weakly-inner [3,9]. At the same time this conclusion tells us that we are not studying the true extension of the Tomita-Takesaki theory. From [10], the automorphisms $A \to \Delta^{it} A \Delta^{-it}$ of R, in that theory, are inner if and only if R is semi-finite; so that the theorem just noted extends only the semi-finite theory.

For the full extension of the Tomita-Takesaki theory by means of the unbounded similarity theory, we must deal with self-adjoint operator algebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$ that are not (necessarily) norm closed. In this case a growth condition on $\|H^n A H^{-n}\|$ is needed. The main result is contained in the:

THEOREM. *If $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are self-adjoint operator algebras, $T\mathfrak{A}_0 T^{-1} = \mathfrak{B}_0$, and $\|H^n A H^{-n}\| \leq k_A^{|n|}$ for each integer n and some constant k_A (depending on A), where UH is the polar decomposition of T, then $U\mathfrak{A}_0'' U^{-1} = \mathfrak{B}_0''$ and $H^z A H^{-z}$ has a (unique) bounded extension in $\mathfrak{A}_0''$ for each complex z and each A in $\mathfrak{A}_0$. In particular $t \to H^{it}$ is a strong-operator-continuous, one-parameter unitary group which gives rise to a one-parameter group of *-automorphisms of $\mathfrak{A}_0''$.*

The proof proceeds by "interpolating" the complex powers $H^z A H^{-z}$ (with estimates on the bounds) between the various $H^n A H^{-n}$ by means of (a variant of) the Hadamard Three Circle Theorem and then using a (Phragmen-Lindelof) function theory argument to show that the bounded extension of $H^z A H^{-z}$ lies in $\mathfrak{A}_0''$.

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Department of Mathematics E1
University of Pennsylvania
Philadelphia, Pennsylvania 19104