NOTES ON THE FERMI GAS (*)

RICHARD V. KADISON

The kinematical structure of an infinite system of identical Fermi particles can be described in terms of the CAR algebra, a C*-algebra $\mathfrak{A}$ whose representations are in one-one correspondence with the representations of Canonical Anticommutation Relations (CAR). In this note and the one that follows (by N. M. Hugenholtz), the methods and steps involved in classifying certain automorphisms of $\mathfrak{A}$ are described. The result concerning these automorphisms is contained in the following theorem.

THEOREM 1: *If $\mathfrak{A}$ is the* CAR *algebra based on the complex Hilbert space $\mathcal{H}$ and α is an automorphism of $\mathfrak{A}$ whose transpose $\hat{\alpha}$ maps the set of pure, gauge-invariant, quasi-free states of $\mathfrak{A}$ onto itself, then, either the Fock vacuum state is mapped onto itself by $\hat{\alpha}$ and there is a unitary operator U on $\mathcal{H}$ such that $\hat{\alpha}(a(f)) = a(Uf)$, or the Fock state is mapped onto the anti-Fock state by α and there is a conjugate-linear, unitary operator W on $\mathcal{H}$ such that $\alpha(a(f)) = a(Wf)^*$.*

The study of equilibrium states of infinite Fermi systems motivates this work. Such states can be labeled by one-parameter, automorphism groups of $\mathfrak{A}$ that commute with the dynamical group. Earlier work [1, 2] on asymptotic orbits of states of such systems indicates that, loosely speaking, the primary stationary states are quasi-free. An automorphism commuting with the free, time evolution will map this set of states into itself. What can be said about such automorphisms? A more primitive problem involves the description of those automorphisms which map the set of gauge-invariant, quasi-free states onto itself. The theorem stated above answers this question.

A development of the theory of the CAR algebra in its Fock representation in the framework of the « exterior calculus » and a corresponding development of gauge-invariant, quasi-free states is critical

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for our arguments. The following result (noticed independently by J. E. Roberts—unpublished—we have learned) is a byproduct of this latter development.

PROPOSITION 2: *If T is a linear transformation of the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{K}$ and $\|T\| \leqslant 1$, then the mapping*

$$a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m) \to a(Tf_n)^* \dots a(Tf_1)^* a(Tg_1) \dots a(Tg_m)$$

extends (uniquely) to a completely-positive, linear mapping ψ_T of the CAR *algebra $\mathfrak{A}_{\mathcal{H}}$ over $\mathcal{H}$ into $\mathfrak{A}_{\mathcal{K}}$ the* CAR *algebra over $\mathcal{K}$.*

The program of this note is to present the development of the Fock representation of $\mathfrak{A}$ and of gauge-invariant, quasi-free states of $\mathfrak{A}$ in the exterior-algebra framework. The note that follows, by N. M. Hugenholtz, will illustrate how this development is used in a combinatorial, counting process that compares dimensions of intersections.

With $\mathcal{H}$ a complex Hilbert space and $\mathcal{H}_n$ the n-fold tensor product, so that, for $x_1, \dots, x_n, y_1, \dots, y_n$ in $\mathcal{H}$,

$$\langle x_1 \otimes \dots \otimes x_n | y_1 \otimes \dots \otimes y_n \rangle = \langle x_1 | y_1 \rangle \dots \langle x_n | y_n \rangle \,,$$

let S_n^- be the projection operator on $\mathcal{H}_n$ which assigns

$$\frac{1}{n!} \sum \chi(\sigma) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$$

to $x_1 \otimes \dots \otimes x_n$, where σ is a permutation of $\{1, \dots, n\}$ and $\chi(\sigma)$ is $+1$ if σ is even, -1 if σ is odd. The range of S_n^- is the space $\mathcal{H}_n^{(a)}$ of antisymmetric tensors. We write $x_1 \wedge \dots \wedge x_n$ for $(n!)^{\frac{1}{2}} S_n^-(x_1 \otimes \dots \otimes x_n)$ (the « antisymmetrized, n-particle state with wave functions $x_1, \dots, x_n$»). We have:

$$\begin{aligned}\langle x_1 \wedge \dots \wedge x_n | y_1 \wedge \dots \wedge y_n \rangle &= n! \langle x_1 \otimes \dots \otimes x_n | S_n^-(y_1 \otimes \dots \otimes y_n) \rangle \\ &= \sum_\sigma \chi(\sigma) \langle x_1 | y_{\sigma(1)} \rangle \dots \langle x_n | y_{\sigma(n)} \rangle = \det \left(\langle x_i | y_j \rangle \right) .\end{aligned}$$

Thus, assuming $x_1 \wedge \dots \wedge x_n$ and $y_1 \wedge \dots \wedge y_n$ are not 0, they are orthogonal if and only if there are scalars $c_1, \dots, c_n$, not all 0, such that

$$0 = \sum_{i=1}^{n} \langle c_i x_i | y_j \rangle = \Big\langle \sum_{i=1}^{n} c_i x_i | y_j \Big\rangle ;$$

that is, if and only if the space, $[x_1, \dots, x_n]$, generated by $x_1, \dots, x_n$, contains a non-zero vector $(\sum c_i x_i)$ orthogonal to $[y_1, \dots, y_n]$. If, in addition, the intersection, $[x_1, \dots, x_n] \cap [y_1, \dots, y_n]$, of the spaces $[x_1, \dots, x_n]$ and $[y_1, \dots, y_n]$ has dimension $n-1$ (in this case, we say that the spaces are « perpendicular »), the projections with ranges $[x_1, \dots, x_n]$ and $[y_1, \dots, y_n]$ commute. It follows that $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}$ is an orthonormal basis for $\mathcal{H}_n^{(a)}$ if $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$. Moreover, $x_1 \wedge \dots \wedge x_n = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent (if and only if $[x_1, \dots, x_n]$ has dimension less than n). Thus $z \in [x_1, \dots, x_n]$, if $z \wedge x_1 \wedge \dots \wedge x_n = 0$ and $x_1 \wedge \dots \wedge x_n \neq 0$. From this, if $x_1 \wedge \dots \wedge x_n = y_1 \wedge \dots \wedge y_n \neq 0$, then $[x_1, \dots, x_n] = [y_1, \dots, y_n]$. On the other hand, if $[x_1, \dots, x_n] = [y_1, \dots, y_n]$, then, expressing each y_j as a linear combination of $x_1, \dots, x_n$, we see that $x_1 \wedge \dots \wedge x_n$ and $y_1 \wedge \dots \wedge y_n$ are scalar multiples of one another. We say that $x_1 \wedge \dots \wedge x_n$ is a *product vector*-the *exterior* (or, *wedge*) *product* of $x_1, \dots, x_n$.

The *antisymmetric Fock space*, $\mathcal{H}_F^{(a)}$, is $\sum_{n=0}^{\infty} \oplus \mathcal{H}_n^{(a)}$. By definition $\mathcal{H}_0^{(a)}$ consists of complex scalar multiples of a single (unit) vector Φ_0, the *Fock vacuum*; and $\mathcal{H}_1^{(a)}$ is $\mathcal{H}$. If $\mathcal{H}$ were finite dimensional, $\mathcal{H}_F^{(a)}$ would be the (finite-dimensional) « exterior » algebra over $\mathcal{H}$. The mapping, $\wedge$, from the n-fold Cartesian product $\mathcal{H} \times \dots \times \mathcal{H}$ to $\mathcal{H}_n^{(a)}$ which assigns $x_1 \wedge \dots \wedge x_n$ to $(x_1, \dots, x_n)$ is an alternating, n-linear mapping. If $\tilde{a}$ is such a mapping of $\mathcal{H} \times \dots \times \mathcal{H}$ into a space $\mathcal{K}$, there is a mapping $\hat{a}$ of $\mathcal{H}_n^{(a)}$ into $\mathcal{K}$ such that $\tilde{a} = \hat{a} \circ \wedge$. In particular if T is a linear mapping of $\mathcal{H}$ into $\mathcal{K}$ then $(x_1, \dots, x_n) \to Tx_1 \wedge \dots \wedge Tx_n$ is an alternating n-linear mapping of $\mathcal{H} \times \dots \times \mathcal{H}$ into $\mathcal{K}_n^{(a)}$; so that there is a linear mapping $\hat{T}$ of $\mathcal{H}_F^{(a)}$ into $\mathcal{K}_F^{(a)}$ such that $\hat{T}(x_1 \wedge \dots \wedge x_n) = Tx_1 \wedge \dots \wedge Tx_n$. If T is a unitary transformation of $\mathcal{H}$ onto $\mathcal{K}$, metric considerations apply and $\hat{T}$ is a unitary transformation of $\mathcal{H}_F^{(a)}$ onto $\mathcal{K}_F^{(a)}$. If $\|T\| \leq 1$, $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, $\tilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$, $P(x, y) = (x, 0)$ and $Q(u, v) = (u, 0)$, with x, y in $\mathcal{H}$ and u, v in $\mathcal{K}$, then there is a unitary transformation U of $\tilde{\mathcal{H}}$ onto $\tilde{\mathcal{K}}$ such that $QUP(x, y) = (Tx, 0)$ for all x, y in $\mathcal{H}$. Then $\hat{U}$ is a unitary transformation of $\tilde{\mathcal{H}}_n^{(a)}$ onto $\tilde{\mathcal{K}}_n^{(a)}$ and $\hat{Q}$ is a projection of $\tilde{\mathcal{K}}_n^{(a)}$ onto $\mathcal{K}_n^{(a)}$. Since $\hat{T}$ is the restriction of $\hat{Q}\hat{U}$ to $\mathcal{H}_n^{(a)}$, we see that $\|\hat{T}\| \leq 1$. If T is a positive operator with pure point spectrum, computing norms with a basis of eigenvectors for T, we find that $\|\hat{T}|\mathcal{H}_n^{(a)}\| = \lambda_1 \dots \lambda_n$, where $\lambda_1, \dots, \lambda_n$ are the n largest eigenvalues of T (multiplicity included). An approximation argument provides the corresponding result for a general positive operator; and polar decomposition provides a norm formula for a general bounded operator. A simple check yields $(\hat{T})^* = \hat{T}^*$.

Since $(f_1, \dots, f_n) \to f \wedge f_1 \wedge \dots \wedge f_n$ is an alternating, n-linear mapping, there is a linear mapping, $a_n(f)^*$, of $\mathcal{H}_n^{(a)}$ into $\mathcal{H}_{n+1}^{(a)}$ with value $f \wedge f_1 \wedge \dots \wedge f_n$ at $f_1 \wedge \dots \wedge f_n$. The family $\{a_n(f)^*\}$ defines a mapping $a(f)^*$ on $\mathcal{H}_F^{(a)}$. With $\{e_j\}$ an orthonormal basis for $\mathcal{H}$ $a(e_1)^*$ maps $\{e_{i_1} \wedge \dots \wedge e_{i_n}: 1 \notin \{i_1, \dots, i_n\}, n = 0, 1, 2, \dots\}$ onto an orthonormal basis for the orthogonal complement, $\mathcal{H}_F^{(a)} \ominus \mathcal{K}$; and $a(e_1)^*$ annihilates this complement. Thus $a(e_1)^*$ is a partial isometry with initial space $\mathcal{K}$ and final space $\mathcal{H}_F^{(a)} \ominus K$. It follows that $I = a(e_1)^* a(e_1) + a(e_1) a(e_1)^*$ $(= \{a(e_1), a(e_1)^*\}_+)$. More generally $a(f)a(f)^* + a(f)^* a(f) = \langle f|f \rangle I$. Polarization of this yields: $\{a(f), a(g)^*\}_+ = \langle f|g \rangle I$. We note that our inner product, $\langle f|g \rangle$, is linear in g and conjugate linear in f. We have $\{a(f)^*, a(g)^*\}_+ = 0$, as well. A conjugate-linear mapping $f \to a(f)$ of $\mathcal{H}$ onto operators $a(f)$ on a Hilbert space satisfying the relations (canonical anticommutation relations)

$$\{a(f), a(g)^*\}_+ = \langle f|g \rangle I, \qquad \{a(f), a(g)\}_+ = 0$$

is said to be a *representation* of the CAR. The particular representation we have exhibited on $\mathcal{H}_F^{(a)}$ is called the *Fock representation*.

We can exhibit the *annihilator* $a(f)$ as explicitly as we described the *creator* $a(f)^*$ by expanding the determinant expression for $\langle f \wedge y_2 \wedge \ldots \wedge y_n | x_1 \wedge \ldots \wedge x_n \rangle$ in terms of its first row:

$$\begin{aligned}
&\langle f \wedge y_2 \wedge \ldots \wedge y_n | x_1 \wedge \ldots \wedge x_n \rangle \\
&\quad = \langle y_2 \wedge \ldots \wedge y_n | a(f)(x_1 \wedge \ldots \wedge x_n) \rangle \\
&\quad = \sum_{j=1}^{n} (-1)^{j+1} \langle f | x_j \rangle \langle y_2 \wedge \ldots \wedge y_n | x_1 \wedge \ldots \wedge x_{j-1} \wedge x_{j+1} \wedge x_{j+1} \wedge \ldots \wedge x_n \rangle ;
\end{aligned}$$

so that

$$a(f)(x_1 \wedge \ldots \wedge x_n) = \sum_{j=1}^{n} (-1)^{j+1} \langle f | x_j \rangle x_1 \wedge \ldots \wedge x_{j-1} \wedge x_{j+1} \wedge \ldots \wedge x_n .$$

With E a projection on $\mathcal{H}$, we denote by $\mathfrak{A}_0(E)$ and $\mathfrak{A}(E)$ the $*$-algebra and C^*-algebra, respectively, on $\mathcal{H}_F^{(a)}$ generated by $\{a(f): Ef = f\}$. We write $\mathfrak{A}_0$ and $\mathfrak{A}$ in place of $\mathfrak{A}_0(I)$ and $\mathfrak{A}(I)$, The C^*-algebra $\mathfrak{A}$ is the CAR algebra and its action on $\mathcal{H}_F^{(a)}$ is called its Fock representation. The state φ_0 of $\mathfrak{A}$ for which $\varphi_0(A) = \langle \Phi_0 | A \Phi_0 \rangle$ is called the *Fock (vacuum) state* of $\mathfrak{A}$. Note that each $a(f)$ is in its left kernel $K\big(\varphi_0(a(f)^* a(f)) = 0\big)$; so that each product of annihilators and creators (*monomial*) in which an annihilator appears to the right is in $\mathcal{K}$. Now each monomial is a sum of monomials in which all creators are to the left of all annihilators (we say that such a monomial is *Wick-ordered*—and *anti-Wick-ordered* if all creators are to the right of all annihilators); so that φ_0 annihilates all Wick-ordered monomials in $\mathfrak{A}_0$ other than I. These monomials span the null space of φ_0 on $\mathfrak{A}_0$. If ϱ is a state of $\mathfrak{A}$ and $\varrho \leqslant 2\varphi_0$, then $a(f)$ is in the left kernel of ϱ. Thus ϱ and φ_0 have the same null space in $\mathfrak{A}_0$ and agree at I. Hence $\varrho = \varphi_0$;

and φ_0 is a pure state of $\mathfrak{A}$. Exactly the same considerations apply to the restriction of φ_0 to $\mathfrak{A}(E)$, for each projection E on $\mathcal{H}$. Thus the restriction of φ_0 to $\mathfrak{A}(E)$ is pure.

The Hilbert space $\bar{\mathcal{H}}$, obtained from $\mathcal{H}$ by assigning an element $\bar{f}$ to each f in $\mathcal{H}$, defining $(\overline{cf+g})$ to be $\bar{c}\bar{f}+\bar{g}$ and $\langle \bar{f}|\bar{g}\rangle$ to be $\langle g|f\rangle$, produces $\bar{\mathcal{H}}_F^{(a)}$, *anti-Fock space*, and $\bar{\Phi}_0$ is the *anti-Fock vacuum*. The mapping $f \to a(\bar{f})^*(=\bar{a}(f))$ is a representation of the CAR (over $\mathcal{H}$), the *anti-Fock representation*; and the mapping $a(f) \to \bar{a}(f)$ extends to a $*$-isomorphism, $A \to \bar{A}$, of the CAR algebra $\mathfrak{A}$ over $\mathcal{H}$ onto the CAR algebra $\bar{\mathfrak{A}}$ over $\bar{\mathcal{H}}$. The state φ_I of $\mathfrak{A}$ defined by $A \to \langle \bar{\Phi}_0|\bar{A}\bar{\Phi}_0\rangle$ is the *anti-Fock state* of $\mathfrak{A}$. Each $a(f)^*$ is in the left kernel of φ_I; so that, replacing $a(f)$ by $a(f)^*$ and using anti-Wick ordered monomials instead of Wick-ordered monomials in the argument above, we have that the restriction of φ_I to each $\mathfrak{A}(E)$ is pure.

Since φ_0 is pure and Φ_0 is cyclic for $\mathfrak{A}$, the weak-operator closure, $\mathfrak{A}^-$, of $\mathfrak{A}$, is $B(\mathcal{H}_F^{(a)})$, the algebra of all bounded operators on $\mathcal{H}_F^{(a)}$. Similarly $\mathfrak{A}(E)^-\tilde{E}_0 = B([\mathfrak{A}(E)\Phi_0])$, where $\tilde{E}_0$ is the projection (in $\mathfrak{A}(E)'$) with range $[\mathfrak{A}(E)\Phi_0]$. If U_E is $\widehat{(I-2E)}$, then $U_E\Phi_0 = \Phi_0$, $a(g)U_E = U_E a(g)$, for each g in $(I-E)(\mathcal{H})$, and $a(f)U_E = -U_E a(f)$, for each f in $E(\mathcal{H})$. If A_0 is an even monomial in $\mathfrak{A}_0(I-E)$ (that is, A_0 is the product of an even total number of annihilators and creators) and A_1 is an odd monomial in $\mathfrak{A}_0(I-E)$, then A_0 and $A_1 U_E$ lie in $\mathfrak{A}(E)'$. Since $\mathfrak{A}_0(E)$ and $\mathfrak{A}_0(I-E)$ generate $\mathfrak{A}_0$ and Φ_0 is cyclic for $\mathfrak{A}_0$;

$$\mathcal{H}_F^{(a)} = [\mathfrak{A}_0\Phi_0] = [\mathfrak{A}(E)\mathfrak{A}(I-E)\Phi_0] = [\mathfrak{A}(E)\mathfrak{A}(E)'\Phi_0]\,.$$

Thus $\tilde{E}_0$ has central carrier I in $\mathfrak{A}(E)^-$; and the mapping ι_E of $\mathfrak{A}(E)^-\tilde{E}_0$ onto $\mathfrak{A}(E)^-$ which assigns A to $A\tilde{E}_0$ is a $*$-isomorphism.

Now, $a(f)\Phi_0 = 0$ and, when $Ef = 0$, $a(f)(\mathfrak{A}_0(E)\Phi_0) = (0)$. Thus $a(f)\tilde{E}_0 = 0$ and $\tilde{E}_0 a(f)^* = 0$, when $Ef = 0$; so that $\tilde{E}_0 A\tilde{E}_0 = \lambda\tilde{E}_0$ when A is in $\mathfrak{A}_0(I-E)$. It follows that $B \to \tilde{E}_0 B\tilde{E}_0$ is a (completely-) positive, linear mapping of $B(\mathcal{H}_F^{(a)})$ onto $\mathfrak{A}(E)^-\tilde{E}_0$. The composition of this mapping with ι_E is a completely-positive, linear mapping, ψ_E, of $B(\mathcal{H}_F^{(a)})$ onto $\mathfrak{A}(E)^-$. By construction of ψ_E,

$$\psi_E(a(x_n)^* \ldots a(x_1)^* a(y_1) \ldots a(y_m)) = a(Ex_n)^* \ldots a(Ex_1)^* a(Ey_1) \ldots a(Ey_m)\,.$$

More generally we have Proposition 2 (stated earlier).

PROOF: If $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, $\tilde{\mathcal{K}} = \mathcal{K} \otimes \mathcal{K}$, $P(h, h') = (h, 0)$ for h, h' in $\mathcal{H}$, $Q(k, k') = (k, 0)$ for k, k' in $\mathcal{K}$, and $\tilde{T}(h, h') = (Th, 0)$, then there is a unitary transformation U of $\tilde{\mathcal{H}}$ onto $\tilde{\mathcal{K}}$ such that $QUP = \tilde{T}$. The mapping $a(f) \to a(Uf)$ extends, uniquely, to a $*$-isomorphism of $\mathfrak{A}_{\tilde{\mathcal{H}}}$ onto $\mathfrak{A}_{\tilde{\mathcal{K}}}$.

The composition of the restriction of this isomorphism to $\mathfrak{A}_{\bar{\mathcal{K}}}(P)$ and ψ_Q is ψ_T.

We note that the characterization of ψ_T as the result of distributing T throughout a Wick-ordered monomial is independent of the ordering *only if* T is an isometry; for $\psi_T(a(f)a(f)^*) = \psi_T(I - a(f)^*a(f)) = I - a(Tf)^*a(Tf) \neq a(Tf)a(Tf)^*$, when $\|f\| = 1$, unless $\langle Tf|Tf\rangle = 1$.

If $A \in B(\mathcal{K})$ and $0 \leqslant A \leqslant I$, we call $\varphi_I \circ \psi_{A^{\frac{1}{2}}}$ the *gauge-invariant, quasi-free state* of $\mathfrak{A}$ with *one-particle operator* A. We write φ_A for this state and note that there is no conflict between this notation and the designation of the Fock and anti-Fock states of $\mathfrak{A}$ by φ_0 and φ_I (i.e. these states are quasi-free with one-particle operators 0 and I, respectively). Note that

$$
\begin{aligned}
&\varphi_A(a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_n)) \\
&= \varphi_I(a(A^{\frac{1}{2}} f_n)^* \dots a(A^{\frac{1}{2}} f_1)^* a(A^{\frac{1}{2}} g_1) \dots a(A^{\frac{1}{2}} g_n)) \\
&= \langle \bar{\Phi}_0 | a(\overline{A^{\frac{1}{2}} f_n}) \dots a(\overline{A^{\frac{1}{2}} f_1}) a(\overline{A^{\frac{1}{2}} g_1})^* \dots a(\overline{A^{\frac{1}{2}} g_n})^* \bar{\Phi}_0\rangle \\
&= \langle \overline{A^{\frac{1}{2}} f_1} \wedge \dots \wedge \overline{A^{\frac{1}{2}} f_n} | \overline{A^{\frac{1}{2}} g_1} \wedge \dots \wedge \overline{A^{\frac{1}{2}} g_n}\rangle \\
&= \det(\langle \overline{A^{\frac{1}{2}} f_i} | \overline{A^{\frac{1}{2}} g_j}\rangle) = \det(\langle g_j | A f_i\rangle) \\
&= \det(\langle g_i | A f_j\rangle) = \langle g_1 \wedge \dots \wedge g_n | A f_1 \wedge \dots \wedge A f_n\rangle .
\end{aligned}
$$

PROPOSITION 3: *If E is a finite-dimensional projection on $\mathcal{K}$ with $\{e_1, \dots, e_n\}$ an orthonormal basis for $E(\mathcal{K})$, then*

$$\varphi_E(T) = \langle e_1 \wedge \dots \wedge e_n | T(e_1 \wedge \dots \wedge e_n)\rangle .$$

PROOF: Let $\{e_j\}$ be an orthonormal basis for $\mathcal{K}$, and T be a Wick-ordered monomial in annihilators and creators corresponding to basis elements. Then $\langle e_1 \wedge \dots \wedge e_n | T(e_1 \wedge \dots \wedge e_n)\rangle$ is 0 unless T has the form $a(e_{i_{\sigma(m)}})^* \dots a(e_{i_{\sigma(1)}})^* a(e_{i_1}) \dots a(e_{i_m})$, with $\{i_1, \dots, i_m\}$ an m-element subset of $\{1, \dots, n\}$, in which case its value and that of $\varphi_E(T)$ is $\chi(\sigma)$. If T does not have this form $\psi_E(T) = 0$, so $\varphi_E(T) = 0$. Thus our equality holds.

It follows that φ_E is pure when E is a finite-dimensional projection on $\mathcal{K}$. More generally, if E is any orthogonal projection on $\mathcal{K}$ and ϱ is a state of $\mathfrak{A}$ such that $\varrho \leqslant 2\varphi_E$ then the restrictions of ϱ to $\mathfrak{A}(E)$ and $\mathfrak{A}(I-E)$ coincide with those of φ_I and φ_0, respectively. Using the fact that monomials A and A' in $\mathfrak{A}_0(E)$ and $\mathfrak{A}_0(I-E)$, respectively, commute or anti-commute and that Wick-ordered monomials are in the left or right kernels of φ_0 while anti-Wick ordered monomials are in the left or right kernels of φ_I, (other than cI, $c \neq 0$), we conclude that $\varrho(AA') = \varrho(A)\varrho(A')$. The same is true for A in $\mathfrak{A}(E)$ and A' in $\mathfrak{A}(I-E)$. Thus $\varrho = \varphi_E$ and φ_E is pure.

If $0 \leqslant A_0 \leqslant I$ with $A_0 (\neq A_0^2)$ in $\mathcal{B}(\mathcal{H})$, using the Spectral Theorem, there is a one-dimensional projection E_1 on $\mathcal{H}$ and a positive number t such that $0 \leqslant A_1 \leqslant I$ and $0 \leqslant A_2 \leqslant I$, where $A_1 = A_0 + tE_1$ and $A_2 = A_0 - tE_1$. Computing with an orthonormal basis $\{e_j\}$ for $\mathcal{H}$ such that $E_1 e_1 = e_1$, we have that $\varphi_{A_0} = \frac{1}{2}(\varphi_{A_1} + \varphi_{A_2})$. To see this, note that

$$\varphi_{A_k}\big(a(e_{i_n})^* \dots a(e_{i_1})^* a(e_{j_1}) \dots a(e_{j_n})\big) = \langle e_{j_1} \wedge \dots \wedge e_{j_n} | A_k e_{i_1} \wedge \dots \wedge A_k e_{i_n} \rangle ,$$

where $k = 0, 1, 2$; and that $A_0 e_j = A_1 e_j = A_2 e_j$, when $j \neq 1$. Thus φ_A is pure if and only if A is a projection.

From the foregoing, if E is a finite-dimensional projection, φ_E is a pure, gauge-invariant, quasi-free state equivalent to the Fock state. Conversely, if E is a projection on one-particle space $\mathcal{H}$ and φ_E is equivalent to the Fock state, then E is a finite-dimensional projection. This follows as a special case of [3; Theorem 2.8]. A direct proof is not difficult. If $\varphi_E = \omega_x | \mathfrak{A}$, for some unit vector x of $\mathcal{H}_F^{(a)}$, then $1 = \varphi_E\big(a(e_j)^* a(e_j)\big) = \omega_x\big(a(e_j)^* a(e_j)\big)$, where $\{e_j\}$ is an orthonormal basis for $E(\mathcal{H})$. Thus $a(e_j)^* x = 0$, for each j. If

$$x = \sum_{i_1 < \dots < i_n ; j_1 < \dots < j_m} c_{i_1 \dots i_n ; j_1 \dots j_m} e_{i_1} \wedge \dots \wedge e_{i_n} \wedge e'_{j_1} \wedge \dots \wedge e'_{j_m} ,$$

where $\{e'_j\}$ is an orthonormal basis for $(I - E)(\mathcal{H})$, then

$$0 = a(e_j)^* x = \sum c_{i_1 \dots i_n ; j_1 \dots j_m} e_j \wedge e_{i_1} \wedge \dots \wedge e_{i_n} \wedge e'_{j_1} \wedge \dots \wedge e'_{j_m} ;$$

so that $j \in \{i_1, \dots, i_n\}$ unless $c_{i_1 \dots i_n ; j_1 \dots j_m} = 0$. If $E(\mathcal{H})$ is infinite-dimensional, we can choose j not in $\{i_1, \dots, i_n\}$; and $x = 0$, contradicting the assumption that x is a unit vector. Thus E is a finite-dimensional projection.

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