4. Operator Algebras

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INTRODUCTION

During this series of lectures, I want to outline for you some of the main results in the theory of von Neumann algebras. There are many subjects, of considerable importance, on which I will not touch. The subjects discussed are what many of us consider to be the core of the theory. These subjects could be classified under three headings: the Basics, Comparison Theory of Projections, and Unitary Equivalence. Under this last heading—and the main part of it—I include the theory of normal states.

It no longer makes very much sense to draw a sharp line between the results and methods of C^*-algebra theory and those of the theory of von Neumann algebras. Nonetheless there are areas of each of these subjects which are unambiguously identified with the one but not the other. For our purpose, we will want some of the tools of C^*-algebra theory. A description of these will provide us with an appropriate introduction.

As excellent general references, we cite the two books of J. Dixmier "Les algèbres d'opérateurs dans l'espace Hilbertien (Algèbres de von Neumann)" Cahiers Scientifiques Fasc. XXV: Gauthier-Villars, Paris, 1957, 2[me] éd. 1969 and "Les C^*-algèbres", Cahiers Scientifiques Fasc. XXIX, Gauthier-Villars, Paris, 1964, 2[me] éd. 1969, (especially pp. 1–55 of "Les C^*-algèbres"). In addition, S. Sakai's "C^*-algebras and W^*-algebras", Ergebnisse der Mathematik und Ihrer Grenzgebiete Bd. 60, Springer-Verlag, Berlin, 1971, gives an excellent account of fundamentals and recent work. The combined bibliography of the Dixmier–Sakai books is comprehensive.

1. SOME C^*-ALGEBRA BASICS

The Hilbert spaces with which we deal are complex (the field of scalars is $\mathbb{C}$). The inner product is denoted by $\langle x, y\rangle$ for a pair of vectors x, y in $\mathscr{H}$. The *length* or *norm* of x is denoted by $\|x\|$ $(= \langle x, x\rangle^{\frac{1}{2}})$. The *operators* on $\mathscr{H}$ are linear transformations of $\mathscr{H}$ into $\mathscr{H}$; and we assume that they are continuous unless otherwise stated. The *bound* or *norm* of an operator T is denoted by $\|T\|$ $(=\sup\{\|Tx\|:\|x\| \leqslant 1\})$, and we recall that the continuity of T is equivalent to its boundedness $(\|T\| < \infty)$. The set of all bounded operators on $\mathscr{H}$ will be denoted by $\mathscr{B}(\mathscr{H})$. It is an algebra under the usual operations of addition, multiplication by scalars, and multiplication (= iteration of transformations) (so $(A + B)(x) = Ax + Bx$, $(aA)x = a(Ax)$, and $(AB)x = A(Bx)$). The function $A \to \|A\|$ is a norm relative to which $\mathscr{B}(\mathscr{H})$ becomes a normed space. It is complete in this norm, so that it is a Banach space; and, indeed, $\|AB\| \leqslant \|A\|\,\|B\|$. Thus $\mathscr{B}(\mathscr{H})$ with the norm $A \to \|A\|$ is a Banach algebra. The metric topology on $\mathscr{B}(\mathscr{H})$ associated with the norm is called the *norm topology*.

The adjoint operation on $\mathscr{B}(\mathscr{H})$ provides an important piece of algebraic structure. Recall that, with A in $\mathscr{B}(\mathscr{H})$ there is associated an A^* in $\mathscr{B}(\mathscr{H})$, called the *adjoint* of A, characterized by the equality $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all x and y in $\mathscr{H}$. One verifies without difficulty that:

(1) $(aA + B)^* = \bar{a}A^* + B^*$

(2) $(AB)^* = B^*A^*$

(3) $(A^*)^* = A$

(4) $\|AA^*\| = \|A\|\,\|A^*\|$

(5) $\|A\| = \|A^*\|$.

An operator A such that $A = A^*$ is said to be *self-adjoint*. A subset $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$ such that $\mathscr{F}^* = \mathscr{F}$ (equivalently, $A^* \in \mathscr{F}$ if $A \in \mathscr{F}$) is said to be self-adjoint. A subalgebra $\mathfrak{A}$ of $\mathscr{B}(\mathscr{H})$ which is both norm closed and self-adjoint is called a *C^*-algebra*.

One of the key initial results of the theory—a slight generalization of a result of Gelfand and Neumark states:

THEOREM 1.1. *If $\mathscr{B}$ is a Banach algebra with an involution $A \to A^*$ satisfying* (1), (2), (3) *and* (4), *above, then there is a Hilbert space $\mathscr{H}$ and a C^*-algebra $\mathfrak{A}$*

acting on it such that $\mathscr{B}$ is algebraically isomorphic to $\mathfrak{A}$ by means of an isomorphism ϕ for which $\phi(B^*) = \phi(B)^*$.

In stating and proving the result it is usual to assume (5) as well as (1)–(4), and to assume that $\mathfrak{A}$ has a unit element. We will denote the unit element of $\mathscr{B}(\mathscr{H})$ by I (so that $Ix = x$) and refer to it as the identity operator. The theorem just noted establishes the "independent algebraic existence" of a C^*-algebra—independent of its action on a particular Hilbert space. It is often useful to think of the C^*-algebra in this way and to speak of its *representations* on a particular Hilbert space $\mathscr{H}$. A representation of the C^*-algebra $\mathfrak{A}$ on the Hilbert space $\mathscr{H}$ is a homomorphism ϕ of $\mathfrak{A}$ into $\mathscr{B}(\mathscr{H})$ such that $\phi(A^*) = \phi(A)^*$ for each A in $\mathfrak{A}$. It is a non-trivial fact that the image $\phi(\mathfrak{A})$ of $\mathfrak{A}$ under this mapping is norm closed—hence, a C^*-algebra. If ϕ is an isomorphism (the kernel of ϕ is (0)), we say that ϕ is a *faithful* representation of $\mathfrak{A}$. When the transforms $\phi(\mathfrak{A})x$ of a vector x in $\mathscr{H}$ by operators in $\phi(\mathfrak{A})$ lie dense in $\mathscr{H}$, we say that ϕ is a *cyclic representation* of $\mathfrak{A}$, and that x is a *cyclic vector* for $\phi(\mathfrak{A})$ (and for ϕ).

The technique of proof of Theorem 1.1, as developed by Segal was especially useful. It involved a construction of representations of $\mathfrak{A}$ based on a special type of linear functional on $\mathfrak{A}$. The functionals are called *states* and the procedure is known as the *GNS* (Gelfand–Neumark–Segal) *construction.* In order to describe this construction, we make use of another essential structure possessed by C^*-algebras—basic to their analysis—the *order structure.* If we think of $\mathfrak{A}$ as acting on $\mathscr{H}$, a *positive operator* in $\mathscr{B}(\mathscr{H})$ is an operator A such that $\langle Ax, x\rangle \geqslant 0$ for all x in $\mathscr{H}$; and the set of positive operators in $\mathfrak{A}$ forms a cone ($A + B \geqslant 0$ if A and B are positive; aA is positive if A is positive and $a \geqslant 0$; $A = 0$ if both A and $-A$ are positive). Relative to this cone, the real linear space of self-adjoint operators in $\mathfrak{A}$ is a partially-ordered vector space. We write "$A \geqslant B$" for "$A - B$ is positive". The unit element I of $\mathfrak{A}$ is an order unit: for each self-adjoint A there are constants a and b such that $aI \leqslant A \leqslant bI$. Moreover $-\|A\| I \leqslant A \leqslant \|A\| I$; and $\|A\|$ is the least non-negative constant for which this inequality is valid. A *state* of $\mathfrak{A}$ is a linear functional ρ on $\mathfrak{A}$ such that

(i) $\rho(A) \geqslant 0$ when $A \geqslant 0$

(ii) $\rho(I) = 1$.

The GNS construction proceeds as follows:

With ρ a state of $\mathfrak{A}$ define an inner product $\{,\}$ on $\mathfrak{A}$ by means of the formula $\{A, B\} = \rho(B^*A)$. As $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geqslant 0$ for each A in $\mathfrak{A}$ (i.e. $A^*A \geqslant 0$ for each A in $\mathfrak{A}$) $\{,\}$ is a positive semi-definite inner product on $\mathfrak{A}$. This is enough in order that the Cauchy–Schwarz inequality should hold and

$$|\{A, B\}| = |\rho(B^*A)| \leqslant \{A, A\}^{\frac{1}{2}}\{B, B\}^{\frac{1}{2}} = \rho(A^*A)^{\frac{1}{2}}\rho(B^*B)^{\frac{1}{2}}.$$

It follows that $\rho(TA) = 0$ for all T in $\mathfrak{A}$ if $\rho(A^*A) = 0$ (of course, $\rho(A^*A) = 0$ if $\rho(TA) = 0$ for each T in $\mathfrak{A}$). The set of such A is a left ideal $\mathscr{K}$ in $\mathfrak{A}$ called the *left kernel* of ρ. It is the set of null vectors with respect to the inner product $\{\ ,\ \}$. If A is self-adjoint $-\|A\|I \leqslant A \leqslant \|A\|I$; so that $-\|A\| \leqslant \rho(A) \leqslant \|A\|$. Thus $|\rho(A)| \leqslant \|A\|$. In general,

$$|\rho(T)| = |\rho(IT)| \leqslant \rho(I)^{\frac{1}{2}}\rho(T^*T)^{\frac{1}{2}} \leqslant \|T^*T\|^{\frac{1}{2}} = \|T\|.$$

Thus states of $\mathfrak{A}$ are bounded linear functionals on $\mathfrak{A}$ of norm 1 (attaining their norm at I). The converse is also valid—functionals ρ on $\mathfrak{A}$ of norm 1 for which $\rho(I) = 1$ are states of $\mathfrak{A}$. This is not difficult to prove but requires some information we have not yet discussed.

The quotient Banach space $\mathfrak{A}/\mathscr{K}$ has a positive definite inner product,

$$\langle A + \mathscr{K}, B + \mathscr{K}\rangle = \rho(B^*A) = \{A, B\},$$

induced on it by $\{\ ,\ \}$. With $\phi_0(A)$ defined on $\mathfrak{A}/\mathscr{K}$ by:

$$\phi_0(A)(B + \mathscr{K}) = AB + \mathscr{K},$$

the resulting mapping is well-defined, since $\mathscr{K}$ is a left ideal and bounded relative to the norm on $\mathfrak{A}/\mathscr{K}$ associated with $\langle\ ,\ \rangle$ for

$$\|\phi_0(A)(B + \mathscr{K})\|^2 = \|AB + \mathscr{K}\|^2 = \rho(B^*A^*AB) \leqslant \|A^*A\|\,\rho(B^*B)$$
$$= \|A\|^2\,\|B + \mathscr{K}\|^2,$$

where we have made use of the fact that $B^*HB \geqslant 0$ if $H \geqslant 0$ (since $\langle B^*HBx, x\rangle = \langle HBx, Bx\rangle \geqslant 0$) so that

$$\rho[B^*(\|A^*A\|I - A^*A)B] \geqslant 0.$$

It follows that $\|\phi_0(A)\| \leqslant \|A\|$ and that $\phi_0(A)$ can be extended to a bounded operator $\phi(A)$ on the completion $\mathscr{H}_\rho$ of $\mathfrak{A}/\mathscr{K}$ relative to the metric deduced from $\langle\ ,\ \rangle$. It is easy to check that ϕ is a homomorphism of $\mathfrak{A}$ into $\mathscr{B}(\mathscr{H}_\rho)$ (and from the preceding, $\|\phi(A)\| \leqslant \|A\|$). That ϕ preserves adjoints follows from:

$$\langle\phi(A)(B + \mathscr{K}), C + \mathscr{K}\rangle = \rho(C^*AB) = \langle B + \mathscr{K}, \phi(A^*)(C + \mathscr{K})\rangle.$$

Thus ϕ is a representation of $\mathfrak{A}$. We say that ϕ is the representation of $\mathfrak{A}$ *engendered* by ρ; and, when it is desirable to indicate the dependence of the representation on ρ, we denote it by π_ρ. The element $I + \mathscr{K}$ in $\mathscr{H}_\rho$, which we denote by x_ρ, for simplicity of notation, has special properties. To begin with

$\|x_\rho\|^2 = \rho(I) = 1$; so that x_ρ is a unit vector. In addition, $\phi(\mathfrak{A})x_\rho = \mathfrak{A}/\mathscr{K}$; so that, by construction, ϕ is a cyclic representation and x_ρ is a cyclic vector for ϕ. Finally, $\rho(A) = \langle\phi(A)x_\rho, x_\rho\rangle$. Note that the functional $\phi(A) \to \langle\phi(A)x_\rho, x_\rho\rangle$ is a state of $\phi(\mathfrak{A})$. We call such a state a *vector state* of $\phi(\mathfrak{A})$ and say that this vector state *represents* ρ.

In the description of the order structure on $\mathfrak{A}$, in particular, when defining positive operators, we assumed that $\mathfrak{A}$ acts on a Hilbert space. If $\mathfrak{A}$ is not so represented, a technique using the spectrum of elements in a Banach algebra allows us to define the order structure. I remind you that a complex number λ is said to lie in the *spectrum* of an element of $\mathfrak{A}$ (relative to $\mathfrak{A}$) when $A - \lambda I$ fails to have a two-sided inverse in $\mathfrak{A}$. The spectrum $\mathrm{sp}(A)$ is a non-empty, closed subset of $\mathbb{C}$ contained in the disc of radius $\|A\|$ (so that $\mathrm{sp}(A)$ is compact). The positive elements of $\mathfrak{A}$ are identified, now, as those self-adjoint elements A of $\mathfrak{A}$ for which $\mathrm{sp}(A)$ consists of non-negative real numbers.

It might be appropriate to pause, here, and note some specific examples of C^*-algebras.

(1) With $\mathscr{H}$ of dimension n, $\mathscr{B}(\mathscr{H})$ is a C^*-algebra, isomorphic to the algebra of $n \times n$ complex matrices when n is a finite cardinal.

(2) If X is a compact Hausdorff space and $C(X)$ is the algebra of complex-valued continuous functions on X (with pointwise operations) then $C(X)$ is a C^*-algebra—where complex conjugation of functions is taken as the involution. In this last case, the C^*-algebra is abelian. A specific example is had by choosing the interval $[0, 1]$ for X. It is worth noting that we have described all commutative C^*-algebras in this example (at least as far as their algebraic structure goes).

THEOREM 1.2. *If $\mathfrak{A}$ is a commutative C^*-algebra there is a compact Hausdorff space X such that $\mathfrak{A}$ is *-isomorphic to $C(X)$.*

This description of commutative C^*-algebras contains the algebraic content of the "spectral theorem". The set of all states of a C^*-algebra is a convex subset of the (continuous) dual space $\mathfrak{A}^\wedge$ of $\mathfrak{A}$. In the topology of convergence on elements of $\mathfrak{A}$, the w^*-topology, this convex set is compact (as a closed subset of the unit ball). The Krein–Milman theorem assures us that it is the closed convex hull of its extreme points—the *pure states* of $\mathfrak{A}$. The pure states of $\mathfrak{A}$ are those states ρ such that

$$\rho = a\rho_1 + (1 - a)\rho_2$$

with $0 < a < 1$ and ρ_1, ρ_2 states, only when $\rho_1 = \rho_2 = \rho$. The pure states of $C(X)$ are the functionals corresponding to evaluation of functions in $C(X)$ at a point of X. Theorem 1.2 can be proved by this technique: examine the pure states of a commutative C^*-algebra, show that they are multiplicative, linear functionals, and that they form a closed subset of the dual. In general the pure states of a C^*-algebra do not form a closed subset of the dual. The vector states of $\mathscr{B}(\mathscr{H})$ are among the pure states of $\mathscr{B}(\mathscr{H})$ but are not all pure states of $\mathscr{B}(\mathscr{H})$. All the others annihilate the compact operators.

If $\mathfrak{A}$ is a C^*-algebra and A is a self-adjoint operator in $\mathfrak{A}$, let $\mathfrak{A}(A)$ denote the C^*-subalgebra of $\mathfrak{A}$ generated by A and I. Since $\mathfrak{A}(A)$ (the norm closure of the polynomials in A) is commutative $\mathfrak{A}(A) \cong C(X)$, for some compact Hausdorff space X. With p a point of X, let ρ_0 be the state of $\mathfrak{A}(A)$ which assigns to each element the value of its corresponding function. The construction of the *-isomorphism of a commutative C^*-algebra with $C(X)$ carries with it the information that the isomorphism preserves order and norm so ρ_0 is a state. Applying the Hahn–Banach theorem, we extend ρ_0 to a functional ρ of norm 1 on $\mathfrak{A}$. Since $\rho_0(I) = 1$, ρ is a state of $\mathfrak{A}$. Now $\rho_0(A^2) = \rho_0(A)^2$ so that $\rho([A - \rho(A)I]^2) = 0$ and $A - \rho(A)I$ is in the left kernel of ρ. Thus $\rho(B(A - \rho(A)I)) = 0$ for each B in $\mathfrak{A}$. That is, $\rho(BA) = \rho(B)\rho(A)$ for each B in $\mathfrak{A}$. Symmetrically, $\rho(AB) = \rho(A)\rho(B)$. It follows that $A - \rho(A)I$ does not have an inverse in $\mathfrak{A}$ and that $\rho(A) \in \mathrm{sp}_{\mathfrak{A}}(A)$. From the outset, $\rho(A) = \rho_0(A) \in \mathrm{sp}_{\mathfrak{A}(A)}(A)$. It follows that $\mathrm{sp}_{\mathfrak{A}}(A)$ and $\mathrm{sp}_{\mathfrak{A}(A)}(A)$ coincide, for a self-adjoint A in $\mathfrak{A}$. What amounts to the same thing, $A - \lambda I$ has an inverse in $\mathfrak{A}$ if and only if it has an inverse in $\mathfrak{A}(A)$. For arbitrary T in $\mathfrak{A}$, if T lies in the C^*-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$, T has an inverse in $\mathfrak{A}_0$ if and only if both T^*T and TT^* have inverses in $\mathfrak{A}_0$ (for then T has both a left and right inverse in $\mathfrak{A}_0$, hence a two-sided inverse). This last occurs if and only if T^*T and TT^* have inverses in $\mathfrak{A}$ which is the case if and only if T has an inverse in $\mathfrak{A}$.

Several useful facts emerge from this discussion:

(i) The spectrum of an element of a C^*-algebra is not dependent on the C^*-subalgebra containing it in which the spectrum is computed.

(ii) A state of a C^*-subalgebra of a C^*-algebra has an extension to the full algebra which is a state.

(iii) If ρ is a state of $\mathfrak{A}$ and A is a self-adjoint element of $\mathfrak{A}$ such that $\rho(A^2) = \rho(A)^2$, then

$$\rho(AB) = \rho(A)\rho(B) = \rho(BA).$$

(iv) If $\mathfrak{A}$ is a commutative C^*-algebra generated by the single self-adjoint element A then $\mathfrak{A} \cong C(\mathrm{sp}(A))$.

If $\mathfrak{A}$ acts on $\mathscr{H}$, and x is a unit vector in $\mathscr{H}$ such that $\langle A^2x, x\rangle = \langle Ax, x\rangle^2$ for some self-adjoint A in $\mathfrak{A}$, then, from the preceding:

$$\langle (A - \langle Ax, x\rangle I)^2 x, x\rangle = 0$$

so that $Ax = \langle Ax, x\rangle\, x$, and x is an eigenvector for A. Let $\mathscr{H}$ be $L_2([0,1])$ relative to Lebesgue measure and let $\mathfrak{A}$ be the C^*-algebra consisting of $\{M_f : f \text{ in } C([0,1])\}$ where $M_f(g) = f \cdot g$. We call M_f the *multiplication operator* corresponding to f. The state ρ_0 of $\mathfrak{A}$ defined by $\rho_0(M_f) = f(0)$

extends to a state ρ of $\mathscr{B}(\mathscr{H})$. Denoting by λ the identity function on $[0, 1]$, $\rho(M_\lambda)^2 = 0 = \rho(M_\lambda^2)$. If ρ were a vector state of $\mathscr{B}(\mathscr{H})$, that vector would be annihilated by M_λ. But no L_2-function on $[0, 1]$ other than 0 is annihilated by multiplication by λ. Thus ρ is not a vector state of $\mathscr{B}(\mathscr{H})$.

If ρ_0 is a state of the C^*-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ the set of all state extensions of ρ_0 to $\mathfrak{A}$ is a convex, w^*-compact set of states of $\mathfrak{A}$. If ρ is one of its extreme points and $\rho = a\rho_1 + (1 - a)\rho_2$, with $0 < a < 1$ and ρ_1, ρ_2 states of $\mathfrak{A}$, then this same relation persists on $\mathfrak{A}_0$. Since ρ_0 is pure, $\rho_0 = \rho_1 | \mathfrak{A}_0 = \rho_2 | \mathfrak{A}_0$; and ρ_1, ρ_2 are extensions of ρ_0. Since ρ is extreme in the set of such extensions, $\rho = \rho_1 = \rho_2$; and ρ is a pure state of $\mathfrak{A}$. Thus pure states of C^*-subalgebras have pure state extensions. In the case of the multiplication algebra, above, and the pure state ρ_0 of $\mathfrak{A}$ described there, if we take for ρ a pure state extension of ρ_0 to $\mathscr{B}(\mathscr{H})$, we have an example of a pure state of $\mathscr{B}(\mathscr{H})$ which is not a vector state.

If ϕ is a representation of the C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$, ϕ is said to be an *irreducible* representation of $\mathfrak{A}$ (equivalently, $\phi(\mathfrak{A})$ is said to act irreducibly on $\mathscr{H}$) when each non-zero vector in $\mathscr{H}$ is a cyclic vector for $\phi(\mathfrak{A})$. In this case no proper closed subspace of $\mathscr{H}$ is invariant under $\phi(\mathfrak{A})$. If $\mathscr{V}$ is a closed subspace of $\mathscr{H}$ the operator E which assigns to a vector its orthogonal projection on $\mathscr{V}$ is a projection (operator) with range $\mathscr{V}$. A check shows that E is self-adjoint and idempotent ($E^2 = E$) and that $\mathscr{V}$ is invariant under an operator A and its adjoint if and only if $AE = EA$. Thus ϕ is irreducible if and only if I and 0 are the only projections commuting with $\phi(\mathfrak{A})$. If ϕ is engendered by the state ρ, ϕ is irreducible if and only if ρ is pure. In effect, a commuting projection different from 0 or I provides a means for decomposing ρ.

If $\mathfrak{A}$ acts on $\mathscr{H}$ and x is a unit cyclic vector for $\mathfrak{A}$ the representation π_x corresponding to the vector state $A \to \langle Ax, x\rangle = \omega_x(A) = \rho(A)$ is unitarily equivalent to the action of $\mathfrak{A}$ on $\mathscr{H}$. The mapping $Ax \to \pi_x(A)x_\rho$ extends to an isomorphism (= unitary transformation) U of $\mathscr{H}$ onto $\mathscr{H}_\rho$ and $UAU^{-1} = \pi_x(A)$ for all A in $\mathfrak{A}$.

2. VON NEUMANN ALGEBRA BASICS

The *strong-operator topology* on $\mathscr{B}(\mathscr{H})$ is the topology for which the net (T_a) is convergent to T when $\|(T_a - T)x\| \to 0$ for each x in $\mathscr{H}$. The *weak-operator topology* on $\mathscr{B}(\mathscr{H})$ is that in which (T_a) converges to T when $\langle T_a x, y\rangle \to \langle Tx, y\rangle$ for each x and y in $\mathscr{H}$. The weak-operator topology is weaker (coarser) than the strong operator topology. Nevertheless

THEOREM 2.1. *The weak- and strong-operator closures of a convex subset of* $\mathscr{B}(\mathscr{H})$ *coincide.*

In essence, if $\mathscr{K}$ is convex its strong-operator closure is contained in its weak-operator closure. Suppose A is in its weak-operator closure but not in its strong-operator closure. Then there are vectors $x_1, \ldots, x_n$ such that $(Ax_1, \ldots, Ax_n)$ is not in the norm closure of $\{(Kx_1, \ldots, Kx_n)\colon K \text{ in } \mathscr{K}\}$ in the direct sum $\mathscr{H} \oplus \ldots \oplus \mathscr{H}$ $(= \tilde{\mathscr{H}})$ of $\mathscr{H}$ with itself n times. The Separation Theorem tells us that there is a linear functional f on $\tilde{\mathscr{H}}$ and a scalar a such that $f(Ax_1, \ldots, Ax_n) > a$ and $f(Kx_1, \ldots, Kx_n) \leqslant a$ for each K in $\mathscr{K}$. But linear functionals on $\tilde{\mathscr{H}}$ arise from vectors; so that there is a vector $(y_1, \ldots, y_n)$ in $\tilde{\mathscr{H}}$ such that $\langle Ax_1, y_1\rangle + \ldots + \langle Ax_n, y_n\rangle > a$ while $\langle Kx_1, y_1\rangle + \ldots + \langle Kx_n, y_n\rangle \leqslant a$—which contradicts the choice of A in the weak-operator closure of $\mathscr{K}$.

It follows from this result that the strong- and weak-operator closures of a subalgebra of $\mathscr{B}(\mathscr{H})$ coincide. Those weak-operator closed subalgebras of $\mathscr{B}(\mathscr{H})$ stable under * are called *von Neumann algebras.* It follows from Theorem 2.1 that a linear functional on a von Neumann algebra $\mathscr{R}$ is weak-operator continuous on a convex subset $\mathscr{K}$ if and only if it is strong-operator continuous on $\mathscr{K}$. By choosing subbasic open sets appropriately in $\mathbb{C}$ it is enough to note that the linear functional has, as inverse image of a convex set, another convex set; which allows us to convert the condition on this inverse image of being strong-operator closed to one of being weak-operator closed. This works as well for a linear mapping η from one von Neumann algebra $\mathscr{R}_1$ into another $\mathscr{R}_2$. Here we assume that η is continuous on $\mathscr{K}$ in the strong-operator topology to $\mathscr{R}_2$ in the weak-operator topology, and conclude that it is continuous on $\mathscr{K}$ in the weak-operator topology to $\mathscr{R}_2$ in this same topology.

The change in closure assumption from norm closed for C^*-algebras to strong-operator closed for von Neumann algebras produces significant structural changes even though it seems like a fine technical distinction. For one thing, the von Neumann algebras have many projection operators while the C^*-algebras may have none. In a deeper sense, the passage from the C^*-algebras to the von Neumann algebras corresponds to the passage from the algebra of continuous functions to the algebra of bounded measurable functions. This correspondence can be made quite formal in the commutative case (Theorem 1.2 is part of the story).

A feature of the weak-operator topology is a certain compactness property it possesses.

THEOREM 2.2. *The unit ball* $(\mathscr{R})_1$ *in* $\mathscr{R}$ *is weak-operator compact, where* $\mathscr{R}$ *is a von Neumann algebra.*

The proof of this proceeds as does the proof that the unit ball in the dual of a normed space is compact—making use of the representation of bounded, conjugate bilinear functionals on $\mathscr{H}$ in terms of bounded operators and the definition of the weak-operator topology.

If $\{H_a\}$ is a monotone increasing net of self-adjoint operators on $\mathscr{H}$ then $\langle H_a x, x\rangle$ is monotone increasing for each x in $\mathscr{H}$. If $H_a \leqslant kI$, for all a, then $\langle H_a x, x\rangle$ converges for each x. By "polarization" $\langle H_a x, y\rangle$ converges for each x, y in $\mathscr{H}$. The resulting limit is a bounded conjugate bilinear functional on $\mathscr{H}$ and corresponds to a self-adjoint operator H on $\mathscr{H}$. Not only is H the weak-operator limit of $\{H_a\}$, but an argument with the Schwarz inequality shows that it is a strong-operator limit of $\{H_a\}$. Of course $H_a \leqslant H$ for all a and H is the least operator with this property. Thus H is characterized as the (unique) least upper bound of $\{H_a\}$. If all the H_a lie in a von Neumann algebra $\mathscr{R}$, then H lies in $\mathscr{R}$.

If $0 \leqslant A \leqslant I$, by passing to the function representation of $\mathfrak{A}(A)$, $(A^{1/n})$ can be seen to be a monotone increasing sequence bounded above by I. It has a least upper bound E which is its strong-operator limit. Then $(A^{2/n})$ has E^2 as its strong-operator limit. But $(A^{1/n}) = (A^{2/2n})$ is a subsequence of $(A^{2/n})$; so that $E = E^2$. One can show, now, that E is the projection on the closure of the range of A. We denote this *range projection* by $R(A)$. As $R(TT^*) = R(T)$ for each bounded T, we conclude that the range projection of each T in a von Neumann algebra $\mathscr{R}$ lies in $\mathscr{R}$. Thus von Neumann algebras have many projections. If $\{E_a\}$ is a family of projections their *union*, $\vee_a E_a$, and their *intersection*, $\wedge_a E_a$, are the projections on the subspace spanned by their ranges and on the intersection of their ranges, respectively. Since $R(E + F) = E \vee F$, we see that $E \vee F \in \mathscr{R}$ if the projections E and F lie in $\mathscr{R}$. If $\{E_a\}$ is a family of projections in the von Neumann algebra $\mathscr{R}$ then unions of finite subfamilies lie in $\mathscr{R}$ and form a monotone increasing net (bounded above by I) with least upper bound (strong-operator limit) $\vee_a E_a$. Thus $(E =)\vee_a E_a \in \mathscr{R}$. Since $E - \vee_a (E - E_a) = \wedge_a E_a, \wedge_a E_a \in \mathscr{R}$. Let P be the union of the range projections of all operators in the von Neumann algebra $\mathscr{R}$; then $PA = A$ for all A in $\mathscr{R}$ so that P is a unit for $\mathscr{R}$. For convenience, when we speak of von Neumann algebras, henceforth, we assume that they contain I.

The algebra $\mathscr{B}(\mathscr{H})$ is an example of a von Neumann algebra. Its centre consists of scalar multiples of I. Those von Neumann algebras with centre consisting of the scalar operators only are called *factors*. Another example is constructed from the algebra of multiplications on $L_2(S, \mu)$ (S a measure space with measure μ) by bounded measurable functions. This is an abelian von Neumann algebra. Recalling that an abelian C^*-algebra is $*$-isomorphic with some $C(X)$ and noting that each von Neumann algebra is a C^*-algebra, one naturally wonders about the special nature of X in the case of an abelian von Neumann algebra.

THEOREM 2.3. *If $\mathscr{A}$ is an abelian von Neumann algebra then $\mathscr{A} \cong C(X)$, with X a compact Hausdorff space in which the closure of each open set is open (as well as closed).*

We say that X is *extremely disconnected* in this case and call the sets which are both closed and open *clopen* sets.

Since each bounded monotone increasing net $\{A_a\}$ in $\mathscr{A}$ has a least upper bound A in $\mathscr{A}$ and since the isomorphism between $\mathscr{A}$ and $C(X)$ is order-preserving, the same is true for each such net $\{f_a\}$ of functions in $C(X)$. That is, there is an f in $C(X)$ which is a least upper bound for $\{f_a\}$. This condition will cause X to be extremely disconnected. From another viewpoint, we have seen that $\mathscr{A}$ has many projections. Each will correspond to an idempotent function in $C(X)$; and such a function is the characteristic function of a clopen set.

If A corresponds to f in $C(X)$ there is a largest clopen set O_λ on which f takes values not exceeding λ. A clopen set on which f takes values not exceeding λ has the closure of the set of points at which f takes values exceeding λ in its complement. This last set and its complement are clopen. The complement contains the first clopen set and is itself a clopen set on which f takes values not exceeding λ. It is O_λ. The characteristic function of O_λ is in $C(X)$ and corresponds to a projection E_λ in $\mathscr{A}$. The characterization of O_λ as the largest clopen set on which f takes values not exceeding λ allows us to conclude that

(1) $E_\lambda \leqslant E_\mu$ when $\lambda \leqslant \mu$,

(2) $\wedge_{\lambda > \lambda_0} E_\lambda = E_{\lambda_0}$,

(3) $\vee_\lambda E_\lambda = I$ and $\wedge_\lambda E_\lambda = 0$.

As a matter of fact, $E_\lambda = 0$ for $\lambda \leqslant -\|A\| - \varepsilon$ for each positive ε and $E_\lambda = I$ for $\lambda \geqslant \| \|A\|$. A family of projections $\{E_\lambda\}$ satisfying (1), (2) and (3) is called a *resolution of the identity*; and the particular one we constructed is called the *resolution of the identity for A*. If we assign to $\int_{-\infty}^{\infty} \lambda \, dE_\lambda$ the meaning of norm convergence of approximating Riemann sums, then

$$\int_{-\infty}^{\infty} \lambda \, dE_\lambda = \int_{-\|A\|-\varepsilon}^{\|A\|} \lambda \, dE_\lambda = A.$$

This last formula is the classical *Spectral Theorem*. We can read out of this discussion the fact that each self-adjoint operator is the norm limit of finite linear combinations of mutually orthogonal "spectral projections" for A with coefficients in $\mathrm{sp}(A)$.

There are two key approximation theorems at the base of the study of von Neumann algebras. If $\mathscr{F} \subseteq \mathscr{B}(\mathscr{H})$ we write

$$\mathscr{F}' = \{T : T \in \mathscr{B}(\mathscr{H}), TA = AT \text{ for all } A \text{ in } \mathscr{F}\}.$$

We call $\mathscr{F}'$ the *commutant* of $\mathscr{F}$.

THEOREM 2.4. (Double Commutant Theorem) *If $\mathscr{R}$ is a von Neumann algebra (containing I) then $(\mathscr{R}')' = \mathscr{R}$.*

Of course $\mathscr{R} \subseteq (\mathscr{R}')'$. Suppose A is in $(\mathscr{R}')'$. To show that A is in the strong-operator closure of $\mathscr{R}$ (hence in $\mathscr{R}$), we must show that given a finite set of vectors $x_1, \ldots, x_n$ there is a T in $\mathscr{R}$ such that $\|(T - A)x_j\|$ is small. For the idea of the argument, we do this for one vector x_0. Let E_0 be the projection with range $[\mathscr{R}x_0]$. Since the range of E_0 is stable under B and B^* for each B in $\mathscr{R}$, $E_0 \in \mathscr{R}'$. Thus A commutes with E_0 and $Ax_0 \in [\mathscr{R}x_0]$, so that there is a T in $\mathscr{R}$ with $\|(T - A)x_0\|$ small. The case of n vectors is handled by using $n \times n$ matrices with entries in $\mathscr{R}$ acting on $\mathscr{H} \oplus \ldots \oplus \mathscr{H}$ (n times) in this same fashion.

The second key approximation result is:

THEOREM 2.5. (Kaplansky Density Theorem) *If $\mathfrak{A}$ is a self-adjoint algebra of operators on a Hilbert space then each operator in the unit ball of the strong-operator closure, $\mathfrak{A}^-$, of $\mathfrak{A}$ is in the strong-operator closure of the unit ball of $\mathfrak{A}$. Moreover, self-adjoint operators in $(\mathfrak{A}^-)_1$ are approximable by self-adjoint operators in $(\mathfrak{A})_1$, positive operators by positive operators; and, if $\mathfrak{A}$ is norm-closed, unitary operators by unitary operators.*

The ingredients of the proof are the following. Suppose H is a self-adjoint operator in $(\mathfrak{A}^-)_1$. If (T_a) is a net of operators in $\mathfrak{A}$ tending to H in the weak-operator topology then $(\frac{1}{2}[T_a + T_a^*])$ tends to H in this topology. Since H is in the weak-operator closure of the set of self-adjoint operators in $\mathfrak{A}$ and this set is convex, H is in the strong-operator closure of this set. Let (H_a) be a net of self-adjoint operators in $\mathfrak{A}$ with strong-operator limit H. With the aid of the function representation of commutative C^*-algebras, we can apply continuous functions defined on the reals to self-adjoint operators. If f is such a function and $f(\lambda) = \lambda$ for λ in $[-1, 1]$ then $f(H) = H$. If, in addition, the range of f is in $[-1, 1]$, then $\|f(K)\| \leqslant 1$ for each self-adjoint K. Finally, if f defines a strong-operator continuous mapping on the self-adjoint operators, then $(f(H_a))$ has $f(H)$ $(=H)$ as strong-operator limit and $\|f(H_a)\| \leqslant 1$. Now $f(H_a)$ is in the norm closure of $\mathfrak{A}$ so that there is some self-adjoint operator in the unit ball of $\mathfrak{A}$ near $f(H_a)$ in norm, hence, strong-operator near $f(H_a)$ (and H).

What can be proved is that each continuous f which vanishes at ∞

defines a strong-operator continuous function on the self-adjoint operators in $\mathscr{B}(\mathscr{H})$. The fact that multiplication is strong-operator continuous on bounded sets yields the result that polynomials are strong-operator continuous on bounded sets of self-adjoint operators. With the Stone–Weierstrass theorem one concludes, now, that all continuous functions are strong-operator continuous on bounded sets. The Cayley Transform $H \to (H - iI)(H + iI)^{-1} = u(H)$ maps self-adjoint operators H into unitary operators and is strong-operator continuous—by inspection. Moreover, $u(H)$ does not have 1 in its spectrum. The function $-i(z + 1)(z - 1)^{-1}$ is an inverse to the Cayley Transform (where $|z| = 1$ and $z \neq 1$). If f is a continuous real-valued function on $\mathbb{R}$ vanishing at ∞, define $g(z)$ to be $f(-i(z + 1)(z - 1)^{-1})$ for z different from 1 and z of modulus 1. Then, letting $g(1)$ be 0, g is continuous on the unit circle (since f vanishes at ∞) and $g(u(H)) = f(H)$. This exhibits f as the composition of two strong-operator continuous mappings, the Cayley Transform and a continuous function g on the bounded set of unitary operators. For arbitrary operators T in $(\mathfrak{A}^-)_1$, we use $\mathfrak{A}_2^-$, the 2×2 matrices over $\mathfrak{A}^-$ acting on $\mathscr{H} \oplus \mathscr{H}$. The operator $\tilde{H}$ with 0 on the diagonal and T, T^* at the off-diagonal positions is self-adjoint, has norm 1 and is in $(\mathfrak{A}_2^-)_1$. It is a strong-operator limit of self-adjoint operators of norm 1 in $\mathfrak{A}_2$. Each entry has norm not exceeding 1 and tends to the corresponding entry of $\tilde{H}$. Thus T is the strong-operator limit of elements in $(\mathfrak{A})_1$.

3. ALGEBRAIC STRUCTURE IN VON NEUMANN ALGEBRAS

The first crude division of von Neumann algebras into distinct algebraic isomorphism classes can be effected in terms of minimal projections. The most forceful use of minimal projections occurs in connection with factors. A projection E in a von Neumann algebra $\mathscr{R}$ is said to be *minimal* (in $\mathscr{R}$) when $E \neq 0$ and $0 < F \leqslant E$ for a projection F in $\mathscr{R}$ only if $F = E$. Clearly the property of being minimal for a projection is preserved under *-isomorphisms. The one-dimensional projections in $\mathscr{B}(\mathscr{H})$ provide examples of minimal projections, and this situation is virtually general.

THEOREM 3.1. *If $\mathscr{M}$ is a factor with a minimal projection, then I is the sum of minimal projections in $\mathscr{M}$. The cardinal number n of all families of minimal projections in $\mathscr{M}$ with sum I is the same; and $\mathscr{M}$ is *-isomorphic to $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is n-dimensional.*

In the situation described in this theorem, $\mathscr{M}$ is said to be a *factor of type I_n*. For a factor with a minimal projection, the theorem stated constitutes a complete description of its algebraic structure (two factors with a minimal projection are *-isomorphic if and only if they have the same cardinal n).

Restricting this discussion to factors is not a serious limitation. Roughly speaking, each von Neumann algebra is a direct sum of factors. More precisely, when $\mathscr{H}$ is separable, a von Neumann algebra is a *direct integral* of factors. The indexing family for the "sum" is a measure space and, instead of summing, we must integrate the component "factors". In any event, the model of a von Neumann algebra as a direct sum of factors is an excellent guide to their structure. It places the proper emphasis on the rôle of factors in the theory.

For a finer analysis of the algebraic structure of von Neumann algebras, it is useful to develop a theory which compares the sizes of the ranges of projections in such an algebra, relative to that algebra.

If E and F are projections in a von Neumann algebra $\mathscr{R}$, we say that E is *equivalent* to F (modulo $\mathscr{R}$), and write $E \sim F$ (mod $\mathscr{R}$), when there is an operator V in $\mathscr{R}$ mapping the range of E isometrically onto that of F.

Replacing V by VE, we can require that V be "normalized" so that it annihilates the range of $I - E$. In this case, we say that V is a *partial isometry* with *initial projection* E and *final projection* F. A computation shows that $V^*V = E$ and $VV^* = F$. Conversely, if V satisifes these equations, it is a partial isometry with initial projection E and final projection F. The projection E is a partial isometry with initial and final projection E; so that $E \sim E$. If V is a partial isometry with initial and final projections E and F, respectively, then V^* has F and E as initial and final projections, respectively. Thus $F \sim E$ if $E \sim F$. If, in addition, W is a partial isometry with initial projection F and final projection G, then WV is a partial isometry with initial projection E and final projection G. Thus $E \sim G$, if $E \sim F$ and $F \sim G$. It follows that $\sim$ is an equivalence relation on the projections of $\mathscr{R}$. It determines when two projections have "the same size" as measured by operators in $\mathscr{R}$.

It may seem like a difficult project to find isometries in $\mathscr{R}$ comparing projections. Actually, arbitrary operators in $\mathscr{R}$ do almost as well. The key to this observation is the "polar decomposition" of operators. Noting that $\| Tx \|^2 = \| (T^*T)^{\frac{1}{2}}x \|^2$, we see that the operator V which maps $(T^*T)^{\frac{1}{2}}x$ onto Tx extends to an isometry of the closure of the range of $(T^*T)^{\frac{1}{2}}$ onto that of T. Extending V by defining it to be 0 on the orthogonal complement of the range of $(T^*T)^{\frac{1}{2}}$ produces a partial isometry with initial space the closure of the range of $(T^*T)^{\frac{1}{2}}$ (which is the closure of the range of T^*); and $T = V(T^*T)^{\frac{1}{2}}$. If $T \in \mathscr{R}$, then $(T^*T)^{\frac{1}{2}}$ is in the C^*-algebra generated by T, hence in $\mathscr{R}$. One shows without difficulty that V commutes with each self-adjoint operator commuting with T; so that $V \in (\mathscr{R}')' = \mathscr{R}$. It follows that $R(T) \sim R(T^*)$. In particular, if T maps some part of the range of E onto some part of the range of F, that is, if $FTE \neq 0$, $R(FTE) \sim R(ET^*F)$ so that E and F have equivalent non-zero subprojections. Now $\{TEx : x \text{ in } H, T \text{ in}$

$\mathscr{R}\}$ is stable under $\mathscr{R}$ and $\mathscr{R}'$, so that the projection Q on the subspace spanned by this set is in $\mathscr{R}'$ and $\mathscr{R}''$ ($=\mathscr{R}$). Thus Q is in the centre, $\mathscr{C}$, of $\mathscr{R}$. If $FTE = 0$ for all T in $\mathscr{R}$ then $FQ = 0$; and $F \leqslant I - Q$. In the situation where $E \leqslant Q$ and $F \leqslant I - Q$, no operator in $\mathscr{R}$ will map a non-zero vector in the range of E onto one in the range of F. We conclude, from this discussion, that E and F fail to have equivalent non-zero subprojections in $\mathscr{R}$ if and only if they are "separated" by a central projection ($E \leqslant Q, F \leqslant I - Q$).

Associated with the equivalence relation $\sim$, there is a partial ordering on the equivalence classes. We write $E \precsim F$ when $E \sim E_0 \leqslant F$. (All the usual notational conventions related to a partial ordering will be used, e.g. $F \succsim E$ as well as $E \precsim F$, etc.). There is no difficulty in showing that $E \precsim E$ or that $E \precsim G$ if $E \precsim F$ and $F \precsim G$. It is true that $E \sim F$ if $E \precsim F$ and $F \precsim E$; but this requires a Hilbert space analogue of the Cantor–Bernstein argument in set theory to establish it. The study of this partial ordering in a von Neumann algebra, is the *comparison theory of projections* in that algebra.

In a factor, there is no possibility of separating non-zero projections by a central projection. Such projections always have equivalent non-zero subprojections. Combining this with the fact that $E \sim F$ if $E = \sum E_a$, $F = \sum F_a$ and $E_a \sim F_a$ for all a, and an exhaustion argument, we have:

THEOREM 3.2. *If E and F are projections in a factor $\mathscr{M}$, then either $E \precsim F$ or $F \precsim E$.*

To parallel this general comparability in factors we have:

THEOREM 3.3. (The Comparison Theorem) *If E and F are projections in a von Neumann algebra $\mathscr{R}$, there is a central projection Q such that $QE \precsim QF$ and $(I - Q)F \precsim (I - Q)E$.*

By analogy with set theory, a projection E in $\mathscr{R}$ equivalent to a proper subprojection is said to be *infinite* (relative to $\mathscr{R}$), otherwise *finite*. A factor $\mathscr{M}$ with a non-zero finite projection but nominimal projection is said to be of *type II*; of *type II_1* if I is finite, of *type II_∞* if I is infinite. If $\mathscr{M}$ has no non-zero finite projection it is said to be of *type III*. Loosely speaking, a von Neumann algebra is of type I_n, II_1, II_∞, or III if all the factors appearing in its decomposition are of that type. It is possible to define von Neumann algebras of various types without reference to the factors appearing in the decomposition—that is, in "global" terms. Each von Neumann algebra is the direct sum of von Neumann of various "pure" types (some of the summands possibly not present).

It is apparent that type is preserved under *-isomorphism, and we have seen that all factors of type I_n (same n) are *-isomorphic. Are there factors not of type I? If G is a countable (discrete) group and $\mathscr{H}$ is $l_2(G)$, the square summable functions on G, and L_{g_0}, R_{g_0} are the operators defined on $\mathscr{H}$ by:

$$(L_{g_0}x)(g) = x(g_0^{-1}g), \qquad (R_{g_0}x)(g) = x(gg_0),$$

then L_g and $R_{g'}$ are commuting unitary operators. Let $\mathscr{L}_G$ and $\mathscr{R}_G$ be the von Neumann algebras generated by $\{L_g: g \text{ in } G\}$ and $\{R_g: g \text{ in } G\}$ respectively. Then $\mathscr{L}_G' = \mathscr{R}_G$ (so that $\mathscr{R}_G' = \mathscr{L}_G$). If each conjugacy class in G is infinite (with the exception of $\{e\}$) then $\mathscr{L}_G$ (and $\mathscr{R}_G$) is a factor of type II_1. Specific examples arise from the group Π of those permutations of the integers which move at most a finite number, and F_n the free (non-abelian) group on n generators ($n \geqslant 2$). It is known that $\mathscr{L}_\Pi$ and $\mathscr{L}_{F_n}$ are not isomorphic. Using weak-commutativity techniques that establish this, a non-denumerable collection of groups were constructed, recently, with associated II_1 factors pairwise non-isomorphic.

If $\mathscr{M}$ is a factor of type II_1, acting on $\mathscr{H}$ and $\tilde{\mathscr{M}}$ is the algebra of all those $\aleph_0 \times \aleph_0$ matrices with entries from $\mathscr{M}$ which, acting on $\mathscr{H}$, the $\aleph_0$-fold direct sum of $\mathscr{H}$ with itself, yield bounded operators, then $\mathscr{M}$ is a factor of type II_∞. Moreover, each factor of type II_∞ arises in this way from a factor of type II_1.

To exhibit factors of type III, we make use of the (unique) C^*-algebra $\mathfrak{A}$ which is generated as a C^*-algebra by an infinite sequence $(\mathscr{N}_j)$ of C^*-subalgebras $\mathscr{N}_j$, mutually commuting, each isomorphic to the algebra of complex 2×2 matrices. There is no difficulty in constructing $\mathfrak{A}$, explicitly, on $L_2(0, 1)$. Representing each element of $\mathscr{N}_j$ as a 2×2 matrix, let ρ_j^λ be the state of $\mathscr{N}_j$ which assigns $\lambda a + (1 - \lambda)b$ to A_j in $\mathscr{N}_j$, where a and b are the diagonal entries of A_j and λ is in $[0, 1]$. There is a state ρ_λ of $\mathfrak{A}$ with the property that

$$\rho_\lambda(A_{j_1} \dots A_{j_n}) = \rho_{j_1}^\lambda(A_{j_1}) \dots \rho_{j_n}^\lambda(A_{j_n}),$$

when $A_{j_1} \in \mathscr{N}_{j_1}, \dots, A_{j_n} \in \mathscr{N}_{j_n}$ and $j_1, \dots, j_n$ are distinct. Applying the GNS constructjon to ρ_λ we construct representations of π_λ of $\mathfrak{A}$ on $\mathscr{H}_\lambda$. Then π_0 and π_1 are irreducible; so that $\pi_0(\mathfrak{A})^- = \mathscr{B}(\mathscr{H}_0)$ and $\pi_1(\mathfrak{A})^- = \mathscr{B}(\mathscr{H}_1)$. In addition $\pi_{\frac{1}{2}}(\mathfrak{A})^-$ is a factor of type II_1 (*-isomorphic to $\mathscr{L}_\Pi$, curiously enough). With λ and λ' in $(0, \frac{1}{2})$, $\pi_\lambda(\mathfrak{A})^-$ and $\pi_{\lambda'}(\mathfrak{A})^-$ are not *-isomorphic and are factors of type III. Thus $\pi_\lambda(\mathfrak{A})^-$ is a factor of type III for $\lambda \neq 0, \frac{1}{2}, 1$; and this family contains a non-denumerable infnity of non-isomorphic factors of type III.

Recent results make deep inroads into the analysis of the structure of type III von Neumann algebras. In essence each such can be realized, in a canonical manner, as the von Neumann algebra generated by a one-parameter group of unitary operators, each inducing an automorphism of a von Neumann algebra $\mathscr{R}_0$ having no summand of type III, and $\mathscr{R}_0$.

4. ACTION OF VON NEUMANN ALGEBRAS ON SPACES

The problem of when two von Neumann algebras act in the same way on their underlying Hilbert spaces can be reduced to a comparison of their algebraic structure. The main result is:

THEOREM 4.1. (Unitary Implementation) *If $\mathscr{R}_1$ and $\mathscr{R}_2$ are von Neumann algebras acting on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, x_1 and x_2 are unit vectors in $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $[\mathscr{R}_1 x_1] = [\mathscr{R}'_1 x_1] = \mathscr{H}_1$ and $[\mathscr{R}_2 x_2] = [\mathscr{R}'_2 x_2] = \mathscr{H}_2$, and ϕ is a *-isomorphism of $\mathscr{R}_1$ onto $\mathscr{R}_2$, then there is a unitary transformation U of $\mathscr{H}_1$ onto $\mathscr{H}_2$ such that $A = U^{-1}\phi(A)U$.*

We say that U *implements* the isomorphism ϕ. Diligent use of this result and the comparison theory of projections reduces the question of action on the space to one of algebraic type for von Neumann algebras.

If the vector state $\omega_{x_1}|\mathscr{R}_1$ can be "transported" to a vector state $\omega_{y_1}|\mathscr{R}_2$ by means of ϕ (that is, if we can find a unit vector y_1 in $\mathscr{H}_2$ such that $\langle Ax_1, x_1\rangle = \langle\phi(A)y_1, y_1\rangle$ for all A in $\mathscr{R}_1$), and if y_1 can be chosen cyclic for $\mathscr{R}_2$ then the mapping $Ax_1 \to \phi(A)y_1$ is an isometric mapping of $\mathscr{R}_1 x_1$ onto $\mathscr{R}_2 y_1$ and extends to a unitary transformation U of $[\mathscr{R}_1 x_1]$ $(= \mathscr{H}_1)$ onto $[\mathscr{R}_2 y_2]$ $(= \mathscr{H}_2)$. There is no difficulty in showing that U implements ϕ.

In any event the functional ω defined by: $\omega(\phi(A)) = \langle Ax_1, x_1\rangle$, is a state of $\mathscr{R}_2$. It has certain continuity properties. The programme outlined above motivates the study of such states.

A state ω of a von Neumann algebra $\mathscr{R}$ is said to be *completely-additive* when $\omega(\sum_a E_a) = \sum_a \omega(E_a)$ for each orthogonal family $\{E_a\}$ of projections in $\mathscr{R}$.

A *-isomorphism of one von Neumann algebra $\mathscr{R}_1$ onto another $\mathscr{R}_2$ preserves order; so that $\sum_a \phi(E_a)$, the smallest projection larger than each $\phi(E_a)$, is the image $\phi(\sum_a E_a)$ of the projection $\sum_a E_a$, when $\{E_a\}$ is an orthogonal family of projections in $\mathscr{R}$. Thus $\omega \circ \phi$ is a completely additive state of $\mathscr{R}_1$ if ω is a completely-additive state of $\mathscr{R}_2$.

An ostensibly more stringent continuity requirement is that $\lim_a \omega(H_a) = \omega(H)$ for each monotone increasing net (H_a) of self-adjoint operators in $\mathscr{R}$. States which satisfy this condition are said to be *normal states* (of $\mathscr{R}$). Finally, there are the assumptions that ω is weak (or strong)-operator continuous on $(\mathscr{R})_1$. We saw that the equivalence of these two assumptions was a consequence of the fact that convex sets of operators have the same weak and strong-operator closures. The main result relating these conditions on ω is:

THEOREM 4.2. *If ω is a state of the von Neumann algebra $\mathscr{R}$ the following conditions are equivalent.*

(a) *There is a countable family $\{y_n\}$ of mutually orthogonal vectors such that $\sum \|y_n\|^2 = 1$ and $\omega = \sum_n \omega_{y_n}|\mathscr{R}$.*

(b) *There is a countable family of vectors* $\{x_n\}$ *such that* $\sum_n \| x_n \|^2 = 1$ *and* $\omega = \sum_n \omega_{x_n} | \mathscr{R}$.

(c) ω *is weak-operator continuous on* $(\mathscr{R})_1$.

(d) ω *is strong-operator continuous on* $(\mathscr{R})_1$.

(e) ω *is normal.*

(f) ω *is completely-additive.*

As for the possibility of realising a normal state ω as a vector state, the condition for this can be described, best, in terms of the *support* of ω. If $I - E$ is the union of all projections in $\mathscr{R}$ which are annihilated by ω, E is said to be the support of ω. It follows from the fact that ω is normal that $\omega(I - E) = 0$. It is not difficult to conclude that $\omega(H) > 0$ unless $R(H) \leqslant I - E$, for a positive H in $\mathscr{R}$.

THEOREM 4.3. *A normal state of a von Neumann algebra* $\mathscr{R}$ *acting on a Hilbert space* $\mathscr{H}$ *is a vector state if and only if its support E is a cyclic projection in* $\mathscr{R}$ *(that is, E has a range* $[\mathscr{R}'z]$ *for some vector z).*

In the circumstances of the Unitary Implementation Theorem, there is a vector x_2 such that $[\mathscr{R}'_2 x_2] = \mathscr{H}_2$. If E is any projection in $\mathscr{R}_2$, then $[\mathscr{R}'_2 E x_2] = [E\mathscr{R}'_2 x_2] = E(\mathscr{H}_2)$; so that the support of each normal state is a cyclic projection in $\mathscr{R}_2$, and each normal state of $\mathscr{R}_2$ is a vector state. An application of comparison theory allows us to choose the vector representing the state as a generator for $\mathscr{H}_2$ under $\mathscr{R}_2$ when the state is separating (since $[\mathscr{R}_2 x_2] = \mathscr{H}_2$), which supplies what is needed for the proof of Theorem 4.1.