19. J. von Neumann, *Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren*, Math. Ann. **102** (1929), 370–427.

20. T. W. Palmer, *Characterizations of C*-algebras*, Bull. Amer. Math. Soc. **74** (1968), 538–540. MR **36** #5709.

21. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, University Series in Higher Math., Princeton, N.J., Reprinted (with corrections) by Robert Krieger Publ. Co., 1974. MR **22** #5903.

22. G. E. Šilov, *On the decomposition of a commutative normed ring in a direct sum of ideals*, Mat. Sb. **32** (74) (1953), 353–364; English transl., Amer. Math. Soc. Transl. (2) **1** (1955), 37–48. MR **14**, 884; **17**, 512.

23. I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260–264. MR **16**, 1125.

24. I. Vidav, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Z. **66** (1956), 121–128. MR **18**, 912.

25. L. Waelbroeck, *Le calcul symbolique dans les algèbres commutatives*, J. Math. Pures Appl. (9) **33** (1954), 147–186. MR **17**, 513.

26. K. Yosida, *On the group embedded in the metrical complete ring*. Japan. J. Math. **13** (1936), 459–472.

The structure of factors, by S. Anastasio and P. M. Willig, Algorithmics Press, New York, 1974, iii+116 pp.

In a paper appearing in the 1929 Mathematische Annalen (*Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren*), von Neumann initiated the study of *Rings of operators* (renamed *von Neumann algebras* in J. Dixmier's classic, *Les algèbres d'opérateurs dans l'éspace Hilbertien*, Paris, 1957). These are algebras, R, of bounded linear transformations (operators) of a Hilbert space H into itself, closed in the strong-operator topology ($A_n \to A$ means that $A_n x \to A x$, for each x in H) and having the property that A^*, the adjoint of A, is in R if A is. Von Neumann saw two motivating forces behind the study of these algebras: applications to the newly emergent Quantum Physics, and application to the study of infinite groups. Quantum Physics, as it was being formulated, was involved with algebraic combinations of (selfadjoint) operators. It was certain to require (at the mathematical level) a deeper understanding of the structure of algebras of operators. The technique of group algebras had been so useful in the study of finite groups that some corresponding construct for infinite groups was certain to be crucial for their analysis.

The detailed study of von Neumann algebras was undertaken in a series of papers written in collaboration with F. J. Murray. The first appeared in the 1936 Annals of Mathematics, *On rings of operators*. Since noncommutativity was the basic technical problem, Murray and von Neumann moved quickly to the study of those von Neumann algebras, *factors*, whose centers consist of scalar multiples of the unit element.

As in much of Functional Analysis, the statements of results in the theory of operator algebras are algebraic in flavor. The ideas, proofs and

difficulties are analytic. The condition that the von Neumann algebra be closed in the strong-operator topology is the first such analytic "subtlety". With no closure assumption, there are no (serious) results (other than those which refer to a closure). Other closure conditions can be (and are) imposed. Norm-closure (closure relative to the norm induced by the operator bound) leads to the (broader) class of *C*-algebras*. All other conditions commonly imposed lead, again, to the von Neumann algebras— though this takes some proving. Loosely speaking a von Neumann algebra bears the same relation to a *C**-algebra that an algebra of bounded Borel measurable functions bears to an algebra of continuous functions. This relation takes explicit form in theorems about commutative operator algebras. In the general case, the spirit of this relationship pervades the subject in the nature of the proofs and the formulation of results. In particular von Neumann algebras are generated by projections (self-adjoint idempotents), in the same way that the characteristic functions of Borel sets generate the algebra of bounded measurable functions.

The basic technique of Murray and von Neumann consists of a method for comparing the "sizes" of projections in a von Neumann algebra R (relative to R). Two projections E and F in R are said to be equivalent modulo R when some operator in R maps the range of E isometrically onto the range of F. When the von Neumann algebra is a factor, M, Murray and von Neumann develop a dimension function, d, on the set of projections in M uniquely characterized (up to a positive multiple) by the properties of taking the same value on equivalent projections, nonzero values on nonzero projections and "additivity" ($d(E+F)=d(E)+d(F)$ when $EF=0$). The value ∞ may be assumed by d (on "infinite" projections—those equivalent to some proper subprojection). The theory indicates that d can have the following ranges (after suitable normalization): $\{0, 1, 2, \cdots, n\}$, $[0, 1]$, $[0, \infty]$, and $\{0, \infty\}$. These possibilities correspond to (define) the classes of factors of type I_n (n can be ∞), type II_1, type II_∞, and type III, respectively. The first class (type I_n) is characterized by having minimal projection. In this case the factor is *-isomorphic to the algebra of all bounded operators on n-dimensional Hilbert space. (The * signifies that the isomorphism preserves the adjoint—equivalently, carries selfadjoint elements onto selfadjoint elements. The isomorphism automatically preserves norms and the strong-operator topology on the unit ball in the algebra.)

The first question that comes to mind is that of the actual existence of the other classes. In their 1936 article Murray and von Neumann construct examples of type II_1 and II_∞ factors by Ergodic Theory techniques. In the third paper (written by von Neumann, Annals of Mathematics, 1940) a more complicated application of this technique was used to construct

factors of type III. The factors of type II_1 are characterized as having no minimal projections and the identity operator finite (i.e., not an infinite projection). Those of type II_∞ have no minimal projection, some nonzero finite projection, and I is infinite. The other factors comprise the type III class (all nonzero projections are infinite).

The next question, in order of importance, is that of "isomorphism". As noted, all factors of type I_n are *-isomorphic. Is the same true for factors of type II_1? In other words, is the job of classification (up to algebraic type) completed by the above separation in terms of the range of the dimension function? Murray and von Neumann answered this question (negatively) in their fourth paper (which appeared in the 1943 Annals of Mathematics) by producing two nonisomorphic factors of type II_1. In the process they construct a class of examples of such factors different from those they described in the 1936 article and closer to the initial motivation of group algebras.

If G is a countable (discrete) group each of whose conjugacy classes (other than that of the identity element) is infinite, H is $l_2(G)$, the Hilbert space of square-integrable, complex-valued functions on G, and L_g is the (unitary) operator corresponding to left translation by g^{-1} on G (that is, $(L_g(\varphi))(g') = \varphi(g^{-1}g')$), then the von Neumann algebra (on H) generated by $\{L_g : g \text{ in } G\}$ is a factor of type II_1. (The infinite conjugacy classes provide the necessary noncommutativity to yield a factor.) Two non-isomorphic factors of type II_1 are obtained by applying this construction to the free (nonabelian) group on two generators and to the group of finite permutations of the integers. In the latter case, given a finite number of group elements, some group element other than the identity commutes with all of them. This translates itself into a corresponding approximate (in the sense of the strong-operator topology) commutativity property for the associated factor. The same property is not enjoyed by the factor associated with the free group; for this property would force more of an approximate, finitely-additive measure on the free group than it can tolerate.

At this point, Murray and von Neumann had clarified the problem. There was no longer a question (in more than a formal sense) of how many *-isomorphism classes of factors of type II_1 there are. There are certainly an infinite number. Moreover it was abundantly clear that refined contortions with groups and approximate commutativity would produce an infinite number of nonisomorphic II_1 factors. It was equally clear (with a small amount of preliminary manipulation) that this route to the examples would be horribly complicated (in a group, combinatorial, analytic sense) and not terribly interesting from the point of view of usable technique or information about the structure of factors. To whom was all

this clear? To those few of us who took up the subject of factors in 1950—
and I daresay, to Murray and von Neumann after they had located the
two isomorphism classes. No one had worked with factors from the time
Murray and von Neumann stopped (1943) until 1950 and no one worked
with factors other than Murray and von Neumann while they were at
work.

Precisely the group, approximate-commutativity route to an infinite
number of examples of nonisomorphic II_1 factors was followed (from
1962 to 1969). It was as complicated and uninteresting as expected (with
one exception to be noted). This is not to say that, as a mathematical
work, it is unimportant. It provided a decent burial for the question still
open in a formal sense. But should it be exhumed, displayed in a book
and extolled in the preface to that book? Much that is important as
mathematical work and must be endured on the route to some goal is
not worth reading. Nonetheless the first half of the book under review
is devoted to a systematization and presentation of this work.

J. T. Schwartz writes a brief preface to the book. It is not carefully
written. He attributes direct integral theory to Murray and von Neumann.
It is work done and published by von Neumann alone (Ann. of Math.,
1949). Schwartz notes that "They [Murray and von Neumann] also
succeeded in giving a complete structural account of a special subclass of
the type II factors, the so-called hyperfinite factors of type II_1." This is
quite misleading. We know that it represents one isomorphism class; but
aside from that we know very little about its structure. Does it contain a
nonhyperfinite factor? Schwartz states that "Investigation of · · · and of
factors continued actively in the period immediately following the von
Neumann-Murray work." Not so—the subject lay fallow until 1950.
He goes on to say "· · · there was little success in analyzing the structure of
factors deeply. In particular, no one was able to decide the fundamental
question of the existence or nonexistence of infinitely many nonisomor-
phic factors of type II_1 and of type III." Everyone was able to decide the
question—not many thought it worth the enormous effort to arrange for
its burial (at least not in the same plot of ground in which Murray and
von Neumann had placed their two examples). Schwartz then claims,
about the eventual solution, that "this work has made possible subsequent
vigorous steps in our understanding of the factor theory and of the struc-
tural invariants of factors." The II_1 examples have done virtually nothing—
except for the third isomorphism class of II_1 factors produced by Schwartz
himself (just prior to the Stockholm Congress in 1962)! Schwartz develops
a technique for projecting the algebra of all operators onto certain factors
which is quite useful in other connections and leads to interesting struc-
tural questions. It is the most important technique to emerge from the

entire study. If Schwartz's statement has any validity, it resides in his having included the II_1 and type III studies in one statement—an unfortunate union, for they are quite different in quality. The type III developments are really important.

It may seem unwarranted to subject Schwartz's short preface to this searching scrutiny. Very likely, he just wants to give the book a good sendoff and stay within reasonable bounds of accuracy. But the danger of convincing the unitiated that there is really much worth detailed study in the II_1 isomorphism classes (as they now stand) is too great to let it pass unnoted. It is all the more misleading because it is difficult work. Complexity is often confused with value in mathematics.

The recent drive to the infinite number of isomorphism classes of II_1 factors began some years after Schwartz's notable contribution. W.-M. Ching constructed a fourth class in his thesis under I. Halperin's direction. The drive ended a year later when, quite appropriately, another research student (this time of G. Reid), D. Mc Duff, constructed an infinite set. During that year, some of the older lions, sensing the kill, moved in to find fifth (Sakai), sixth and seventh (Dixmier-Lance), and eighth and ninth (Zeller-Meier) classes of II_1 factors.

The situation for type III factors was very different. Murray and von Neumann had constructed many examples but had not distinguished isomorphism classes of type III factors. (It is likely from their line of work that they had not made a serious attempt to do so.) Employing the approximate-commutativity technique of Murray and von Neumann, modified for type III factors, L. Pukanszky constructed two nonisomorphic factors of type III (Math. Debrecen 1956). Shortly after his construction of the third isomorphism class of II_1 factors, Schwartz extended his method to yield a third isomorphism class of type III factors.

The physicists' interest in the von Neumann algebra development was far from negligible. R. Haag's "local ring" formulation of quantum field theory (associating von Neumann algebras with regions of Minkowski space-time) had opened a promising line of investigation. H. Araki had established (1963) that type III von Neumann algebras were present in the description of the free field. Further work indicated strongly that most of what would be found in the way of von Neumann algebras associated with the quantum physics of infinitely extended systems would be type III factors (and probably hyperfinite, at that—that is, generated by an ascending union of algebras *-isomorphic to finite-dimensional matrix algebras each containing I).

Working on his Ph.D. thesis (as a physics student under Arthur Wightman), R. T. Powers studied representations of canonical anticommutation relations (CAR). He proved (not in, but as an offshoot of, his thesis)

that there is a one parameter family of such representations generating mutually nonisomorphic factors of type III. From their very nature, they are hyperfinite factors. Thus Powers had answered another important question. There are nonisomorphic, hyperfinite, type III factors (unlike the II_1 situation, where Murray and von Neumann had proved them isomorphic). The interplay between mathematics and physics was vital in this work. The methods Powers used were inspired by physical structure (notably the "cluster decomposition property" of states—the tendency for their correlations to diminish far off in space—or time), but he made critical use of the methods that had been developed in C^*-algebra theory (notably Glimm's results on UHF algebras—again, thesis work).

At this same time, M. Tomita proposed a solution to an old question raised by Murray and von Neumann concerning type III factors. M. Takesaki systematized, extended and developed the ideas of Tomita, fashioning a powerful tool for the study of type III von Neumann algebras. He, H. Araki and A. Connes have used this tool, over the past four years, in an awe inspiring analysis of such algebras. If anything can be described as "vigorous steps in our understanding of the factor theory and of the structural invariants of factors," this is it. In effect they have established that type III von Neumann algebras are crossed products of the reals with a type II von Neumann algebra and proved a uniqueness for this decomposition. The problem returns, then, to the structure of factors of type II_1. Despite the infinite set of nonisomorphic examples, we know precious little about this structure.

The second half of the book under review devotes itself to type III factors. It begins with a discussion of infinite tensor products and a presentation of Powers's result on nonisomorphic, hyperfinite, type III factors. They take an effective route to this result. Unfortunately their route avoids one of the most striking and important results Powers obtains in his proof—the transitivity of pure states of a UHF algebra (in particular, of the CAR) under *-automorphisms (Corollary 3.8 of Powers's 1967 Annals of Mathematics article). After this they present an incomplete though useful, account of the Araki-Woods extension of Powers's work. This is followed by J. J. Williams' version of the construction of a continuum of nonisomorphic, nonhyperfinite type III factors. Again, we are back at a worthy result not worth detailed exposition in a book.

It would be unreasonable to expect the authors to record the truly important developments centering on the analysis of type III von Neumann algebras. These results are too recent for that. I would advise the reader interested in studying this area to wait for such a book and to hope that it is written by one of the heroes of that victory.

RICHARD V. KADISON