A NOTE ON DERIVATIONS OF OPERATOR ALGEBRAS

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1. *Introduction*

In [**1, 2, 3, 5, 6**] proofs are to be found of the assertion that derivations of von Neumann algebras are inner. All involve some elements of complexity. The present note is devoted to an especially simple proof of this fact. Our point of departure is Theorem 4 of [**3**]; or, rather, that part of it which asserts that a derivation of a von Neumann algebra is implemented by some (bounded), operator on the Hilbert space on which it acts. This fact depends on only a few ideas and simple operator-algebra techniques. In this note, we complete the argument by showing that an operator with "minimal norm" which implements the derivation lies in the factor.

For greater clarity, we present in §2 the argument for the case of a factor. For the case of the general von Neumann algebra, we work with an operator having minimal norm over each central projection (as in [**4**; Theorem 3.1]). The modifications necessary to extend the argument of §2 to the general von Neumann algebra appear in §3. The "minimal norm" technique is, in fact, inherent in the argument of [**3, 5**].

2. *The argument*

If d is a derivation of $\mathcal{M}$ (factor or von Neumann algebra) acting on the Hilbert space $\mathcal{H}$, and T is an operator which implements d on $\mathcal{M}$ (that is, $d(A) = AT - TA$ for each A in $\mathcal{M}$), then the set of operators in the closed ball of radius $\|T\|$ and centre 0 which implement d is a weak-operator-closed (hence, -compact) non-null subset of that ball. From the "finite-intersection property" (compactness), there is a non-null set of operators of minimal norm implementing d.

LEMMA 1. *If $\mathcal{M}$ is a factor acting on the Hilbert space $\mathcal{H}$ and d is a derivation of $\mathcal{M}$ into $\mathcal{M}$ then $\|T\| = \|E'TF'\|$ for each pair of projections E', F' in $\mathcal{M}'$ such that $0 < F' \leqslant E'$ and each T of minimal norm in $\mathcal{B}(\mathcal{H})$ implementing d.*

Proof. There is a projection F_1' in $\mathcal{M}'$ such that $F_1' \leqslant F'$ and $I = F_1' + F_2' + \ldots + F_n'$ where F_1' and F_j' are equivalent in $\mathcal{M}'$ (n may be infinite). Since

$$\|F_1'TF_1'\| = \|F_1'E'TF'F_1'\| \leqslant \|E'TF'\| \leqslant \|T\|, \tag{2.1}$$

it will suffice to prove that $\|F_1'TF_1'\| = \|T\|$.

Suppose T_0 is an operator in $\mathcal{B}(F_1'(\mathcal{H}))$ having minimal norm among those operators which implement the derivation $AF_1' \to d(A)F_1'$ of $\mathcal{M}F_1'$ into itself. Since

$$d(A)F_1' = F_1'd(A)F_1' = F_1'(AT - TA)F_1' = AF_1'TF_1' - F_1'TF_1'A,$$

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we have that $\|T_0\| \leqslant \|F_1'TF_1'\| \leqslant \|T\|$. If $T_1 = \sum_{j=1}^{n} V_j'^* T_0 V_j'$, where V_j' is a partial isometry in $\mathscr{M}'$ with initial projection F_j' and final projection F_1', then

$$AT_1 - T_1 A = \sum_{j=1}^{n} F_j' A T_1 - T_1 A F_j' = \sum_{j=1}^{n} F_j' V_j'^* A T_0 V_j' - V_j'^* T_0 A V_j' F_j'$$

$$= \sum_{j=1}^{n} V_j'^* (AT_0 - T_0 A) V_j' = \sum_{j=1}^{n} V_j'^* d(A) F_1' V_j'$$

$$= \sum_{j=1}^{n} d(A) F_j' = d(A). \tag{2.2}$$

Since $T_1 = \sum_{j=1}^{n} F_j' V_j'^* T_0 V_j' F_j'$ and $\{F_j'\}$ is an orthogonal family of projections (in $\mathscr{M}'$), $\|T_1\| = \sup_j \{\|V_j'^* T_0 V_j'\|\} = \|T_0\|$. As T_1 implements d, $\|T\| \leqslant \|T_1\| = \|T_0\|$. Thus $\|T\| = \|T_0\|$; and $\|T\| = \|F_1'TF_1'\|$.

THEOREM 2. *If d is a derivation of a factor $\mathscr{M}$ acting on the Hilbert space $\mathscr{H}$ and T is an operator of minimal norm implementing d then T is in $\mathscr{M}$.*

Proof. From Lemma 1, $\|E'TF'\| = \|T\|$ for each pair of projections E', F' in $\mathscr{M}'$ such that $0 < F' \leqslant E'$. We show that $(I-E')TE' = 0$ for each projection E' in $\mathscr{M}'$. Hence $T = E'TE' + (I-E')T(I-E')$ so that $TE' = E'T$ for each projection E' in $\mathscr{M}'$; and T is in $\mathscr{M}''$ $(= \mathscr{M})$.

Suppose that $(I-E')TE' \neq 0$. Since

$$(I-E')(E'T - TE') = -(I-E')TE' \in \mathscr{M}'$$

(see the observation [3; Lemma 5] that T induces a derivation of $\mathscr{M}'$), there is a projection F' in $\mathscr{M}'$ such that $0 < F' \leqslant E'$ and $aF' \leqslant F'T^*(I-E')TF'$ for some positive scalar a. It follows that

$$\|F'T^*E'TF'\| = \|E'TF'\|^2 = \|T\|^2 = \|TF'\|^2 = \|F'T^*TF'\|$$

$$= \|F'T^*(I-E')TF' + F'T^*E'TF'\| \geqslant \|F'T^*E'TF' + aF'\|$$

$$> \|F'T^*E'TF'\|;$$

a contradiction. Thus $(I-E')TE' = 0$ and $T \in \mathscr{M}$.

COROLLARY 3. *Each derivation of a factor is inner.*

Proof. From [3; Theorem 4], each derivation of a factor is implemented by a bounded operator. From the discussion at the beginning of this section, there is an operator of minimal norm implementing the derivation. From Theorem 2, this operator lies in the factor.

3. *The general case*

We formalize the discussion of minimal-norm operators necessary for general von Neumann algebras in the lemma that follows. If T is such that, for each central projection Q in the von Neumann algebra $\mathscr{R}$, TQ is an operator of minimal norm implementing the derivation d of $\mathscr{R}$ restricted to $\mathscr{R}Q$, we say that T has *totally minimal norm* (implementing d).

LEMMA 4. *If d is a derivation of the von Neumann algebra $\mathscr{R}$ there is an operator T with totally minimal norm implementing d.*

Proof. From [3; Theorem 4], there is an operator B implementing d. Let $\mathscr{F}$ be the class of finite orthogonal families $\{Q_1, \ldots, Q_n\}$ of central projections in $\mathscr{R}$ with sum I. Partially order $\mathscr{F}$ by "refinement" (so that $\{P_1, \ldots, P_m\} \leqslant \{Q_1, \ldots, Q_n\}$ when each P_j is a sum of Q_k's). Let $T_{(Q_1, \ldots, Q_n)}$ be $\sum_{k=1}^{n} T_k$ where T_k implements the restriction of d to $\mathscr{R}Q_k$ and has minimal norm. Of course, $\|T_k\| \leqslant \|BQ_k\|$ so that $\|T_{(Q_1, \ldots, Q_n)}\| \leqslant \|B\|$ and the net $(T_{(Q_1, \ldots, Q_n)} : \{Q_1, \ldots, Q_n\}$ in $\mathscr{F})$ is contained in the (weak-operator-compact) closed ball of radius $\|B\|$ and centre 0 in $\mathscr{B}(\mathscr{H})$. Since each $T_{(Q_1, \ldots, Q_n)}$ implements d, T does, where T is the limit of some cofinal subnet.

If Q is a central projection in $\mathscr{R}$ and $\{Q_1, \ldots, Q_n\}$ is a refinement of $\{Q, I-Q\}$, by renumbering, we may suppose $Q = Q_1 + \ldots + Q_k$. If B_0 in $\mathscr{B}(Q(\mathscr{H}))$ implements the restriction of d to $\mathscr{R}Q$, then $\|T_j\| \leqslant \|B_0 Q_j\|$ for $j = 1, \ldots, k$ by choice of T_j. As $T_1 + \ldots + T_k = QT_{(Q_1, \ldots, Q_n)}$, we have $\|T_1 + \ldots + T_k\| = \|QT_{(Q_1, \ldots, Q_n)}\| \leqslant \|B_0\|$. Since T is the limit of a cofinal subnet of $T_{(Q_1, \ldots, Q_n)}$ it is the limit of the (cofinal) subnet of that subnet consisting of refinements of $\{Q, I-Q\}$. Thus $\|QT\| \leqslant \|B_0\|$, and T has totally minimal norm.

The argument of Lemma 4 is an adaptation of that of [4; Theorem 3.1(b)] to the present situation. Lemma 1 can be extended, now, to a von Neumann algebra $\mathscr{R}$ by asserting that $\|TQ\| = \|E'TF'\|$, where Q is the central carrier of F' and T has totally minimal norm. In the proof, we choose F_1' so that $F_1' + F_2' + \ldots + F_n' = Q_0$ with Q_0 a non-zero central projection and F_1' equivalent to F_j' in $\mathscr{M}'$. Each occurrence of $\|T\|$ is replaced by one of $\|TQ_0\|$ and $\|E'TF'\|$ is replaced by $\|E'TF'Q_0\|$ in (2.1). The conclusion from (2.1), in this case, is that it will suffice to prove $\|F_1'TF_1'\| = \|TQ_0\|$ in order to show that $\|E'TF'Q_0\| = \|TQ_0\|$ for some non-zero central projection Q_0 of the central carrier of F'. Using a maximal orthogonal family of such central projections Q_0, we conclude that $\|E'TF'\| = \|TQ\|$. The computation of $AT_1 - T_1 A$ leads to $d(A)Q_0$ rather than $d(A)$ in (2.2); so that T_1 implements the restriction of d to $\mathscr{R}Q_0$ rather than d.

In Theorem 2 each occurrence of $\|T\|$ is replaced by one of $\|TQ\|$ where Q is the central carrier of F'; and the extended theorem asserts that an operator implementing d, with totally minimal norm lies in $\mathscr{R}$.

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