ALGEBRAIC AUTOMORPHISMS OF OPERATOR ALGEBRAS

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1. *Introduction*

In [9] we initiated a study of the group of $*$-automorphisms of a C*-algebra in its norm topology proving, there, that if $\|\alpha-\iota\| < 2$ for a $*$-automorphism α of a C*-algebra $\mathfrak{A}$ acting on $\mathscr{H}$, there is a unitary operator U in $\mathfrak{A}^-$, the weak-operator closure of $\mathfrak{A}$, such that $\alpha(A) = UAU^*$ for all A in $\mathfrak{A}$. Using the holomorphic function calculus together with the techniques of C*- and von Neumann algebras, we showed that, in this case, α lies on a norm-continuous, one-parameter group of $*$-automorphisms of $\mathfrak{A}$ and is, therefore, the exponential of a derivation of $\mathfrak{A}$. The derivation theorem [**2**; p. 312, Corollaire] completes the argument. We noted, as a consequence of those results, that the connected component of the identity in the group of $*$-automorphisms of $\mathfrak{A}$ is open and (algebraically) generated by its norm-continuous, one-parameter subgroups.

Since a $*$-automorphism α is an isometry, $\|\alpha\| = 1$; so that, in any event, $\|\alpha-\iota\| \leqslant 2$. One of the main points of [9] is that as soon as α leaves the surface of the ball (of radius 2 about ι) in which it must lie, it becomes weakly-inner. Nevertheless, for the structural results about the (norm) topological groups of $*$-automorphisms, it would suffice to establish our results for a $*$-automorphism α such that $\|\alpha-\iota\| < a$, with a any positive constant. J. Dixmier gives such a proof [**2**; pp. 313–315], which he ascribes to J. P. Serre, where $a = \sqrt{3}$. By applying arguments similar to ours directly to an automorphism α of an arbitrary Banach algebra (rather than to an implementing operator), G. Zeller-Meier [**14**] noted that α is the exponential of a derivation when its spectrum lies in the open right half-plane. S. Sakai [**12**; 4.1.19] concludes from this that an automorphism α (not necessarily adjoint-preserving) of a C*-algebra $\mathfrak{A}$ acting on a Hilbert space is implemented by an invertible operator in $\mathfrak{A}^-$ if $\|\alpha-\iota\| < 1$.

If α is an automorphism of $\mathfrak{A}$, it may happen that $\|\alpha\| > 1$ when α is not assumed to be adjoint-preserving. In this case, the condition $\|\alpha-\iota\| < 2$ no longer has the topological significance of moving α off the surface of a ball in which it has to lie. Despite this, it turns out that (once again) α is weakly inner when it is interior to this ball. The purpose of this note is to prove that fact (and related results) for arbitrary automorphisms of C*-algebras; (see Theorem 5).

2. *Automorphisms*

With $\mathfrak{A}$ a Banach algebra, we denote by $\mathrm{sp}(A)$ the spectrum of an element A of $\mathfrak{A}$. Our automorphisms of C*-algebras are not, in general, adjoint-preserving; those which preserve adjoints are described as $*$-automorphisms. As usual, $\mathscr{B}(\mathscr{H})$ denotes the algebra of all bounded linear operators acting on the Hilbert space $\mathscr{H}$.

If $\mathfrak{A}$ is a C*-algebra with centre $\mathscr{C}$, each automorphism α of $\mathfrak{A}$ leaves $\mathscr{C}$ invariant. An element C of $\mathscr{C}$ is normal, and therefore is self-adjoint if and only if it has real spectrum. From this, and since α preserves spectrum, it follows that the restriction

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$\alpha|\mathscr{C}$ is a $*$-automorphism of $\mathscr{C}$. Uniqueness of the norm topology in a C*-algebra ([**10**; p. 188, Theorem 4.1.20]; for a more general result see [**8**]) implies that α (and α^{-1}) is continuous.

For completeness, we include a short proof of this last result. With A self-adjoint in $\mathfrak{A}$, there is a real number λ in $\mathrm{sp}(A)$ such that $|\lambda| = \|A\|$. Since $\lambda \in \mathrm{sp}(\alpha(A))$,

$$\|\alpha(A)\| \geqslant |\lambda| = \|A\|;$$

so α is norm increasing on self-adjoint elements. We prove next that the set $\mathscr{M}$ of self-adjoint elements of $\mathfrak{A}$ has image $\alpha(\mathscr{M})$ closed in $\mathfrak{A}$. For this, suppose that

$$A,\ A_n \in \mathfrak{A}, \quad A_n = A_n^* \ (n = 1, 2, \ldots) \quad \text{and} \quad \|\alpha(A_n) - \alpha(A)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Then

$$\|A_n - A\|^2 = \|(A_n - A)(A_n - A^*)\| \leqslant \big\|\alpha\big((A_n - A)(A_n - A^*)\big)\big\|$$
$$\leqslant \|\alpha(A_n) - \alpha(A)\| \, \|\alpha(A_n) - \alpha(A^*)\| \to 0,$$

since

$$\|\alpha(A_n) - \alpha(A)\| \to 0 \quad \text{and} \quad \|\alpha(A_n) - \alpha(A^*)\| \to \|\alpha(A) - \alpha(A^*)\|.$$

Thus A ($= \lim A_n$) is self-adjoint, and $\alpha(A) \in \alpha(\mathscr{M})$.

Since $\mathscr{M}$ and $\alpha(\mathscr{M})$ are (real) Banach spaces, and the restriction $\alpha|\mathscr{M}$ is norm increasing, the Closed Graph Theorem shows that $\alpha|\mathscr{M}$ is continuous. Continuity of α on $\mathfrak{A}$ now follows from the equation

$$\alpha(A) = \alpha\big(\tfrac{1}{2}(A + A^*)\big) + i\alpha\big(\tfrac{1}{2}i(A^* - A)\big).$$

We summarise in the following result the conclusions from the preceding paragraphs.

LEMMA 1. *If $\mathfrak{A}$ is a* C**-algebra with centre $\mathscr{C}$, and α is an automorphism of $\mathfrak{A}$, then α is bicontinuous and the restriction $\alpha|\mathscr{C}$ is a $*$-automorphism of $\mathscr{C}$.*

LEMMA 2. *If $\mathfrak{A}$ is a* C**-algebra acting on a Hilbert space $\mathscr{H}$, α is an automorphism of $\mathfrak{A}$ and $\|\iota - \alpha\| < 2$, then α extends to an ultraweakly bicontinuous automorphism $\bar{\alpha}$ of $\mathfrak{A}^-$, which leaves the centre of $\mathfrak{A}^-$ elementwise fixed, and $\|\iota - \bar{\alpha}\| = \|\iota - \alpha\|$.*

Proof. With π the universal respresentation of $\mathfrak{A}$, there is a central projection P in $\pi(\mathfrak{A})^-$, and a $*$-isomorphism ϕ from $\pi(\mathfrak{A})^- P$ onto $\mathfrak{A}^-$, such that $\phi\big(\pi(A)P\big) = A$ for each A in $\mathfrak{A}$ (see, for example, the first paragraph of the proof of [**11**; Theorem 4]). By [**2**; p. 54, Corollaire 1], ϕ is ultraweakly bicontinuous. The automorphism $\pi\alpha\pi^{-1}$ of $\pi(\mathfrak{A})$ is bicontinuous in the norm topology, and is therefore bicontinuous also relative to the ultraweak topology, since this is the weak topology on the Banach space $\pi(\mathfrak{A})$ [**3**; p. 237, proof of Corollaire 12.1.3.]. By means of the Kaplansky density theorem and the (separate) ultraweak continuity of operator multiplication, it follows (as in [**9**; p. 40, proof of Lemma 3]) that $\pi\alpha\pi^{-1}$ extends to an ultraweakly bicontinuous automorphism β of $\pi(\mathfrak{A})^-$, and that

$$\|\iota - \beta\| = \|\iota - \pi\alpha\pi^{-1}\| = \|\pi(\iota - \alpha)\,\pi^{-1}\| = \|\iota - \alpha\| < 2.$$

With Q a central projection in $\pi(\mathfrak{A})^-$, $\beta(Q)$ is another such projection by Lemma 1.

Since Q, $\beta(Q)$ are commuting projections, and

$$\|\beta(Q)-Q\| = \tfrac{1}{2}\|\beta(2Q-I)-(2Q-I)\| \leqslant \tfrac{1}{2}\|\beta-\iota\|\,\|2Q-I\| = \tfrac{1}{2}\|\beta-\iota\| < 1,$$

it follows that $\beta(Q) = Q$. By continuity, and the spectral theorem, β fixes each element in the centre of $\pi(\mathfrak{A})^-$. Since $\beta(P) = P$, the restriction $\gamma = \beta|\pi(\mathfrak{A})^- P$ is an ultraweakly bicontinuous automorphism of the von Neumann algebra $\pi(\mathfrak{A})^- P$, and leaves its centre elementwise fixed. With A in $\mathfrak{A}$,

$$\begin{aligned}\phi\gamma\phi^{-1}(A) &= \phi\gamma\phi^{-1}\phi(\pi(A)P) = \phi\gamma(\pi(A)P) = \phi\big(\beta(\pi(A)P)\big)\\ &= \phi(\beta\pi(A)\,\beta(P)) = \phi(\pi\alpha\pi^{-1}\pi(A)\,P) = \phi(\pi\alpha(A)\,P) = \alpha(A).\end{aligned}$$

With $\bar{\alpha} = \phi\gamma\phi^{-1}$, $\bar{\alpha}$ is an ultraweakly bicontinuous automorphism of $\mathfrak{A}^-$, which extends α and leaves the centre of $\mathfrak{A}^-$ elementwise fixed. Since the $*$-isomorphism ϕ is isometric,

$$\|\iota-\bar{\alpha}\| = \|\phi(\iota-\gamma)\phi^{-1}\| = \|\iota-\gamma\| = \|(\iota-\beta)\,|\,\pi(\mathfrak{A})^-P\| \leqslant \|\iota-\beta\| = \|\iota-\alpha\|;$$

so $\|\iota-\bar{\alpha}\| = \|\iota-\alpha\|$.

For the next two lemmas, we shall employ the concept of the *numerical range*

$$W(T) = \{\langle Tx, x\rangle : x\in\mathscr{H},\quad \|x\| = 1\}$$

of a bounded linear operator T acting on a Hilbert space $\mathscr{H}$. We shall need the following properties:

(a) $W(T)$ is convex;

(b) the closure $W(T)^-$ contains $\operatorname{sp}(T)$;

(c) if $\mathscr{H}$ is the direct sum $\sum\oplus\,\mathscr{H}_a$ of a family $(\mathscr{H}_a)$ of Hilbert spaces, and $T = \sum\oplus\,T_a$ (where $T_a\in\mathscr{B}(\mathscr{H}_a)$ for each a, and $\sup\|T_a\| < \infty$), then $W(T)^-$ is the closed convex hull of $\bigcup W(T_a)$.

For (a), which is the infinite-dimensional version of the Toeplitz-Hausdorff Theorem, we refer to [**13**]. For (b), it suffices (after (a)) to note that $W(T)^-$ contains the approximate point spectrum of T, and so contains the boundary of $\operatorname{sp}(T)$. A Banach space generalization of (b) is proved in [**1**; p. 88, Theorem 1]. Finally, (c) is an immediate consequence of (a), together with the definition of $W(T)$.

LEMMA 3. *If $\mathscr{H}$ is a Hilbert space and α is an automorphism of $\mathscr{B}(\mathscr{H})$, there is an invertible element T of $\mathscr{B}(\mathscr{H})$ such that*

$$\alpha(A) = TAT^{-1}\,(A\in\mathscr{B}(\mathscr{H})),\quad \|T\| = 1,\quad \|T^{-1}\| = \|\alpha\|.$$

If, further, $\|\iota-\alpha\| < 2$, T can be chosen so that

$$W(T) \subseteq \{z : \operatorname{Re} z \geqslant \tfrac{1}{6}[4-\|\iota-\alpha\|^2]^{\frac{1}{2}}\}.$$

Proof. By [**6**; p. 100, Theorem 2] there is an invertible element T of $\mathscr{B}(\mathscr{H})$, such that $\alpha(A) = TAT^{-1}$ for each A in $\mathscr{B}(\mathscr{H})$. Upon replacing T by $\|T\|^{-1}\,T$, we may suppose that $\|T\| = 1$; and it is apparent that, then, $\|\alpha\| \leqslant \|T^{-1}\|$. We can now choose sequences (x_n), (y_n) of unit vectors in $\mathscr{H}$, so that $\|Tx_n\| \to \|T\| = 1$ and $\|T^{-1}y_n\| \to \|T^{-1}\|$. For each n, choose an operator V_n, in the unit ball of $\mathscr{B}(\mathscr{H})$, which maps $T^{-1}y_n$ to $\|T^{-1}y_n\|\,x_n$. Then

$$\|\alpha\| \geqslant \|\alpha(V_n)\| = \|TV_nT^{-1}\| \geqslant \|TV_nT^{-1}y_n\| = \|T^{-1}y_n\|\,\|Tx_n\|;$$

whence $\|\alpha\| \geqslant \lim \|T^{-1}y_n\|\,\|Tx_n\| = \|T^{-1}\|$, and so $\|\alpha\| = \|T^{-1}\|$.

Now suppose that $\|\iota-\alpha\| < 2$, and let $r_0 = \frac{1}{6}[4-\|\iota-\alpha\|^2]^{\frac{1}{2}}$. We shall prove that $|\langle Tx, x\rangle| \geqslant r_0$ whenever x is a unit vector in $\mathscr{H}$. This implies that the point a, closest to 0 in the compact convex set $W(T)^-$, satisfies $|a| \geqslant r_0$. Elementary geometrical considerations then show that $W(T)^-$ is contained in the half-plane

$$\{z : \mathrm{Re}\,|a^{-1}|\bar{a}z \geqslant r_0\};$$

and all the conclusions of the lemma are satisfied when T is replaced by $|a^{-1}|\bar{a}T$.

It remains to show that $|\langle Tx, x\rangle| \geqslant r_0$ when $x \in \mathscr{H}$ and $\|x\| = 1$. With E the one-dimensional projection whose range contains x,

$$\begin{aligned}\|\iota-\alpha\| &\geqslant \|(I-2E)-\alpha(I-2E)\| = 2\|\alpha(E)-E\| = 2\|TET^{-1}-E\|\\ &\geqslant 2\|Tx\|^{-1}\,\|(TET^{-1}-E)\,Tx\| = 2\|Tx\|^{-1}\,\|TEx-ETx\|\\ &= 2\|Tx\|^{-1}\,\|Tx-\langle Tx, x\rangle x\| = 2\|Tx\|^{-1}\,(\|Tx\|^2-|\langle Tx, x\rangle|^2)^{\frac{1}{2}}\\ &= 2[1-|\langle Tx, x\rangle|^2/\|Tx\|^2]^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\begin{aligned}|\langle Tx, x\rangle| &\geqslant \tfrac{1}{2}\|Tx\|\,[4-\|\iota-\alpha\|^2]^{\frac{1}{2}} \geqslant \tfrac{1}{2}\|T^{-1}\|^{-1}\,[4-\|\iota-\alpha\|^2]^{\frac{1}{2}}\\ &= \tfrac{1}{2}\|\alpha\|^{-1}\,[4-\|\iota-\alpha\|^2]^{\frac{1}{2}} \geqslant \tfrac{1}{6}[4-\|\iota-\alpha\|^2]^{\frac{1}{2}} = r_0,\end{aligned}$$

since $\|\alpha\| \leqslant 1+\|\iota-\alpha\| < 3$.

LEMMA 4. *If α is an automorphism of a C*-algebra $\mathfrak{A}$ and $\|\alpha-\iota\| < 2$, then there is a derivation δ of $\mathfrak{A}$ for which $\alpha = \exp\delta$.*

Proof. Since the assertion of this lemma is independent of the representation of $\mathfrak{A}$, we may assume, without loss of generality, that $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$ is the reduced atomic representation (see, for example, [9; p. 35]). In this case, there is an orthogonal family of projections commuting with $\mathfrak{A}$, having ranges $\mathscr{H}_a$, such that $\mathfrak{A}$ acts irreducibly on each $\mathscr{H}_a$, that these subrepresentations are inequivalent for distinct a, and that each irrreducible representation of $\mathfrak{A}$ is equivalent to one of these subrepresentations. Writing " $\mathfrak{A}_a$ " for the restriction of $\mathfrak{A}$ to $\mathscr{H}_a$, it follows, from [7; Corollary 4] and the irreduciblity of $\mathfrak{A}_a$, that

$$\mathfrak{A}^- = \textstyle\sum_a \oplus\, \mathfrak{A}^-{}_a = \sum_a \oplus\, \mathscr{B}(\mathscr{H}_a).$$

As $\|\alpha-\iota\| < 2$, Lemma 2 applies; and α extends to an automorphism $\bar{\alpha}$ of $\mathfrak{A}^-$ such that $\|\bar{\alpha}-\iota\| = \|\alpha-\iota\| < 2$. Moreover, $\bar{\alpha}$ leaves the centre of $\mathfrak{A}^-$ elementwise fixed so that the restriction $\bar{\alpha}_a$ of $\bar{\alpha}$ to each $\mathscr{B}(\mathscr{H}_a)$ is an automorphism for which $\|\bar{\alpha}_a-\iota\| \leqslant \|\bar{\alpha}-\iota\| = \|\alpha-\iota\| < 2$. From Lemma 3, there is an invertible operator T_a of norm 1 on $\mathscr{H}_a$ such that $\|T_a{}^{-1}\| = \|\bar{\alpha}_a\|$, $\bar{\alpha}_a(A) = T_aAT_a{}^{-1}$ $(A \in \mathscr{B}(\mathscr{H}_a))$ and $W(T_a) \subseteq \{z; \mathrm{Re}\,z \geqslant r_0\}$, where $r_0 = \frac{1}{6}[4-\|\alpha-\iota\|^2]^{\frac{1}{2}}$. If $T = \sum_a \oplus T_a$ then $\|T\| = 1$, $\sum_a \oplus T_a{}^{-1}$ is the inverse T^{-1} of T and $\|T^{-1}\| \leqslant \|\alpha\|$. In addition, from the comments preceding Lemma 3, $\mathrm{sp}(T) \subseteq W(T)^- \subseteq \{z : \mathrm{Re}\, z \geqslant r_0\}$.

Let $\mathbb{C}'$ be the plane of complex numbers with 0 and the negative real axis deleted. Let $\mathrm{Log}\, z$ be $\ln r + is$ with $-\pi < s < \pi$, where $z = r \exp is$, for z in $\mathbb{C}'$. From the holomorphic function calculus for elements of a complex Banach algebra and the Spectral Mapping Theorem [5; VII, 3.11] $B = \mathrm{Log}\, T \in \mathscr{B}(\mathscr{H})$, $T = \exp(B)$, and

$$\mathrm{sp}(B) \subseteq \{r+is : \ln r_0 \leqslant r \leqslant 0, \quad -\arccos r_0 \leqslant s \leqslant \arccos r_0\}.$$

We now change our notation, and denote by $\bar{\alpha}$ the automorphism $A \to TAT^{-1}$ of $\mathscr{B}(\mathscr{H})$. Expansion in powers of B shows that $\bar{\alpha} = \exp(\mathrm{ad}\, B)$, where

$$\mathrm{ad}\, B(A) = BA - AB$$

(see, for example, [9; Lemma 2]). As $\mathrm{ad}\, B$ is the difference of two commuting operators on $\mathscr{B}(\mathscr{H})$ (namely, left and right multiplication by B), and each of these has the same spectrum as B, we have

$$\mathrm{sp}(\mathrm{ad}\, B) \subseteq \{r+is : \ln r_0 \leqslant r \leqslant -\ln r_0, \quad -2\arccos r_0 \leqslant s \leqslant 2\arccos r_0\}.$$

Thus

$$\mathrm{sp}(\bar{\alpha}) \subseteq \{r \exp is : r_0 \leqslant r \leqslant r_0^{-1}, \quad -\pi < -2\arccos r_0 \leqslant s \leqslant 2\arccos r_0 < \pi\}.$$

For each real r, let $\tilde{r}$ denote multiplication by r. Let g_r be $\exp \circ \tilde{r} \circ \mathrm{Log}$ on $\mathbb{C}'$. On the strip $\{z : |\mathrm{Im}\, z| < \pi\}$, $g_r \circ \exp = \exp \circ \tilde{r}$. Since ad B has spectrum in that strip and exp and $\tilde{r}$ are holomorphic (see [5; VII, 3.12]),

$$g_r(\bar{\alpha}) = g_r(\exp[\mathrm{ad}\, B]) = \exp(r\, \mathrm{ad}\, B) = \exp(\mathrm{ad}\, rB).$$

Defining $\bar{\alpha}^r$ to be $g_r(\bar{\alpha})$, $\bar{\alpha}$ lies on the norm-continuous, one-parameter group, $r \to \bar{\alpha}^r$ of automorphisms of $\mathscr{B}(\mathscr{H})$.

When we prove that each $\bar{\alpha}^r$ leaves $\mathfrak{A}$ invariant, we will have that $\alpha (= \bar{\alpha} \mid \mathfrak{A})$ lies on the norm-continuous, one-paramater group of automorphisms $r \to \bar{\alpha}^r \mid \mathfrak{A}$, so that $\alpha = \exp \delta$ for some derivation δ of $\mathfrak{A}$ (as in [9; Lemma 2]). To prove that $\bar{\alpha}^r$ leaves $\mathfrak{A}$ invariant, note that $\bar{\alpha}^r = (2\pi i)^{-1} \oint_C g_r(z)(z-\bar{\alpha})^{-1}\, dz$, where C is a (rectifiable, Jordan) curve with $\mathrm{sp}(\bar{\alpha})$ in its interior. As the integral is formed in the sense of norm-convergence and $\mathfrak{A}$ is norm closed, it will suffice to show that $(z_0-\bar{\alpha})^{-1}$ leaves $\mathfrak{A}$ invariant for each z_0 on an appropriate curve C. With the limitation noted earlier on the location of $\mathrm{sp}(\bar{\alpha})$, we can choose a compact set K with $\mathrm{sp}(\bar{\alpha})$ in its interior K_0, and a curve C in $\mathbb{C}'$ with K in its bounded component, such that $(z_0-z)^{-1}$ is holomorphic in z on K_0. By Runge's Theorem, $(z_0-z)^{-1}$ is uniformly approximable by polynomials on compact subsets of K_0. Thus $(z_0-\bar{\alpha})^{-1}$ is a uniform limit of polynomials in $\bar{\alpha}$ (see [5; VII, 3.13]), for z_0 on C; and it leaves $\mathfrak{A}$ invariant.

Recalling that [2; p. 312, Corollaire] each derivation δ of a C*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ has the form $\delta = \mathrm{ad}\, B \mid \mathfrak{A}$, for some B in $\mathfrak{A}^-$, as an immediate consequence of Lemma 4, we have

THEOREM 5. *If α is an automorphism of the* C*-*algebra $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$, and $\|\alpha - \iota\| < 2$, there is an invertible operator T in $\mathfrak{A}^-$ such that*

$$\alpha(A) = TAT^{-1}$$

for all A in $\mathfrak{A}$.

Remark A. If α is an automorphism of the von Neumann algebra $\mathscr{R}$ such that $\|\alpha - \iota\| \leqslant 2$, then α is inner.

Remark B. As there are $*$-automorphisms of von Neumann algebras (factors) which are outer [2; p. 288, Exercise 15], and for $*$-automorphisms α, $\|\alpha - \iota\| \leqslant 2$, we see that the constant 2 cannot be improved in Theorem 5.

Remark C. With $\mathfrak{A}$ a C*-algebra and aut $(\mathfrak{A})$ the group of automorphisms of $\mathfrak{A}$ provided with its norm topology, we denote by $\text{aut}_\iota(\mathfrak{A})$ the connected component of the identity. Employing the argument of [9; Theorem 7], we deduce, from Lemma 4 and Theorem 5, that $\text{aut}_\iota(\mathfrak{A})$ is an open subgroup of aut $(\mathfrak{A})$ which is generated (algebraically) by its norm-continuous, one-parameter subgroups. If $\mathfrak{A}$ acts on $\mathscr{H}$ and $\alpha \in \text{aut}_\iota(\mathfrak{A})$ then $\alpha(A) = TAT^{-1}$ for all A in $\mathfrak{A}$ and some T in $\mathfrak{A}^-$.

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