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MAPPINGS OF OPERATOR ALGEBRAS

by Richard V. KADISON *

Let $\mathcal{H}$ be a Hilbert space over the complex numbers and $\mathcal{B}(\mathcal{H})$ the family of bounded (continuous) linear operators on $\mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$ is an algebra under the usual operations of addition and multiplication of transformations ; and the adjoint $(*)$ operation $A \to A^*$ is an involutory anti-automorphism of $\mathcal{B}(\mathcal{H})$. With the norm of an operator A in $\mathcal{B}(\mathcal{H})$ defined as its bound $\|A\|$, $\mathcal{B}(\mathcal{H})$ becomes a Banach space and the $*$ operation is an isometry. The weak-operator topology, defined as the weakest topology on $\mathcal{B}(\mathcal{H})$ in which the functionals $A \to (Ax, y)$ are continuous, will be needed along with the norm topology associated with the operator-bound norm.

The subalgebras of $\mathcal{B}(\mathcal{H})$ stable under the $*$ operation and closed in the norm topology – the C^*-algebras, as well as their special subclass consisting of those closed in the weak-operator topology, the von Neumann algebras, are the principal objects of attention in this report. The main purpose of this exposition is to describe the developments which have occurred over the past five years in the study of special classes of mappings of such algebras. The primary concern is with the $(*)$ automorphisms and derivations ; but, as an outgrowth of these considerations, the recent work on cohomology of these algebras will be discussed.

A $(*)$ automorphism α of a C^*-algebra $\mathfrak{U}$ is an algebraic automorphism of $\mathfrak{U}$ such that $\alpha(A^*) = \alpha(A)^*$. If U is a unitary operator in $\mathfrak{U}$, $A \to UAU^*$ is an automorphism of $\mathfrak{U}$. Such automorphisms are said to be *inner*. Automorphisms tend to be *outer* (i.e., not inner). If $\mathcal{C}$ is the compact operators on $\mathcal{H}$ and $\mathfrak{U}$ is the C^*-algebra generated by $\mathcal{C}$ and I, each U in $\mathcal{B}(\mathcal{H})$ induces an automorphism of $\mathfrak{U}$, though many unitary operators are not the sum of a scalar and a compact operator. These last automorphisms are *spatial* – induced by a unitary operator in $\mathcal{B}(\mathcal{H})$.

In general automorphisms of C^*-algebras will not be spatial. Homeomorphisms of locally compact measure spaces which don't preserve null sets of the measure produce automorphisms of the C^*-algebra of multiplication operators by continuous functions which are not spatial. Automorphisms of von Neumann algebras, on the other hand, tend to be spatial – provided that their action on the center respects certain elementary numerical invariants. In the case of the *factors*-the von Neumann algebras with center consisting of scalars – automorphisms will be spatial (with a possible exception [14] in the case of a factor of type II_∞ with II_1 commutant).

Though spatial, in general, automorphisms of von Neumann algebras tend not to be inner. If G is a countable (discrete) group with conjugate classes infinite

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and $\mathcal{H}$ is the Hilbert space of complex-valued, square-integrable functions on G, then the weak-operator closed algebra generated by the unitary operators U_a defined by $(U_a f)(g) = f(a^{-1} g)$ is a factor $\mathcal{M}$ (of type II_1). Each automorphism of G induces a spatial automorphism of $\mathcal{M}$. If G is the free group on two generators a and b, the automorphism interchanging a and b will be outer [2 : Ex. 15, p. 288]. If G is the group of those permutations of the integers which move at most a finite set then each locally compact group with a countable base has a (faithful, strong-operator-continuous) representation on $\mathcal{H}$ by unitary operators which (with the exception of I) induce outer automorphisms of $\mathcal{M}$ [1].

An automorphism α of a C^*-algebra is an isometry ; for A and $\alpha(A)$ have the same spectrum. Thus $\|\alpha(A)\| = \|A\|$ when A is self-adjoint. For arbitrary T in the algebra, $\|T\|^2 = \|T^*T\| = \|\alpha(T^*T)\| = \|\alpha(T)\|^2$. Hence α is, in particular, a bounded operator on $\mathfrak{U}$ (as a Banach subspace of $\mathcal{B}(\mathcal{H})$) ; and $\|\alpha\| = 1$. If ι denotes the identity automorphism of the von Neumann algebra $\mathfrak{R}$, $\|\alpha - \iota\| \leqslant 2$. While outer automorphisms of von Neumann algebras abound, if $\|\alpha - \iota\| < 2$ then α is inner [11 : Theorem 7]. This theorem is established by C^*-and von Neumann algebra techniques combined with analytic methods. The proof is directed toward showing that α lies on a oneparameter group of automorphisms of the form $\exp(t\delta)$, where δ is a bounded linear operator on $\mathfrak{R}$. Because the mappings $\exp(t\delta)$ are automorphisms, δ is a derivation of $\mathfrak{R}$

$$\text{(i.e. } \delta(AB) = \delta(A)B + A\delta(A)\text{)}.$$

The theorem that derivations of von Neumann algebras are inner [7, 10, 15, 19] applies ; and there is an iH in $\mathfrak{R}$ such that $\delta(A) = i(HA - AH)$ for each A in $\mathfrak{R}$. Since α preserves adjoints, the same is true for δ ; and H may be chosen self-adjoint. The automorphism α, with which we started, is induced by the unitary operator $\exp(iH)$ (in $\mathfrak{R}$).

The development leading up to the theorem that derivations of von Neumann algebras are inner began with the observation that this is true for type I von Neumann algebras [16]. The prototype of these algebras is $\mathcal{B}(\mathcal{H})$. There is a group $\mathcal{U}$ of unitary operators in $\mathcal{B}(\mathcal{H})$ whose linear span has norm closure a C^*-algebra $\mathfrak{U}$ with weak-operator closure $\mathcal{B}(\mathcal{H})$; and $\mathcal{U}$ is the ascending union of finite groups. Choosing an orthonormal basis for $\mathcal{H}$, $\mathcal{U}$ can be taken as the group generated by those unitary operators which either permute or reflect through 0 a finite number of basis elements and fix the others. Since $\mathcal{U}$ is an ascending union of finite groups, $\mathcal{U}$ has a (two-sided, invariant) mean μ. If φ is a bounded function from $\mathcal{U}$ into $\mathcal{B}(\mathcal{H})$, meaning $U \to (\varphi(U)x, y)$, for each pair of vectors x, y in $\mathcal{H}$, leads to a bounded bilinear functional on $\mathcal{H}$ and, thence, to an operator $\mu(\varphi)$ in $\mathcal{B}(\mathcal{H})$. If $\varphi(U) = U^*\delta(U)$, then $\delta(V) = VT - TV$ for V in $\mathcal{U}$, where $T = \mu(\varphi)$. This follows from meaning φ_V, where

$$\varphi_V(U) = \varphi(UV) = (UV)^*\delta(UV) = V^*U^*[U\delta(V) + \delta(U)V] =$$
$$= V^*\delta(V) + V^*U^*\delta(U)V.$$

From the properties of the mean, $T = V^*\delta(V) + V^*TV$. By linearity and norm continuity $\delta(A) = AT - TA$ for each A in $\mathfrak{U}$. At this point, we can make use of special (automatic) continuity properties of derivations [15 : Lemma 3], to conclude that $\delta(A) = AT - TA$ for all A in $\mathcal{B}(\mathcal{H})$. This continuity results from the observation that, if $I \geqslant A \geqslant 0$, then $\delta(A) = \delta(A^{1/2})A^{1/2} + A^{1/2}\delta(A^{1/2})$; so that, if (Ax, x) $(= \|A^{1/2}x\|^2)$ is small, $(\delta(A)x, x)$ is small.

The same argument, slightly embellished, will prove that derivations of type I von Neumann algebras are inner. More general results can also be proved by this method. If $\mathfrak{R}$ is a von Neumann algebra, $\mathfrak{M}$ is a two-sided (unital) $\mathfrak{R}$-module which is the dual of a Banach space $\mathfrak{M}_*$, and if the bilinear mappings $(A, m) \to Am$ and $(A, m) \to mA$ are bounded and w^* continuous in m, $\mathfrak{M}$ is said to be a *dual (Banach)* $\mathfrak{R}$*-module*. If these mappings are ultraweakly continuous in A (i.e. weak-operator continuous in A on bounded subsets of $\mathfrak{R}$), $\mathfrak{M}$ is said to be *normal*. The argument just sketched will show that a derivation of a type I von Neumann algebra $\mathfrak{R}$ into a normal dual $\mathfrak{R}$-module $\mathfrak{M}$ (i.e. a linear mapping δ of $\mathfrak{R}$ into $\mathfrak{M}$ such that $\delta(AB) = \delta(A)B + A\delta(B)$), has the form $A \to Am - mA$, for some m in $\mathfrak{M}$ [9 : Cor. 5.4]. In particular, if δ is a derivation of $\mathfrak{R}$ into $\mathcal{B}(\mathcal{H})$ there is a T in $\mathcal{B}(\mathcal{H})$ such that $\delta(A) = AT - TA$.

This module formulation of derivation results lends itself, at once, to considerations of cohomology of C^*-algebras with coefficients in a module [4,5]. With $\mathfrak{A}$ a C^*-algebra and $\mathfrak{M}$ a Banach $\mathfrak{A}$-module, let $C_c^n(\mathfrak{A}, \mathfrak{M})$ be the linear space of bounded n-linear mappings of $\mathfrak{A}$ into $\mathfrak{M}$. The coboundary operator Δ is defined [4] by :

$$(\Delta\rho)(A_1, \ldots, A_{n+1}) = A_1\rho(A_2, \ldots, A_{n+1}) - \rho(A_1A_2, A_3, \ldots, A_{n+1}) + \\ + \ldots \pm \rho(A_1, \ldots, A_{n-1}, A_nA_{n+1}) \mp \rho(A_1, \ldots, A_n)A_{n+1},$$

for ρ in $C_c^n(\mathfrak{A}, \mathfrak{M})$. Such mappings ρ are the (bounded) *n-cochains*. Those ρ for which $\Delta\rho = 0$ are the (bounded) *n-cocycles*. They form a subspace $Z_c^n(\mathfrak{A}, \mathfrak{M})$ of $C_c^n(\mathfrak{A}, \mathfrak{M})$. Since $\Delta\Delta = 0$; the n-cochains of the form $\Delta\xi$ with ξ an n-1-cochain are cocycles. They are the *n-coboundaries*. The factor space of $Z_c^n(\mathfrak{A}, \mathfrak{M})$ by the space $B_c^n(\mathfrak{A}, \mathfrak{M})$ of (bounded) n-coboundaries is the n-th *cohomology group* $H_c^n(\mathfrak{A}, \mathfrak{M})$ of $\mathfrak{A}$ with coefficients in $\mathfrak{M}$. Note that the 1-cocycles are those linear mappings δ of $\mathfrak{A}$ into $\mathfrak{M}$ such that

$$(\Delta\delta)(A, B) = A\delta(B) - \delta(AB) + \delta(A)B = 0$$

– that is to say, the derivations of $\mathfrak{A}$ into $\mathfrak{M}$. When $\mathfrak{M}$ is $\mathfrak{A}$ (with action given by the multiplication on $\mathfrak{A}$) the 1-cocycles become the standard derivations of $\mathfrak{A}$ into $\mathfrak{A}$. The 0-cochains are the constant mappings – the elements of $\mathfrak{M}$; and the coboundary of m is $Am - mA$ (at A). To say that a derivation δ of $\mathfrak{A}$ into $\mathfrak{M}$ cobounds is to say, then, that, for some m in $\mathfrak{M}$, $\delta(A) = Am - mA$, for each A in $\mathfrak{A}$. The theorem that the derivations of a von Neumann algebra $\mathfrak{R}$ (into itself) are inner is the assertion that $H_c^1(\mathfrak{R}, \mathfrak{R}) = 0$. In this framework, it is known that $H_c^n(\mathfrak{R}, \mathfrak{M}) = 0$ when $\mathfrak{R}$ is a type I von Neumann algebra and $\mathfrak{M}$ is a normal dual $\mathfrak{R}$-module. The same is true for all hyperfinite von Neumann algebras $\mathfrak{R}$ [6, 9, 12, 13].

If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{K}$ is a (norm-closed) two-sided ideal in $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{K}$ is, again, a C^*-algebra [3 : Prop. 1.8.2, p. 17]. The problem of "lifting" a derivation δ of $\mathfrak{A}/\mathfrak{K}$ to $\mathfrak{A}$ leads to considerations of 2-cohomology of $\mathfrak{A}$ with coefficients in $\mathfrak{K}$. If ξ is a (norm-continuous) linear mapping of $\mathfrak{U}$ into $\mathfrak{U}$ which lifts δ then $A\xi(B) - \xi(AB) + \xi(A)B(= \rho(A, B))$ is in $\mathfrak{K}$ for all A and B in $\mathfrak{U}$, Moreover $\Delta\rho = 0$; so that ρ is in $Z_c^2(\mathfrak{U}, \mathfrak{K})$. If $\rho = \Delta\eta$ with η in $C_c^1(\mathfrak{U}, \mathfrak{K})$, then $\xi - \eta$ lifts δ (as does ξ). As $\Delta(\xi - \eta) = \rho - \rho = 0$, $\xi - \eta$ is a derivation of $\mathfrak{U}$ into $\mathfrak{U}$.

A new element of difficulty enters the higher cohomology arguments by virtue of the fact that higher order cocycles do not enjoy the automatic continuity properties of derivations. Derivations of a C^*-algebra are norm continuous [8, 18] and ultraweakly continuous [15]. If ξ is a (norm) discontinuous linear mapping of $\mathfrak{R}$ into $\mathfrak{R}$, $\Delta\xi$ is a 2-coboundary (hence, 2-cocycle) which is not norm continuous (in general). Similarly, starting with ξ norm but not ultraweakly continuous, $\Delta\xi$ *must* fail to be ultraweakly continuous. A Tauberian result to the effect that if the 2-coboundary $\Delta\xi$ is ultraweakly continuous (in its first argument), then ξ is ultraweakly continuous [9 : Lemma 4.7], governs this situation. A sketch of the proof follows.

It suffices [22] to show that $\xi(\Sigma E_j) = \xi(I) = \Sigma\xi(E_j)$ (ultraweak convergence), where $\{E_j\}$ is a family of orthogonal projections in the von Neumann algebra $\mathfrak{R}$. By ultraweak continuity of $A \to (\Delta\xi)(A, B) = A\xi(B) - \xi(AB) + \xi(A)B$, with E_j for A and E_k for B, summing over j, we have

$$\xi(I)E_k = (\Delta\xi)(I, E_k) = (\Delta\xi)(\Sigma E_j, E_k) = (\Sigma E_j)\xi(E_k) - \xi(E_k) + \\ + (\Sigma\xi(E_j))E_k = (\Sigma\xi(E_j))E_k .$$

As this holds for each E_k and $\Sigma E_k = I$, $\Sigma\xi(E_j) = \xi(I)$. The same is not true for 3-cocycles ; for a (discontinuous) 2-coboundary can always be added to a 2-cochain without changing its coboundary.

The evidence is every strong that $H_c^n(\mathfrak{R}, \mathfrak{R}) = 0$ for a general von Neumann algebra $\mathfrak{R}$, but this remains to be completed. Although derivations of C^*-algebras are not inner, in general (the algebra generated by the compact operators and I illustrates this), there are special instances in which they are. The most striking of these is the case of simple C^*-algebras with a unit. For such algebras, all derivations are inner [17, 20, 21]. It may well be the case that all cohomology groups vanish for such algebras ; but this, too, awaits further study.

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