PURE STATES AND APPROXIMATE IDENTITIES[1]

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1. **Introduction.** In this note we show that each norm separable C^*-algebra has an increasing abelian approximate identity and that, if the algebra has an identity, each pure state is multiplicative on some maximal abelian subalgebra.

Let A be a C^*-algebra without an identity element. A net $\{u_i\}_{i\in I} \subseteq A$, where I is a directed index set, is called an *approximate identity* for A if $\|u_i\| \leqq 1$ for all $i \in I$, and $\|u_i x - x\| \to 0$; $\|x u_i - x\| \to 0$ for all $x \in A$. We say that $\{u_i\}_{i\in I}$ is *increasing* if $u_i \geqq 0$ and $i \leqq j \Rightarrow u_i \leqq u_j$ for all $i, j \in I$. With u_i selfadjoint, if one of the limits exists, so does the other. Each C^*-algebra has an increasing approximate identity [2, 1.7.2]. An approximate identity $\{u_i\}_{i\in I}$ is *countable* if I is countable. It is *abelian* if u_i and u_j commute for all $i, j \in I$.

An element $x \in A$ is said to be *strictly positive* if $\rho(x) > 0$ for each nonzero positive linear functional ρ on A. A strictly positive element is positive [2, 2.6.2].

We use the following notation: If M is a collection of vectors in a Hilbert space H, and $\mathfrak{F}$ is a family of bounded linear operators on H, then $[\mathfrak{F}M]$ is the closed linear span of the set $\{F\xi : F \in \mathfrak{F}, \xi \in M\}$.

2. **Results.**

LEMMA 1. *If $x \in A$ is strictly positive, and π is a nondegenerate representation of A on a Hilbert space H, then $[\pi(x)H] = H$.*

PROOF. Suppose $0 \neq \xi \in [\pi(x)H]^\perp$. Since π is nondegenerate there is an element $a \in A$ such that $\pi(a)\xi \neq 0$. Let $\rho = \omega_\xi \circ \pi$, where ω_ξ is the positive linear functional $y \to (y\xi, \xi)$. Then $\rho(a^*a) = (\pi(a^*a)\xi, \xi) = \|\pi(a)\xi\|^2 \neq 0$, so ρ is a positive, nonzero linear functional on A. But $\rho(x) = (\pi(x)\xi, \xi) = 0$ which contradicts the assumption that x is strictly positive. Hence $[\pi(x)H]^\perp = (0)$ and the lemma follows.

THEOREM 1. *A C^*-algebra A has a countable increasing abelian approximate identity if and only if A contains a strictly positive element.*

PROOF. If $x_0 \in A$ is strictly positive, we may take x_0 with norm equal to 1. Let $u_i = x_0^{1/i}$, and observe that $u_i \geqq 0$, $\|u_i\| = 1$; $i \geqq j \Rightarrow u_i \geqq u_j$ and u_i and u_j commute for all $i, j \in I$. We want to show that for any

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$x \in A$, $\|xu_i - x\| \to 0$. It is sufficient to do this for $x \geqq 0$. (Using Theorem 5.1 of Akemann [1], the present proof could be somewhat shortened, but we prefer to give a complete argument.)

Let $x \geqq 0$, and let y be the unique positive square root of x. Let $\tilde{A}$ be the C^*-algebra obtained by adjoining an identity e to A. Then $u_i \leqq e$ for all i, so that $0 \leqq yu_iy \leqq yey = x$, and $yu_iy \leqq yu_jy$ if $i \leqq j$. Hence, if $z_i = x - yu_iy$, $\{z_i\}$ becomes a monotone decreasing sequence of positive elements in A. We claim that $\|z_i\| \to 0$. Let

$$S = \{\rho \in A^* : \rho \geqq 0; \|\rho\| \leqq 1\}.$$

S is compact in the w^*-topology [2, 2.5.5]. We may regard each z_i as a continuous function on S by the evaluation map. Since $z_i \geqq 0$, $\|z_i\| = \sup\{\rho(z_i) : \rho \in S\}$ [2, 2.7.3]; so that it suffices to show that z_i converges uniformly to 0 on S. As the sequence z_i is monotone, this will follow from Dini's theorem once we know that $\rho(z_i) \to 0$ for each $\rho \in S$.

Let π be a nondegenerate representation of A on a Hilbert space H. $\pi(u_i) = \pi(x_0^{1/i}) = \pi(x_0)^{1/i}$, which by spectral theory converges strongly to the range projection of $\pi(x_0)$. Since x_0 is strictly positive it follows by Lemma 1 that $\pi(u_i) \to I$ strongly on H, where I is the identity operator on H.

Let $\rho \neq 0$ be an arbitrary element of S and π_ρ be the associated representation of A on the Hilbert space H_ρ. Then π_ρ is nondegenerate with a cyclic vector ξ_ρ and

$$\begin{aligned} \rho(z_i) &= (\pi_\rho(z_i)\xi_\rho, \xi_\rho) \\ &= ((\pi_\rho(x) - \pi_\rho(yu_iy))\xi_\rho, \xi_\rho) \\ &= (\pi_\rho(y)\xi_\rho - \pi_\rho(u_i)\pi_\rho(y)\xi_\rho, \pi_\rho(y)\xi_\rho) \end{aligned}$$

which converges to zero since $\pi_\rho(u_i) \to I$ strongly. Hence $\|z_i\| \to 0$.

Working in $\tilde{A}$ (as Akemann does in [1]), let v_i be the positive square root of $e - u_i$. Then $\|yv_i\|^2 = \|yv_iv_iy\| = \|y(e-u_i)y\| = \|x - yu_iy\| \to 0$, and hence $\|xu_i - x\| = \|y^2v_i^2\| \leqq \|y\| \cdot \|yv_i\| \cdot \|v_i\| \leqq \|y\| \cdot \|yv_i\| \to 0$. Thus $\{u_i\}$ is an approximate identity.

Conversely, suppose $\{u_i\}$ is an increasing abelian approximate identity, and let $x = \sum_{n=1}^\infty 2^{-n}u_n$. If ρ is a nonzero positive linear functional on A, we know that $\rho(u_n) \to \|\rho\|$ [2, 2.1.5]. Hence $\rho(u_n) > 0$ for some n, so $\rho(x) = \sum_{n=1}^\infty 2^{-n}\rho(u_n) > 0$. This shows that x is strictly positive, and the proof is complete.

Observe that if A is separable, then A has a strictly positive element. Indeed, if $\{y_n\}$ is dense in A, then $\{x_n = y_n^* y_n\}$ is dense in $A^+ = \{x \in A : x \geqq 0\}$. Clearly $x = \sum_{n=1}^\infty (2^n\|x_n\|)^{-1}x_n$ is strictly positive in A.

COROLLARY 1. *Any separable C^*-algebra has a countable increasing abelian approximate identity.*

REMARK. Let X be a locally compact Hausdorff space, $A=\mathcal{C}^0(X)$, the C^*-algebra of all continuous complex functions on X vanishing at infinity. It is easily verified that A contains a function f which is everywhere positive if and only if X is σ-compact. Since each state on A may be represented by a Borel measure on X, we see that such a function f is a strictly positive element of A. Evidently X may be σ-compact without having a countable base for its topology, so A may have a strictly positive element without being separable. Needless to say, A will not always have strictly positive elements. An example is $\mathcal{C}^0(\boldsymbol{R})$, when $\boldsymbol{R}$ is given the discrete topology.

A positive linear functional ρ on a C^*-algebra A is a *state* if $\|\rho\|=1$. If A has an identity e, this is equivalent to the condition $\rho(e)=1$. We say ρ is *pure* if $\rho\neq 0$ and each positive, linear functional γ on A such that $0\leqq\gamma\leqq\rho$, is of the form $\gamma=a\rho$; $0\leqq a\leqq 1$.

THEOREM 2. *Let A be a separable C^*-algebra with identity. If ρ is a pure state on A, then there is a maximal abelian C^*-subalgebra B of A such that $\rho|B$ is multiplicative.*

PROOF. Let N_ρ be $\{x\in A:\rho(x^*x)=0\}$ and A_0 be $N_\rho\cap N_\rho^*$. A_0 is a C^*-subalgebra of A and is therefore separable. Hence A_0 contains a strictly positive element x_0. Let B_0 be a maximal abelian C^*-subalgebra of A_0 containing x_0 and B be $B_0+\boldsymbol{C}\cdot e$. Then B is an abelian C^*-subalgebra of A. Since ρ vanishes on B_0, $\rho|B$ is multiplicative of norm 1. To show that B is a maximal abelian C^*-subalgebra of A, it suffices to show that if a selfadjoint x in A commutes with B, then $x\in B$. Now, $x\in B$ if and only if $x-\rho(x)e\in B$; so we may assume that $\rho(x)=0$. Let π_ρ be the irreducible representation of A associated with ρ on the Hilbert space H_ρ, with cyclic vector ξ_ρ [2, 2.5.4]. Let $H_0=[\xi_\rho]^\perp$. We claim that $[\pi_\rho(A_0)H_\rho]=H_0$. Indeed, if $y\in A_0$ then $\|\pi_\rho(y)\xi_\rho\|^2=(\pi_\rho(y^*y)\xi_\rho,\ \xi_\rho)=\rho(y^*y)=0$; so that $(\pi_\rho(y)\pi_\rho(x)\xi_\rho,\ \xi_\rho)=0$ for all x in A. On the other hand, let ξ in H_0 be arbitrary. By the transitivity theorem [2, 2.8.3] there is a selfadjoint element $y\in A$ such that $\pi_\rho(y)\xi_\rho=0$ and $\pi_\rho(y)\xi=\xi$. But then $y\in A_0$ and the claim follows. Hence $\pi_\rho|A_0$ is a nondegenerate representation on H_0. Let E be the orthogonal projection of H_ρ onto H_0. By Lemma 1, $[\pi_\rho(x_0)H_0]=H_0$. Now x_0 and x commute; so that E and $\pi_\rho(x)$ commute. Hence H_0 and $[\xi_\rho]$ are invariant under $\pi_\rho(x)$. This means that ξ_ρ is an eigenvector for $\pi_\rho(x)$; so $\pi_\rho(x)\xi_\rho=a\xi_\rho$ for some real a. Now $0=\rho(x)=(\pi_\rho(x)\xi_\rho,\ \xi_\rho)=a(\xi_\rho,\ \xi_\rho)=a$. Hence $\pi_\rho(x)\xi_\rho=0$ so $x\in A_0$. Since B_0 is maximal abelian in A_0 it follows that $x \in B_0 \subseteq B$. The proof is complete.

REFERENCES

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