Lectures on Operator Algebras

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The notes which follow present a survey of the initial portions of the theory of self-adjoint operator algebras (especially, C^* algebras). This survey is incomplete though sufficiently representative and detailed, we feel, to give some idea of the subject and its techniques. The material discussed is, to a large extent, contained in the first few chapters of *Les C^* algèbres et leurs représentations*, J. Dixmier, Paris, Gauthier-Villars 1964, and the first chapter of *Introduction to Hilbert Space*, P. Halmos, New York, Chelsea 1951. We let these books and their bibliographies suffice for our reference list.

There are some small contributions to the state of the subject in these notes. To the best of our knowledge the remark that extreme points not only exist but generate compact convex sets in the more general case where one assumes only that a separating family of continuous linear functionals exists has not previously been made. The proof that a functional of norm 1 assuming its norm at the identity is a state is more streamlined than usual. Our transitivity theorem for irreducible C^* algebras is proved here with the additional information that the operator in the algebra having desired action on the given finite set of vectors can be chosen so that its norm is as small as any operator having the same effect on these vectors.

I. Fundamentals of Hilbert Space and Bounded Operators

1. *Review of the Geometry of Hilbert Space*

We denote by $\mathscr{H}$ a Hilbert space and by (x,y) the inner (or dot or scalar) product of two vectors x and y in $\mathscr{H}$. To recall, $\mathscr{H}$ is a linear space over the complex numbers $\mathbf{C}$ together with a positive-definite, conjugate-bilinear form $x,y \to (x,y)$ (so that $(x,x) = \|x\|^2 > 0$ unless $x=0$, $(\alpha x+y,z) = \alpha(x,z)+(y,z)$, and $(x,y) = \overline{(y,x)}$); and is complete relative to the norm $x \to \|x\|$ associated with this form (that is, if $\|x_n - x_m\| \to 0$ as n, $m \to \infty$, we say (x_n) is Cauchy convergent in this case, then there is an x in $\mathscr{H}$ such that $\|x_n - x\| \to 0$ as $n \to \infty$).

We recall the Cauchy-Schwarz inequality

$$|(x,y)| \leqslant \|x\| \cdot \|y\|,$$

which is proved by considering the coefficients of $(x+\lambda y, x+\lambda y)$ as a quadratic polynomial in λ and exploiting the fact that it is real and non-negative for each value of λ. Note especially that it is valid without the assumption that $x,y \to (x,y)$ is definite (that is, we could allow $(x,x)=0$ for non-zero x) and that $\mathscr{H}$ is complete. The fact that equality holds if and only if x and y are linearly dependent does, however, require the definiteness of the inner product. The "triangle inequality", $\|x+y\| \leqslant \|x\| + \|y\|$, follows by squaring both the sides and using the Schwarz inequality. This together with the fact that $\|\alpha x\| = |\alpha| \cdot \|x\|$ and that $\|x\| > 0$ if $x \neq 0$ shows that $x \to x$ is a "norm" on $\mathscr{H}$ (we say that the pair $\mathscr{H}$, $\| \;\; \|$ is a "normed space"). Having assumed that $\mathscr{H}$ is complete relative to $\| \;\; \|$, it is in particular a "Banach space" (i.e. a complete normed space). One checks, from the definition, that the norm on our Hilbert space $\mathscr{H}$ satisfies the "parallelogram law" $\|x+y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (in fact this equality characterizes those norms on a Banach space which are associated with a Hilbert space inner product).

If $\mathscr{M}$ is a closed subspace of $\mathscr{H}$ (i.e. if x and y are in $\mathscr{M}$ then $\alpha x + y$ is in $\mathscr{M}$, and if $\|x_n - x\| \to 0$ as $n \to \infty$ with x_n in $\mathscr{M}$ then x is in $\mathscr{M}$) and y is a vector in $\mathscr{H}$, let d be the distance from y to $\mathscr{M}$ (i.e. $d = \text{infimum } \{\|y-x\| \colon x \text{ in } \mathscr{M}\}$ and choose x_n in $\mathscr{M}$ such that $\|y - x_n\| \to d$. Using the parallelogram law, one shows that (x_n) is Cauchy convergent; so that (x_n) has a limit $\mathscr{M}(y)$ (in $\mathscr{M}$, since $\mathscr{M}$ is closed) from the completeness of $\mathscr{H}$. Of course, $\|y - \mathscr{M}(y)\| = d$. With x a unit vector in $\mathscr{M}$,

$$(y-\mathscr{M}(y)+\alpha x, y-\mathscr{M}(y)+\alpha x)$$
$$= d^2+2\,\mathrm{Re}\,\alpha(x, y-\mathscr{M}(y))+|\alpha|^2 \geqslant d^2,$$

for each α in $\mathbf{C}$; so that $(x,y-\mathscr{M}(y))=0$ and $y-\mathscr{M}(y)$ is "orthogonal" to $\mathscr{M}$. For z in $\mathscr{M}$ distinct from $\mathscr{M}(y)$, $\|y-z\|^2 = \|y-\mathscr{M}(y)\|^2+\|z-\mathscr{M}(y)\|^2$ (by orthogonality of $y-\mathscr{M}(y)$ and $z-\mathscr{M}(y)$); so that $\mathscr{M}(y)$ is the *unique* vector in $\mathscr{M}$ with the norm minimizing (or the orthogonality) property. We call $\mathscr{M}(y)$ "the orthogonal projection of y on $\mathscr{M}$".

If ϕ is continuous linear functional on $\mathscr{H}$ (i.e. ϕ is a mapping of $\mathscr{H}$ into $\mathbf{C}$ such that $\phi(\alpha x+y)=\alpha\phi(x)+\phi(y)$ and $|\phi(x_n)-\phi(x)| \to 0$ if $\|x_n-x\| \to 0$ as $n \to \infty$) and $\mathscr{M}$ is the null space of ϕ (i.e. the set of x in $\mathscr{H}$ such that $\phi(x)=0$), then $\mathscr{M}$ is a closed subspace of $\mathscr{H}$. If $\phi \neq 0$ there is a unit vector y in $\mathscr{H}$ not in $\mathscr{M}$; so that $y-\mathscr{M}(y) \neq 0$. The continuous linear functional $x \to (x,y-\mathscr{M}(y))$ is non-zero and, has $\mathscr{M}$ in its null space. Thus ϕ is a scalar multiple of this functional. From this we conclude the so-called Riesz Representation theorem for linear functionals (continuous) on a Hilbert space, viz.—for each such ϕ there is a (unique) vector z in $\mathscr{H}$ such that $\phi(x)=(x,z)$ for each x in $\mathscr{H}$.

An orthonormal set $\{x_\gamma\}$ of vectors in $\mathscr{H}$ is one such that $(x_\gamma, x_{\gamma'})=0$ if $x_{\gamma'} \neq x_\gamma$ and $(x_\gamma, x_\gamma)=1$. Zorn's lemma permits us to assert the existence of a maximal such orthonormal set (one contained properly in no larger one). If $\{x_\gamma\}$ is such a maximal orthonormal set and $\mathscr{M}$ is the smallest closed subspace of $\mathscr{H}$ containing it (we say $\mathscr{M}$ is the closed subspace generated by $\{x_\gamma\}$), then $\mathscr{M}=\mathscr{H}$, for if y in $\mathscr{H}$ is not in $\mathscr{M}$, $y-\mathscr{M}(y)$ is a non-zero vector orthogonal to each x_γ contradicting the maximality of $\{x_\gamma\}$. Conversely if $\{x_\gamma\}$ is an orthonormal set generating $\mathscr{H}$ then any vector orthogonal to each x_γ is orthogonal to all of $\mathscr{H}$, hence to itself, and is therefore 0; so that $\{x_\gamma\}$ is a maximal orthonormal set in this case. There is no difficulty now in proving the Bessel inequality: $\|x\|^2 \geqslant \Sigma_\gamma |(x,x_\gamma)|^2$ for each orthonormal set $\{x_\gamma\}$, or the Parseval equality: $\|x\|^2=\Sigma_\gamma |(x,x_\gamma)|^2$ when $\{x_\gamma\}$ is a maximal orthonormal set. The expansion formula $x=\Sigma_\gamma (x,x_\gamma)\,x_\gamma$ also follows when $\{x_\gamma\}$ is a maximal orthonormal set (where the summation is understood to mean that for each $\epsilon > 0$ there is a finite set F_0 of γ's such that if F is a finite set of γ's containing F_0, $\|x - \Sigma_{\gamma \text{ in } F}(x, x_\gamma)x_\gamma\| < \epsilon$), in view of which $\{x_\gamma\}$ is referred to as an *orthonormal basis* for $\mathscr{H}$ (it is easy to establish the uniqueness of this expansion, $x = \Sigma_\gamma a_\gamma x_\gamma$ implies $a_\gamma = (x, x_\gamma)$ for all γ). Two orthonormal bases for $\mathscr{H}$ have the same number of elements (i.e. their elements can be put into one to one correspondence). In case one of the bases has a finite number of elements, this basic fact follows from the existence of a non trivial solution to a finite number of simultaneous linear equations in more unknowns than equations. For infinite bases $\{x_\gamma\}$ and $\{y_\delta\}$, this follows from the fact that a sum of positive numbers can converge only if it has at most a countable number of terms (for otherwise some infinite subset of them exceed $1/n$ for some positive integer n) and the fact that $\aleph_0 \cdot \aleph = \aleph$ for each infinite cardinal $\aleph$ (where $\aleph_0$ is the cardinality of the set of integers). From the first of these facts and the Parseval equality it follows that the set S_γ of δ's with $(x_\gamma, y_\delta) \neq 0$ is countable (has cardinality not exceeding $\aleph_0$); and, since no y_δ is orthogonal to each element of the basis $\{x_\gamma\}$, each δ lies in some S_γ. Thus the set of δ's is contained in $U_\gamma S_\gamma$ which has cardinality not exceeding $\aleph_0 \cdot \aleph$ $(= \aleph)$, where $\aleph$ is the cardinality of the set of γ's. By symmetry the set of γ's has cardinality not exceeding that of the set of δ's; and $\{x_\gamma\}$, $\{y_\delta\}$ have the same number of elements. This cardinal number common to all bases in $\mathscr{H}$ is called "the dimension of $\mathscr{H}$".

Two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ which have the same dimension admit (orthonormal) bases $\{x_\gamma\}$, $\{y_\gamma\}$, respectively, which can be indexed by the same set. Each element of $\mathscr{H}_1$ has the form $\Sigma_\gamma a_\gamma x_\gamma$ with $\Sigma_\gamma |a_\gamma|^2 < \infty$, and each such sum represents an element of $\mathscr{H}_1$. The mapping U carrying $\Sigma a_\gamma x_\gamma$ in $\mathscr{H}_1$ onto $\Sigma a_\gamma y_\gamma$ in $\mathscr{H}_2$ is linear (i.e. $U(\alpha x + y) = \alpha Ux + Uy$) and $(Ux, Ux') = (x, x')$ for each x, x' in $\mathscr{H}_1$; moreover U maps $\mathscr{H}_1$ onto $\mathscr{H}_2$ in a one to one manner. We say U is a unitary transformation of $\mathscr{H}_1$ onto $\mathscr{H}_2$ (it is a structure-preserving isomorphism of $\mathscr{H}_1$ onto $\mathscr{H}_2$). Thus Hilbert spaces are characterized by a single cardinal number, their dimension—two such are isomorphic if and only if they have the same dimension.

The case where the basis has cardinality $\aleph_0$ (so-called "separable Hilbert space") is of special interest. It can also be characterized as not being finite dimensional but having a countable dense subset (i.e. a countable set whose closure is $\mathscr{H}$). Separable Hilbert space can be represented in several (apparently distinct) ways: l_2, the space of square summable sequences (a_n) with the inner product $((a_n),(b_n)) = \sum_{n=1}^{\infty} a_n \overline{b_n}$; $L_2([0,1])$, $L_2(\mathbf{R})$, and $L_2(\mathbf{R}^3)$—the spaces of (classes) of square integrable functions relative to Lebesgue measures on $[0,1]$, the real line $\mathbf{R}$, and Euclidean 3-space $\mathbf{R}^3$, respectively, each provided with inner product $\int f(x)\overline{g(x)}\,dx = (f,g)$ integration being taken over the space relative to the indicated measure on it). A given mathematical or physical question may call for more than just the Hilbert space structure, in which case, the distinct representations of separable Hilbert space may play essentially different rôles. In any event, one representation may be more convenient than another for dealing with a particular problem.

2. *Linear Transformations and Operators*

A linear transformation T from one normed space $\mathscr{H}_1$ to another $\mathscr{H}_2$ will be continuous ($\|Tx_n - Tx\| \to 0$ when $\|x_n - x\| \to 0$) if and only if $\sup\{\|Tx\| : \|x\| = 1, x \text{ in } \mathscr{H}_1\}$ $(=\|T\|)$ is finite. In this case we say that T is bounded. We denote by $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ the family of all such transformations. Note that with T bounded, $\|Tx_n - Tx\| \leqslant \|T\| \cdot \|x_n - x\|$ so that $\|Tx_n - Tx\| \to 0$ if $\|x_n - x\| \to 0$. On the other hand, if T is not bounded, there are unit vectors x_n with $n \leqslant \|Tx_n\|$ so that $\|y_n\| \to 0$ while $\|Ty_n\| = 1$, for all n, where $y_n = x_n / \|Tx_n\|$. Provided with the usual addition and multiplication by scalar operations for linear transformations $((T_1 + T_2)(x) = T_1 x + T_2 x$ and $(\alpha T)(x) = \alpha(Tx))$ $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ becomes a complex linear space. It is not difficult to check that $T \to \|T\|$ is a norm on $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and that $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ is complete relative to this norm (i.e. is a Banach space) if and only if $\mathscr{H}_2$ is complete.

The unitary transformations discussed in the preceding section are a special class of bounded linear transformations, as are the orthogonal projections on a closed subspace. In this last case, we are dealing with bounded linear transformations from a space $\mathcal{H}$ into itself. We call these (bounded) *operators* on $\mathcal{H}$ and denote the (Banach) space of such by $\mathcal{B}(\mathcal{H})$. The standard product of operators $((T_1T_2)(x) = T_1(T_2x))$ provides $\mathcal{B}(\mathcal{H})$ with the structure of an algebra; and the norm satisfies the inequality $\|T_1T_2\| \leqslant \|T_1\| \cdot \|T_2\|$. It follows that the product of operators is jointly continuous; so that $\mathcal{B}(\mathcal{H})$ is a *Banach algebra* (an algebra with a norm in which it is a Banach space and for which multiplication is jointly continuous).

With $\mathcal{H}_1$ and $\mathcal{H}_2$ Hilbert spaces and T in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the mapping $x \to (Tx, y)$ is a continuous linear functional on $\mathcal{H}_1$ for each y in $\mathcal{H}_2$, whence, by the Riesz representation theorem for such functionals there is a vector T^*y in $\mathcal{H}_1$ (and this vector is unique) such that $(Tx, y) = (x, T^*y)$. One checks easily that T^* is a linear transformation of $\mathcal{H}_2$ into $\mathcal{H}_1$, that $\|T^*\| = \|T\|$; so that T^* lies in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. The transformation T^* is called the *adjoint* of T. Concerning the adjoint operation $T \to T^*$, one notes that $(\alpha T_1 + T_2)^* = \bar{\alpha} T_1^* + T_2^*$ (we say, * is *conjugate linear*), $(T^*)^* = T$ (* is *involutory*), and $(T_2T_1)^* = T_1^*T_2^*$ where T_1 is in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and T_2 in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ (in particular, when T_1 and T_2 are in $\mathcal{B}(\mathcal{H})$). Moreover $\|T^*T\| = \|T^*\| \cdot \|T\|$.

We say that T in $\mathcal{B}(\mathcal{H})$ is *self-adjoint* when $T = T^*$ and note that for an arbitrary bounded operator A on $\mathcal{H}$, $\frac{1}{2}(A + A^*)$ and $(1/2i)(A - A^*)$ are self-adjoint; so that each such A has the decomposition $\frac{1}{2}(A + A^*) + i(A - A^*)/2i$. Note that, for a unitary transformation U, we have U^*U is the identity operator I on $\mathcal{H}_1$, while UU^* is the identity operator on $\mathcal{H}_2$ (we use the same symbol I for the identity operator on different Hilbert spaces, when no confusion can arise). We note also that an orthogonal projection operator E is self-adjoint and idempotent ($E^2 = E$) and that these two properties characterize such operators. In fact, if E projects onto $\mathcal{M}$ and $\mathcal{M}^\perp$ is its orthogonal complement (the closed subspace of $\mathcal{H}$ consisting of vectors orthogonal to all vectors in $\mathcal{M}$), then, since each vector is the sum of its orthogonal projection onto $\mathcal{M}$ and onto $\mathcal{M}^\perp$, this last orthogonal projection operator is $I - E$. Thus $(Ex, y) = (Ex, Ey + (I - E)y) = (Ex, Ey) = (x, Ey)$. It follows that

$(Ex,x) = (E^2x,x) = (Ex,Ex) \geqslant 0$ for each x in $\mathscr{H}$. Operators T with this property ($(Tx,x) \geqslant 0$ for all x in $\mathscr{H}$) are said to be *positive*.

A useful extension of the Riesz representation theorem for linear functionals on a Hilbert space describes bounded conjugate bilinear forms on $\mathscr{H}_1, \mathscr{H}_2$ in terms of bounded linear transformations. Such a form is a mapping $x,y \to B(x,y)$ from $\mathscr{H}_1 \times \mathscr{H}_2$ to $\mathbf{C}$ such that $B(\alpha x_1 + x_2, y) = \alpha B(x_1,y) + B(x_2,y)$, $B(x, \alpha y_1 + y_2) = \bar{\alpha} B(x,y_1) + B(x,y_2)$ and $\sup\{|B(x,y)| : \|x\| = \|y\| = 1\}$ ($= a$) is finite. The result in question states that there is a linear transformation T from $\mathscr{H}_1$ to $\mathscr{H}_2$ such that $(Tx,y) = B(x,y)$, for each x in $\mathscr{H}_1$ and y in $\mathscr{H}_2$, and $\|T\| = a$. Moreover, if $\mathscr{H}_2 = \mathscr{H}_1$ and B is Hermitean (i.e. $B(x,y) = \overline{B(y,x)}$), then T is self-adjoint. If B is positive (with $\mathscr{H}_2 = \mathscr{H}_1$, $B(x,x) \geqslant 0$ for each x in $\mathscr{H}_1$) then T is positive. The proof proceeds by noting that $x \to B(x,y)$ is a continuous linear functional on $\mathscr{H}_1$ so that there is a T^*y in $\mathscr{H}_1$ with $(x,T^*y) = B(x,y)$ for all x in $\mathscr{H}_1$. That T, the adjoint of T^*, has the desired properties follows easily from the successive hypotheses.

3. *The Spectral Theorem. Heuristics*

A particularly natural class of operators on $\mathscr{H}$ consists of those operators A for which there is an orthonormal basis $\{x_n\}$ (depending on A) such that $Ax_n = \alpha_n x_n$, for each n. With α_m real, A is a superposition of "real stretches" in the orthogonal directions x_m. If $\sup_m \alpha_m$ ($= \alpha$) is finite, then A is bounded, and $\|A\| = \alpha$. In rough form, the essence of the spectral theorem is a characterization of the set of self-adjoint operators as being precisely the norm limits of such real stretch operators (i.e. A is a bounded self-adjoint operator if and only if there is a sequence of real stretch operators (A_n) such that $\|A - A_n\| \to 0$). The norm limits of the stretch operators allowing complex stretches (α_m possibly complex) are the *normal operators*—those operators A such that $A^*A = AA^*$. We recall that a unitary operator U on $\mathscr{H}$ satisfies $U^*U = UU^* = I$; so that U is, in particular, normal. The unitary operators are norm limits of stretch operators corresponding to complex stretches of modulus 1. Projections are themselves stretch operators corresponding to (at most) the stretches 0 and 1. Grouping together those basis elements corresponding to the same stretch, we can express each stretch operator as a linear combination of projections; so that each self-adjoint operator is a norm limit of such combinations. More strongly, it suffices to deal with stretch operators having at most a finite number of stretches; so that each self-adjoint operator is a norm limit of finite linear combinations of mutually orthogonal projections.

It is natural to inquire as to whether or not the norm limits of stretch operators have themselves "stretches" (or *eigenvalues* as they are technically termed) apparent in their action. In general these norm limits will not have eigenvalues; but there will be "generalized stretches", the so-called *spectral values* apparent. These are the complex numbers λ corresponding to the operator A such that $A-\lambda I$ does not have a two-sided inverse in $\mathscr{B}(\mathscr{H})$. The set of such λ, denoted by $\sigma(A)$, is called the *spectrum* of A. This same definition of spectrum applies to arbitrary (complex) Banach algebras (with a unit element) as well as to $\mathscr{B}(\mathscr{H})$; and it is generally valid that the spectrum of an element in such an algebra is a non-void, bounded closed (i.e. compact) subset of $\mathbf{C}$. In fact $\sigma(A)$ is contained in the closed disk of radius $\|A\|$ with center 0; for if $\lambda > \|A\|$ then $\|B\| < 1$, where $B = A/\lambda$, so that $I-B$ has the inverse $C = \sum_{n=0}^{\infty} B^n$ which converges to an element of the Banach algebra by virtue of completeness and the inequality $\|B^n\| \leqslant \|B\|^n$. Thus $A-\lambda I$ has the inverse $-\lambda^{-1}C$ and λ is not in $\sigma(A)$. This argument shows that the open ball of radius 1 with the center at I consist of invertible elements; and, since multiplication by an invertible element is a bicontinuous transformation of the algebra onto itself, each invertible element is contained in an open ball consisting of invertible elements. Thus the non-invertible (*singular*) elements form a closed subset of the algebra, and $\sigma(A)$ is a closed set. If $\sigma(A)$ were void, then for each x and y in $\mathscr{H}$, $((A-\lambda I)^{-1}x, y)$ would be a bounded entire function which tends to 0 as $\lambda \to \infty$, hence is 0. But then $(A^{-1}x, y)$ would be 0 for each x, y in $\mathscr{H}$; and A^{-1} would be 0, an impossibility. Thus $\sigma(A)$ is not void.

There are numerous important results relating the spectrum of an operator to the operator's structure. We conclude this chapter by noting one such—*the spectral radius formula*: $(r(A)=)$ $\sup\{|\lambda| : \lambda \text{ in } \sigma(A)\} = \lim_{n\to\infty} \|A^n\|^{1/n}$ (which asserts, in particular, the existence of this limit). The existence of the limit follows by establishing that $\overline{\lim}\,\|A^n\|^{1/n} \leqslant r(A) \leqslant \underline{\lim}\,\|A^n\|^{1/n}$ (so that $\underline{\lim}\,\|A^n\|^{1/n} = \overline{\lim}\,\|A^n\|^{1/n}$), which proves the formula at the same time. We note that if λ is in $\sigma(A)$ then λ^n is in $\sigma(A^n)$, which we have seen implies $|\lambda^n| \leqslant \|A^n\|$. Hence $|\lambda| \leqslant \underline{\lim}\,\|A^n\|^{1/n}$, and $r(A) \leqslant \underline{\lim}\,\|A^n\|^{1/n}$. Now $(I - zA)^{-1}$ is an analytic, Banach-algebra-valued function of the complex variable z where defined, and, as we noted, this inverse exists for $\|zA\| < 1$. Thus, for small z, $(I - zA)^{-1}$ is defined and represented by the power series $\sum_{n=0}^{\infty} z^n A^n$; so that this series represents it in its circle of convergence. As in standard complex variable theory this circle is seen to have radius $(\overline{\lim}\,\|A^n\|^{1/n})^{-1}$. Hence if $r < \overline{\lim}\,\|A^n\|^{1/n}$ there is a λ with $r < |\lambda|$ such that $I - A/\lambda$ (and, so, $A - \lambda I$) does not have a two-sided inverse, i.e. with λ in $\sigma(A)$. Thus $r(A)$ exceeds each r smaller than $\lim \|A^n\|^{1/n}$, and the formula follows.

II. The General Theory of Operator Algebras

1. *C* Algebras and States*

We noted in the preceding chapter that $\mathscr{B}(\mathscr{H})$ is a Banach algebra with an adjoint operation satisfying (1) $(\alpha A + B)^* = \bar{\alpha} A^* + B^*$ (2) $A^{**} = A$ (3) $(AB)^* = B^* A^*$ and (4) $\|A^* A\| = \|A^*\| \cdot \|A\|$ (as well as (5) $\|A^*\| = \|A\|$).

Definition 2.1.1. The pair consisting of a Banach algebra $\mathfrak{A}$ and a mapping $A \to A^*$ of $\mathfrak{A}$ onto itself satisfying (1)–(4) is called a B^* algebra. Those norm closed subalgebras of $\mathscr{B}(\mathscr{H})$ stable under the adjoint operation (i.e. containing A^* when they contain A) are called C^* algebras (so that each C^* algebra is a B^* algebra).

The fundamental general problem of the subject is the classification of these algebras—first algebraically and then with regard to their action on the underlying Hilbert space (in the case of C^* algebras). In more detail, one would like to associate with each such algebra (or with those in a special class) a family of invariants such that two algebras are * isomorphic (i.e. algebraically isomorphic under a mapping ϕ for which $\phi(A^*) = \phi(A)^*$) if and only if they have the same invariants. With regard to the action on the underlying space, we would want the invariants to determine when there is an isomorphism (unitary transformation) of the underlying Hilbert space carrying one algebra onto another. More generally, it is desirable to have invariants which determine when * *representations* of B^* algebras are *unitarily equivalent.*

Definition 2.1.2. A * representation of a B^* algebra $\mathfrak{A}$ is an algebraic homomorphism ϕ of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ such that $\phi(A^*) = \phi(A)^*$. Two * representations ϕ and ψ of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ are said to be unitarily equivalent when there is a unitary transformation U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $\phi(A) = U^{-1}\psi(A)U$ for each A in $\mathfrak{A}$.

More ambitiously one could ask for "canonical forms" for the B^* algebras with regard to their algebraic structure and spatial action (i.e. canonically constructed representatives, one from each of the algebraic isomorphism or unitary equivalence classes). Of course other types of questions about operator algebras are of importance and are studied; though, in principle, if the "canonical form problem" has been completely settled (say for a special class of B^* algebras), all other questions should be referred to the canonical forms and become special questions in some other subject. For example, we shall note that the study of commutative B^* algebras is precisely the same as the study of the algebra $C(X)$ of continuous complex-valued functions (under the operations of pointwise multiplication, addition, and multiplication by scalars) on the various compact-Hausdorff spaces (these are the topological spaces for which each family of open sets with union the whole space has a finite sub-family with this same union-*compactness*, and each pair of distinct points is contained in a pair of disjoint open sets—the *Hausdorff property*). Thus the question of whether a commutative C^* algebra contains a projection different from 0 and I becomes the question of whether $C(X)$ contains an element not 0 or 1 equal to its own square. This is equivalent to X being disconnected, so that a C^* algebra associated with the unit interval $[0,1]$ (in the usual metric topology) will not contain a projection different from 0 and I.

The precise statement of the theorem asserting the existence of the isomorphism between a commutative C^* algebra and $C(X)$ and its proof are the subject of the next section. It is profitable for us to note here that the principal formal problem of describing the points of X, the proposed compact-Hausdorff space, is effectively dealt with by observing that the linear functional on $C(X)$ corresponding to a given point p in X (viz. $f \to f(p)$) has very special properties, and then working with the class of functionals on the commutative C^* algebra having analogous properties in place of X. Again, to analyze * representations of C^* algebras, we shall want to develop techniques for constructing such representations. As we shall see in §3, the use of special classes of functionals on a C^* algebra provides us with such a technique. In both these instances where linear functionals are used, the functionals involved have special properties with respect to the order and linear structure of the C^* algebra in question. Denoting by $\mathfrak{A}_*$ the set of self-adjoint operators in a family $\mathfrak{A}$ of operators, if $\mathfrak{A}$ is a C^* algebra, $\mathfrak{A}_*$ is a real linear space in which the set $\mathscr{P}$ of positive operators forms a "cone" (i.e. (1) A and $-A$ in $\mathscr{P}$ imply $A=0$ (2) A in $\mathscr{P}$ and $\alpha \geqslant 0$ imply αA is in $\mathscr{P}$ and (3) if A and B lie in $\mathscr{P}$ so does $A+B$). The pair consisting of a real linear space and the partial ordering induced by such a cone ($A \geqslant B$ when $A-B$ lies in $\mathscr{P}$) is called *a partially ordered linear space*. Moreover, I is an *order unit* for $\{\mathfrak{A}_*, \mathscr{P}\}$ (i.e. for each A in $\mathfrak{A}_*$ there is a positive α such that $-\alpha I \leqslant A \leqslant \alpha I$); for $-\|A\| I \leqslant A \leqslant \|A\| I$, since $|(Ax,x)| \leqslant \|A\|(x,x)$ by virtue of the Schwarz inequality.

Definition 2.1.3. A *state* of a partially ordered linear space with order unit is a linear functional on the space, 1 at the order unit, and taking non-negative values on the cone of positive elements. A *pure state* is a state which is not a proper convex combination of two distinct states (i.e. if $\rho = a\tau + (1-a)\eta$ with $0 < a < 1$ and ρ pure then $\tau = \eta$).

If $\mathfrak{A}$ is a C^* algebra, we shall also refer to a linear functional whose restrictions to $\mathfrak{A}_*$ is a state as *a state of* $\mathfrak{A}$ (and one whose restriction is a pure state as a *pure state of* $\mathfrak{A}$). We shall note in the next section that the pure states of a commutative C^* algebra provide us with the points of the associated compact-Hausdorff space and in §3 that, for general C^* algebras they give rise to the important class of * representations having the "irreducibility" property. We conclude this section with some basic information about states.

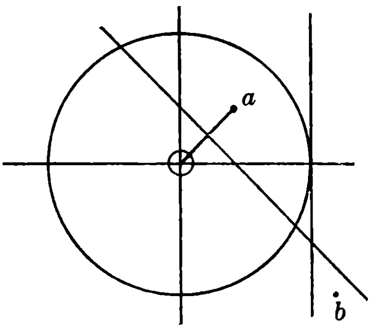
If $\mathfrak{A}$ is a partially-ordered linear space with order unit I, and $\mathfrak{A}_0$ is a linear subspace of $\mathfrak{A}$ containing I, then $\mathfrak{A}_0 \cap \mathscr{P}$ is a positive cone in $\mathfrak{A}_0$; and each state ρ_0 of $\mathfrak{A}_0$ (relative to this cone $\mathscr{P}_0$) has an extension to a state ϕ of $\mathfrak{A}$. This is established by noting that if A in $\mathfrak{A}$ is not in $\mathfrak{A}_0$, ρ_0 can be extended to be a state of the linear space generated by $\mathfrak{A}_0$ and A (define the extension at A to be any value between $\inf\{\rho_0(B): B$ in $\mathfrak{A}_0$ and $B \geqslant A\}$ and $\sup\{\rho_0(C): C$ in $\mathfrak{A}_0$ and $C \leqslant A\}$—this inf and sup being finite by the assumption that $\mathfrak{A}_0$ contains an order unit for $\mathfrak{A}$; and extend linearly to the space generated by $\mathfrak{A}_0$ and A) and then using Zorn's lemma to select a maximal extension ρ of ρ_0 which, by virtue of the ability to extend just noted and its maximality, will be state of $\mathfrak{A}$. This argument indicates that the extension of ρ_0 to a state of $\mathfrak{A}$ is not unique in general (A can be assigned any value between sup and inf). A moment's further reflection shows that the equality of "inf" and "sup" for each A not in $\mathfrak{A}_0$ is a necessary and sufficient criterion for the state extension of ρ_0 from $\mathfrak{A}_0$ to $\mathfrak{A}$ to be unique. If ρ_0 is a pure state of $\mathfrak{A}_0$ then its set of state extensions $\mathscr{E}$ is a convex set each extreme point ρ of which (i.e. point of $\mathscr{E}$ which is not a proper convex combination of two distinct points of $\mathscr{E}$) is a pure state of $\mathfrak{A}$. If $\rho = a\rho_1 + (1-a)\rho_2$ with ρ_1 and ρ_2 states of $\mathfrak{A}$ and $0 < a < 1$, then the same relation holds for the restrictions $\rho|\mathfrak{A}_0$, $\rho_1|\mathfrak{A}_0$ and $\rho_2|\mathfrak{A}_0$ of ρ, ρ_1 and ρ_2 to $\mathfrak{A}_0$. But $\rho|\mathfrak{A}_0 = \rho_0$ and both $\rho_1|\mathfrak{A}_0$ and $\rho_2|\mathfrak{A}_0$ are states of $\mathfrak{A}_0$; so that $\rho_1|\mathfrak{A}_0 = \rho_2|\mathfrak{A}_0 = \rho_0$, since ρ_0 is a pure state of $\mathfrak{A}_0$. Thus ρ_1 and ρ_2 are in $\mathscr{E}$; and $\rho_1 = \rho_2 = \rho$, since ρ is an extreme point of $\mathscr{E}$. It remains to note that $\mathscr{E}$ has extreme points; and this follows from two important general principles in functional analysis.

If V is a vector space over the reals or complex numbers, we can provide the space $\tilde{V}$ of linear functionals on V, its *dual space*, with a special topology called, variously, the weak V topology, the weak predual (or preconjugate) topology, the point-open topology or the w^* topology (we employ this last). It is the topology in which convergence of (ρ_n) in $\tilde{V}$ to ρ means $(\rho_n(v))$ converges to $(\rho(v))$ for each v in V. More precisely, the w^* topology in $\tilde{V}$ is given by taking as the open sets all unions of sets of the form $\{\rho:\rho \text{ in } \tilde{V}, |(\rho-\rho_0)(v_j)|<1, v_1, \ldots, v_n \text{ in } V \text{ and } \rho_0 \text{ in } \tilde{V}\}$. If $\mathfrak{A}_*$ is a partially ordered linear space with order unit I and positive cone $\mathscr{P}$, and $\mathscr{P}_1$ is the set of elements in $\mathscr{P} \leqslant I$, the set X of all functions on $\mathscr{P}_1$ to the unit interval $[0,1]$ topologized with the point-open topology (the open sets in X are unions of sets of the form $\{f:f \text{ in } X, |f(A_j)-f_0(A_j)|<\epsilon, A_1, \ldots, A_n \text{ in } \mathscr{P}_1, f_0 \text{ in } X, \text{ and } \epsilon>0\}$—this is the so-called *product topology* on X, the "cartesian product" of copies of $[0,1]$ indexed by elements of $\mathscr{P}_1$) is compact (a consequence of a basic theorem in general topology known as "Tychonoff's theorem"). The mapping taking a state of $\mathfrak{A}_*$ onto its restriction to $\mathscr{P}_1$ is a one-to-one mapping of the set of states into X which is trivially seen to be a homomorphism of the *state space* of $\mathfrak{A}_*$ in its w^* topology into X in its product topology (from the definitions of these topologies). There is no difficulty in seeing that the image is closed and hence compact in X; so that the set $\mathscr{S}(\mathfrak{A}_*)$ of states of $\mathfrak{A}_*$ form a compact convex subset of the dual of $\mathfrak{A}_*$ in its w^* topology. A slight modification of this argument yields the Alaoglu–Bourbaki theorem: The unit ball of the dual of a normed space (i.e. those functionals with bound not exceeding 1) is compact in the w^* topology.

The dual $\tilde{V}$ of a linear space V (over the reals or complex numbers) is such that addition and multiplication by scalars are both jointly continuous operations in their variables relative to the w^* topology. We say that the pair consisting of a linear space and a topology on it for which this is true is a *topological linear space*. Moreover $\tilde{V}$ in its w^* topology has a separating family of continuous linear functionals (i.e. any two elements of $\tilde{V}$ which have the same value at each such functional must coincide), namely the functionals arising from evaluation of elements (functionals) in $\tilde{V}$ at elements of V (the w^* topology on $\tilde{V}$ is chosen precisely so that the functionals $\phi \to \phi(v)$ on $\tilde{V}$ are continuous). The final fact we need, to establish that the set of extensions $\mathscr{E}$ of ϕ has an extreme point so that a pure state of $\mathfrak{A}_0$ has a pure state extension to $\mathfrak{A}$, is the Krein-Milman theorem: A compact convex subset K of a topological linear space V which possesses a separating family of continuous linear functionals is the closed convex hull of (i.e. smallest closed convex set containing) its extreme points. Note that $\mathscr{E}$ is a w^* closed set of states, hence w^* compact. For the demonstration, we may assume that K is non-null and define a *face* of K to be a subset F which contains $ax+(1-a)y$ for all a in $[0,1]$ if it contains $ax+(1-a)y$ for one a in $(0,1)$ when x and y lie in K (geometrically, F contains each line segment in K which meets F in an interior point of the segment). One notes that a face F of K is convex and that each face of F is a face of K. Using the facts that intersections of faces are faces and that if a family of compact sets is such that each finite subfamily has a non-null intersection then the full family has a non-null intersection (called *the finite intersection property*) in conjunction with Zorn's lemma (recalling that K is assumed to be compact and is a face of itself), we conclude the existence of a minimal non-null compact face F_0 of K contained in any given face F of K. Since each continuous function on a non-null compact set attains its maximum on that set and since each continuous linear functional attains its maximum on a convex set at a set of points which form a face of it, each such functional on V attains its maximum on F_0 at each point of F_0 (by minimality of F_0), hence is constant on F_0. These functionals form a separating family for V, so that F_0 consists of a single point. But one-point faces are extreme points. For the remainder of the proof of the Krein-Milman theorem, we draw on the basic "separation theorem" for topological linear spaces: if N is a convex set containing a non-null open subset of V and K_0 is a non-null convex set having null intersection with the interior of N (the union of all the open sets in N) then there is a non-zero continuous linear functional ϕ on V and a constant α such that ϕ is not less than α on N and not greater than α on K_0. Suppose, now, that K_0 is the closed convex hull of the extreme points of K (hence, non-null, from the foregoing) and that x is a point of K not in K_0. For each x_0 in K_0 we can find a continuous linear functional ϕ on V such that $\phi(x) > \phi(x_0)$. With β strictly between $\phi(x)$ and $\phi(x_0)$ the sets N_{x_0} and M_{x_0} at which ϕ are, respectively, greater and less than β are open, disjoint, convex subsets of V containing x and x_0. The sets $\{M_{x_0}: x_0 \text{ in } K_0\}$ form an open covering of K_0 from which we can select the finite sub-covering $M_{x_1}, \ldots, M_{x_n}$, since K_0, being a closed subset of the compact set K, is itself compact. The intersection N of $N_{x_1}, \ldots, N_{x_n}$ is an open convex set containing x and disjoint from the union of $M_{x_1}, \ldots, M_{x_n}$, hence from K_0. With ϕ and α as in the separation theorem just noted, we can choose v in V such that $\phi(v) \neq 0$ and, by multiplication by a suitably small scalar, so small that $x+v$ and $x-v$ are in N. Thus $\phi(x) \pm \phi(v) \geqslant \alpha$; and since $\phi(v) \neq 0$, $\phi(x) > \alpha$. The set F at which ϕ attains its maximum on K does not, therefore, meet K_0, since ϕ is not greater than α on K_0. Yet F is a closed face of K and contains an extreme point of K, contradicting the fact that K_0 contains all the extreme points of K. Thus $K_0 = K$, and K is the closed convex hull of its extreme points.

Returning to states ρ of a C^* algebra, we have noted earlier that $-\|A\|I \leqslant A \leqslant \|A\|I$ for each self-adjoint A in $\mathfrak{A}$, so that $-\|A\|\rho(I) \leqslant \rho(A) \leqslant \|A\|\rho(I)$ and $|\rho(A)| \leqslant \|A\|$ for such A. For arbitrary B in $\mathfrak{A}$, we have $|\rho(B)| = |\rho(I \cdot B)| \leqslant \rho(B^*B)^{1/2}\rho(I^*I)^{1/2} = \rho(B^*B)^{1/2} \leqslant \|B^*B\|^{1/2} = \|B\|$, from the Schwarzine quality applied to the positive semi-definite inner product $T,S \rightarrow \rho(S^*T)$ defined by ρ on $\mathfrak{A}$. (We shall have much use for this inner product in constructing * representations of C^* algebras from states.) It follows that each state ρ of a C^* algebra has norm 1 and assumes its norm at I. A (partial) converse of this fact is also valid, viz. a functional ρ of norm 1 on a C^* algebra $\mathfrak{A}$ such that $\rho(I) = 1$ is a state of $\mathfrak{A}$. To prove this one must show that $\rho(A) \geqslant 0$ for each positive A in $\mathfrak{A}$. Since the restriction of ρ to the (commutative) C^* subalgebra $\mathfrak{A}_0$

of $\mathfrak{A}$ generated by A and I satisfies the same hypotheses as does ρ on $\mathfrak{A}$, we may assume that $\mathfrak{A}$ is commutative. When we establish as we shall in the next section, that a commutative C^* algebra is algebraically, order and norm isomorphic to a complex function algebra $C(X)$, the problem will be reduced to showing that a norm 1 functional ρ on $C(X)$ which is 1 at the constant function 1 is real and non-negative on positive functions. Assuming the contrary, $\rho(f)=a$, with $0 \nleq a$, $0<f<1$ and f in $C(X)$. In this case, the open half-plane containing 0 determined by the perpendicular bisector of the segment joining it to a intersects the half-plane $\text{Re}\, z>1$. With b in this intersection, $\|b-f\| \leqslant |b| < |b-a| = |b-\rho(f)| = |\rho(be-f)| \leqslant \|b-f\|$—a contradiction.



As an immediate consequence of the Krein-Milman theorem, we observe that the state space $\mathscr{S}(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is the closed convex hull of its extreme points, the pure states of $\mathfrak{A}$; so that the pure states of $\mathfrak{A}$ form a separating family of functionals for $\mathfrak{A}$ once we show that $\mathscr{S}(\mathfrak{A})$ does.

2. *Commutative C^* Algebras and the Spectral Theorem*

The algebraic content of the spectral theorem is contained in the following result.

Theorem 2.2.1. If $\mathfrak{A}$ is a commutative B^* algebra, there is a compact-Hausdorff space X (the set of non-zero multiplicative linear functionals on $\mathfrak{A}$ in the w^* topology) and an isomorphism ϕ defined by $\phi(A)\,(\rho)=\rho(A)$ of the algebra $\mathfrak{A}$ onto the algebra of complex-valued continuous functions on X such that $\phi(A^*) = \overline{\phi(A)}$, $\|A\| = \|\phi(A)\|$ ($= \sup\{|\phi(A)(\rho)| : \rho \text{ in } X\}$), $A \geqslant 0$ if and only if $\phi(A) = 0$ (i.e. $\phi(A)(\rho) \geqslant 0$ for each ρ in X), and $\sigma(A)$ is the range of $\phi(A)$. If $\mathfrak{A}$ is generated by a single self-adjoint element A (i.e. is the norm closure of polynomials in A) then X can be taken as $\sigma(A)$ in its usual metric topology; and, with p a polynomial, $\phi(p(A))$ is the restriction of p to $\sigma(A)$.

Before discussing the proof of this theorem, we note its relation to the heuristic discussion of "stretch" operators in §3 of chapter I. We commented that if A is a self-adjoint operator it is a norm limit of stretch operators. With $\mathfrak{A}$ the (commutative) C^* algebra generated by A and I, each unit eigenvector x of A corresponds to a pure state of $\mathfrak{A}$ defined by $B \rightarrow (Bx, x)$, B in $\mathfrak{A}$ (we shall see shortly that such functionals—obviously states of $\mathfrak{A}$—are pure states on the assumption that x is an eigenvector for A). The other pure states of $\mathfrak{A}$ correspond to the case when no actual stretch is present (and, so, correspond to the "generalized stretches").

For the proof of the theorem, we denote by X the set of non-zero multiplicative linear functionals on $\mathfrak{A}$. With ρ in X, $\rho(I^2) = \rho(I)^2 \neq 0$ so that $\rho(I) = 1$. Hence $\rho(A)$ is in $\sigma(A)$, for each A in $\mathfrak{A}$; since $\rho(B(A - \rho(A)I)) = 0$ for each B in $\mathfrak{A}$, so that $B(A - \rho(A)I)$ is not I. Thus $|\rho(A)| \leqslant \|A\|$, and $\|\rho\| \leqslant 1$. It follows from the Alaoglu–Bourbaki theorem that X, being w^* closed in the dual of $\mathfrak{A}$, is w^* compact. With A a self-adjoint element of $\mathfrak{A}$ (i.e. $A = A^*$) and ρ in X, if $\rho(A) = \alpha + i\beta$, α and β real; then $\rho(B + i\gamma I) = i(\beta + \gamma)$, where $B(= A - \alpha I)$ is also self-adjoint. Thus $i(\beta + \gamma)$ is in $\sigma(B + i\gamma I)$ and $-i(\beta + \gamma)$ is in $\sigma((B + i\gamma I)^*)$, since * is a conjugate-linear, (anti-) automorphism of $\mathfrak{A}$. Now $|(\beta + \gamma)^2| = \beta^2 + 2\beta\gamma + \gamma^2 \leqslant \|B + i\gamma I\| \cdot \|(B + i\gamma I)^*\| = \|(B + i\gamma I)(B - i\gamma I)\| \leqslant \|B^2\| + \gamma^2$, for all real γ of the same sign as β. It follows that $\beta = 0$ and $\rho(A)$ is real. Since, for arbitrary T in $\mathfrak{A}$, $\rho(T) = \rho(T_1) + i\rho(T_2)$ with $\rho(T_1)$ and $\rho(T_2)$ real, where T_1 ($= (T + T^*)/2$) and T_2 ($= (T - T^*)/2i$) are self-adjoint, and $T^* = T_1 - iT_2$; $\rho(T^*) = \overline{\rho(T)}$.

If λ is in $\sigma(A)$ then $(A - \lambda I)\mathfrak{A}$ is a proper ideal in $\mathfrak{A}$ and, by Zorn's lemma, is contained in a maximal ideal $\mathcal{M}$ of $\mathfrak{A}$. Since the norm closure of $\mathcal{M}$ is again an ideal of $\mathfrak{A}$ and proper (for $\|I - B\| \geqslant 1$ for each B in $\mathscr{M}$, otherwise $I-(I-B)=B$ has the inverse $\sum_{n=0}^{\infty}(I-B)^n$), $\mathscr{M}$ must be closed (by maximality). Now the quotient algebra $\mathfrak{A}/\mathscr{M}$ (i.e. the set of classes $\{B+\mathscr{M}:B \text{ in } \mathfrak{A}\}$ with the sum $A+\mathscr{M}+B+\mathscr{M}$ is $(A+B)+\mathscr{M}$ and product $(A+\mathscr{M})(B+\mathscr{M}) = AB+\mathscr{M}$) provided with the norm $\inf\{\|A+B\|:B \text{ in } \mathscr{M}\}$ for $\|A+\mathscr{M}\|$ is again Banach algebra. Moreover, each element of $\mathfrak{A}/\mathscr{M}$ other than 0 has an inverse (since $\mathscr{M}$ is a maximal ideal and $\mathfrak{A}$ is commutative). But since each $A+\mathscr{M}$ has non-void spectrum, $A+\mathscr{M}-(\lambda I+\mathscr{M})$ must be 0 for some scalar λ, so that $A-\lambda I$ is in $\mathscr{M}$. (We have just observed that each Banach field over the complex numbers coincides with the complex numbers.) The mapping ρ of $\mathfrak{A}$ into $\mathbf{C}$ defined by $A-\rho(A)I$ is in $\mathscr{M}$ is then multiplicative; and since $A-\rho(A)I-(A-\lambda I)$ $(=-(\rho(A)-\lambda)I)$ is in $\mathscr{M}$, a proper ideal in $\mathfrak{A}$, $\rho(A)=\lambda$.

The mapping $A \to \phi(A)$ of $\mathfrak{A}$ into $C(X)$ defined by $\phi(A)(\rho)=\rho(A)$, for each ρ in X (that $\phi(A)$ is continuous on X is an immediate consequence of the definition of the w^* topology) is an isometric (norm-preserving) algebraic isomorphism of $\mathfrak{A}$ into $C(X)$ such that $\phi(A^*)=\overline{\phi(A)}$ for each A in $\mathfrak{A}$. For this, it remains to note that, with B self-adjoint in $\mathfrak{U}$, $\|B^{2n}\|=\|B\|^{2n}$ (from the assumption that $\|A^*A\|=\|A^*\|\cdot\|A\|$ for each A in $\mathfrak{A}$); so that $\|B\|$ is the spectral radius of B (see §3 of chapter I). From the foregoing, there is a ρ in X with $\|A^*\|\cdot\|A\|=\|A^*A\|=|\rho(A^*A)|=|\rho(A^*)|\cdot|\rho(A)|\leqslant\|A^*\|\cdot\|A\|$. Since $|\rho(A^*)|\leqslant\|A^*\|$ and $|\rho(A)|\leqslant\|A\|$, we must have equality in both cases; so that $\|A\|=\sup\{|\phi(A)(\rho)|:\rho \text{ in } X\}$ $(=\|\phi(A)\|)$. The image $\phi(\mathfrak{A})$ of $\mathfrak{A}$ in $C(X)$ is closed, since $\mathfrak{A}$ is complete and ϕ is an isometry. Moreover, $\overline{\phi(A)}$ $(=\phi(A^*))$ is in $\phi(\mathfrak{A})$, for each $\phi(A)$ in $\phi(\mathfrak{A})$, 1 $(=\phi(I))$ is in $\phi(\mathfrak{A})$, and $\phi(\mathfrak{A})$ separates points of X (i.e. with ρ_1 and ρ_2 distinct in X there is an A in $\mathfrak{A}$ such that $\rho_1(A)=\phi(A)(\rho_1) \neq\phi(A)(\rho_2)=\rho_2(A)$). These are the conditions under which the Stone-Weierstrass theorem tells us that $\phi(\mathfrak{A})=C(X)$.

Suppose now that $\mathfrak{A}$ is generated by the set of elements $\{A_\alpha\}$. With Y the Cartesian product of the sets $\sigma(A_\alpha)$ taken in the product topology (i.e. coordinate-wise convergence), the mapping which assigns to each point ρ in X the point of Y with α-coordinate $\rho(A_\alpha)$

is continuous, since each coordinate function $\phi(A_\alpha)$ of this mapping is continuous on X. If ρ_1 and ρ_2 map into the same point of Y then ρ_1 and ρ_2 agree on all polynomials in the A_α, a norm dense subset of $\mathfrak{A}$, and, by norm continuity of ρ_1 and ρ_2, they agree on $\mathfrak{A}$, i.e. $\rho_1 = \rho_2$. Since the mapping is one-one, continuous, X is compact, and Y is Hausdorff, it is a homeomorphism of X with a compact subset of Y_0 of Y. In particular, if $\mathfrak{A}$ is generated by a single self-adjoint element A, X is homeomorphic with the compact subset $\sigma(A)$ of the reals. In general the homeomorphism induces an isomorphism of $C(X)$ and, hence, $\mathfrak{A}$ onto $C(Y_0)$ which maps A_α onto the α-coordinate function of Y restricted to Y_0; and, with the single generator A, $p(A)$ is mapped onto the restriction of the polynomial p to $\sigma(A)$.

We have noted that λ is in $\sigma(A)$ if and only if there is a ρ in X such that $\rho(A) = \lambda$; so that $\sigma(A)$ is the range of $\phi(A)$. Since ϕ is an isomorphism and $\phi(A^*) = \overline{\phi(A)}$, we have A self-adjoint if and only if $\phi(A)$ is a real-valued function. Moreover, A is positive (i.e. self-adjoint with non-negative spectrum) if and only if $\phi(A) \geqslant 0$. In case $\mathfrak{A}$ is a C^* algebra acting on the Hilbert space $\mathscr{H}$, the functional $A \to (Ax,x)$ with x a unit vector in $\mathscr{H}$ maps, under ϕ, onto a functional of norm 1 which is 1 at $\phi(I)$ $(=1)$; so that this functional is a state of $C(X)$. Thus if A is self-adjoint with positive spectrum $(Ax,x) \geqslant 0$ for all x in $\mathscr{H}$. For each self-adjoint B in $\mathfrak{A}$, $\phi(B) = \phi(B)^+ - \phi(B)^-$ where $\phi(B)^+(p) = \max\{\phi(B)(p), 0\}$ and $\phi(B)^-(p) = -\min\{\phi(B)(p), 0\}$. Now $\phi(B)^+$ $(=(|\phi(B)| + \phi(B))/2)$ and $\phi(B)^-$ $(=(|\phi(B)| - \phi(B))/2)$ are positive functions in $C(X)$ such that $\phi(B)^+\phi(B)^- = 0$ and, therefore, correspond to positive operators B^+, B^- in $\mathfrak{A}$ such that $B = B^+ - B^-$ and $B^+B^- = 0$. If (Ax,x) is real for each x in $\mathscr{H}$, A is self-adjoint since (Az,y) is expressible as a linear combination of numbers of the form (Ax,x) $(=(x,Ax))$ so that $(Az,y) = (z,Ay)$. Thus if $(Ax,x) \geqslant 0$ for each x in $\mathscr{H}$, A is self-adjoint and $A = A^+ - A^-$ with $A^+A^- = 0$ and A^+, A^- self-adjoint with positive spectrum. Since $(A^-)^3$ is self-adjoint with positive spectrum, $((A^-)^3x,x) \geqslant 0$ for each x in $\mathscr{H}$. But $-(A^-)^3 = A(A^-)^2$, and $(A(A^-)^2x,x) = (AA^-x, A^-x) \geqslant 0$. Thus $((A^-)^3x,x) = 0$ for each x in $\mathscr{H}$, $(A^-)^3 = 0$, and $A^- = 0$. Thus A is self-adjoint with positive spectrum if and only if $(Ax,x) \geqslant 0$ for each x in $\mathscr{H}$—identifying the two apparently distinct notions of "positivity" for operators.

If $\mathfrak{A}$ is a Banach algebra and $\mathfrak{A}_0$ a closed subalgebra, one should make clear whether the spectrum of an element A in $\mathfrak{A}_0$ is being discussed relative to $\mathfrak{A}_0$ or to $\mathfrak{A}$. After all, the question of whether λ is in the spectrum of A relative to $\mathfrak{A}_0$ is one of the existence of an inverse to $A-\lambda I$ in $\mathfrak{A}_0$ which may be quite different from its existence in $\mathfrak{A}$. In general, $\sigma_{\mathfrak{A}}(A) \leqslant \sigma_{\mathfrak{A}_0}(A)$, with the obvious notation. In case $\mathfrak{A}$ is a B^* algebra and $\mathfrak{A}_0$ a B^* subalgebra this distinction disappears (i.e. $\sigma_{\mathfrak{A}}(A) = \sigma_{\mathfrak{A}_0}(A)$ in this case). This amounts to showing that if A in $\mathfrak{A}_0$ has an inverse in $\mathfrak{A}$ this inverse lies $\mathfrak{A}_0$ (equivalently, that A has *an* inverse in $\mathfrak{A}_0$). If A is self-adjoint with inverse A^{-1} in $\mathfrak{A}$, then A^{-1} is self-adjoint and commutes with A. Let $\mathfrak{A}_1$ be the (commutative B^*) subalgebra of $\mathfrak{A}$ generated by A, A^{-1} and I. Say $\mathfrak{A}_1$ is isomorphic to $C(X)$ with ϕ the isomorphism. Then $\phi(A)$ is an invertible, continuous, real-valued function on the compact space X; so that its range S is a compact subset of the reals bounded away from 0. Thus $\lambda \rightarrow \lambda^{-1}$ is continuous on S and, therefore (from the Weierstrass polynomial approximation theorem) the uniform limit of a sequence of restrictions of polynomials p_n to S. Since S is bounded away from 0, we may choose p_n without constant term. But then $p_n(\phi(A)) = \phi(p_n(A))$ tends in norm to $\phi(A)^{-1}$; so that, since ϕ is an isometry, $p_n(A)$ tends in norm to A^{-1}. Thus A^{-1} lies in the B^* algebra generated by A. If we replace $\lambda \rightarrow \lambda^{-1}$ in the preceding argument by an arbitrary continuous function f defined on the range of $\phi(A)$ (i.e. on $\sigma(A)$), we see that there is an element, the norm limit of $p_n(A)$, of the B^* algebra generated by A and I (we cannot assume, in general, that p_n is without constant term) corresponding to $f(\phi(A))$ in $C(X)$. We denote this element by $f(A)$, so that $\phi(f(A)) = f(\phi(A))$. What we have observed is that the process of taking a *continuous* function f of a self-adjoint element A of a B^* algebra $\mathfrak{A}$ is independent of the commutative B^* algebra containing A which is used and always results in an element $f(A)$ in the B^* algebra generated by A and I.

Suppose, now, that B is an arbitrary element of $\mathfrak{A}_0$ having an inverse, B^{-1}, in $\mathfrak{A}$. Then B^*B is a positive, self-adjoint, invertible element of $\mathfrak{A}_0$. From the foregoing, $(B^*B)^{1/2}$ lies in $\mathfrak{A}_0$ as does $(B^*B)^{-1/2}$, its inverse. Thus $B(B^*B)^{-1/2}$ $(=U)$ lies in $\mathfrak{A}_0$ as does $U^*(=(B^*B)^{-1/2}B^*)$. But $UU^*=U^*U=I$, and $(B^*B)^{-1/2}U^*$ is *an* inverse to B in $\mathfrak{A}_0$. Thus $B^{-1}=(B^*B)^{-1/2}U^*$; and B^{-1} lies in $\mathfrak{A}_0$.

By virtue of our theorem, to identify the pure states of $\mathfrak{A}$, a commutative B^* algebra (relative to the cone of positive elements), amounts to describing the pure states of $C(X)$ (order isomorphic with $\mathfrak{A}$ under the algebraic isomorphism ϕ). Note that each multiplicative linear functional on $C(X)$ (hence on $\mathfrak{A}$) is a state for it has the norm 1 and takes the value 1 at 1 (see the preceding section). With ρ multiplicative on $\mathfrak{A}$, A self-adjoint, and $\rho=\frac{1}{2}(\rho_1+\rho_2)$, we have

$$\rho(A^2) = \tfrac{1}{2}(\rho_1(A^2)+\rho_2(A^2)) = \rho(A)^2 = \tfrac{1}{4}(\rho_1(A)+\rho_2(A))^2;$$

so that

$$\begin{aligned} 0 &= [\rho_1(A^2)-\rho_1(A)^2]+[\rho_2(A^2)-\rho_2(A)^2]+\rho_1(A^2)- \\ &\quad -2\rho_1(A)\rho_2(A)+\rho_2(A^2) \\ &\geqslant [\rho_1(A)-\rho_2(A)]^2 \end{aligned}$$

(for $\rho_1(A^2)\geqslant\rho_1(A)^2$ and $\rho_2(A^2)\geqslant\rho_2(A)^2$ by applying the Schwarz inequality to the positive semi-definite inner products induced on $\mathfrak{A}$ by the states ρ_1 and ρ_2). Thus $\rho(A)=\rho_1(A)=\rho_2(A)$ for each self-adjoint A in $\mathfrak{A}$; and ρ is a pure state of $\mathfrak{A}$. On the other hand, if ρ is a pure state of $\mathfrak{A}$ and A is a positive self-adjoint element of $\mathfrak{A}$ less than I such that $0\neq\rho(A)\neq 1$ then ρ_1 and ρ_2 defined by $\rho_1(B)=\rho(AB)/\rho(A)$ and $\rho_2(B)=\rho((I-A)B)/\rho(I-A)$ are states of $\mathfrak{A}$ (for, with B positive, $\phi(AB)=\phi(A)\phi(B)$ is a positive function in $C(X)$ so that $AB\geqslant 0$). Since $\rho=\rho(A)\rho_1+\rho(I-A)\rho_2$ and ρ is pure, $\rho=\rho_1$; and $\rho(A)\rho(B)=\rho(AB)$ for the given A and all B in $\mathfrak{A}$. If $A\geqslant 0$ and $\rho(A)=0$ then

$$|\rho(AB)| = |\rho(A^{1/2}A^{1/2}B)| \leqslant \rho(A)^{1/2}\rho(B^*AB)^{1/2} = 0,$$

so that $0=\rho(AB)=\rho(A)\rho(B)$. If $A\leqslant I$ and $\rho(A)=1$ then $\rho(I-A)=0$ and $I-A\geqslant 0$; so that $\rho((I-A)B)=\rho(I-A)\rho(B)$, for all B in $\mathfrak{A}$ and $\rho(AB)=\rho(A)\rho(B)$. Thus $\rho(AB)=\rho(A)\rho(B)$ for each B in $\mathfrak{A}$ and each A with $I \geqslant A \geqslant 0$ in $\mathfrak{A}$. The same now holds for all $A \geqslant 0$, all $A \leqslant 0$ and, finally all A $(=\|A\|I-(\|A\|I-A))$. Thus the pure states of a commutative B^* algebra $\mathfrak{A}$ are precisely the non-zero multiplicative linear functionals on $\mathfrak{A}$. The basic question—which compact-Hausdorff spaces arise as the pure state space of some commutative B^* algebra?—has the simple answer—All. We note several useful facts and constructions preliminary to establishing this. We note first that each * representation or * antirepresentation ϕ (i.e. $\phi(AB)=\phi(B)\phi(A)$) of a B^* algebra $\mathfrak{A}$ preserves order, for A is positive if and only if $A=B^2$ with B a self-adjoint element of $\mathfrak{A}$; so that $\phi(A)=\phi(B)^2 \geqslant 0$ (this holds more generally for maps ϕ which preserve squares or, equivalently, the Jordan product $\frac{1}{2}(AB+BA)$). It follows that such maps ϕ are norm decreasing, for $-\|A\|I \leqslant A \leqslant \|A\|I$; so that $-\|A\|I \leqslant \phi(A) \leqslant \|A\|I$ and $\|\phi(A)\| \leqslant \|A\|$ for self-adjoint A, from the function algebra representation of commutative B^* algebras. For more general B in $\mathfrak{A}$, we have $\|B\|^2=\|BB^*\| \geqslant \|\phi(BB^*)\|=\|\phi(B)\|^2$, using the fact that $\|B\|=\|B^*\|$ in a B^* algebra which we know in C^* algebras and which is true—though somewhat complicated to prove in B^* algebras. In addition, if ϕ is a * isomorphism or anti-isomorphism of the B^* algebra $\mathfrak{A}_1$, into the B^* algebra $\mathfrak{A}_2$ then ϕ is an order isomorphism (for, if $\phi(A) \geqslant 0$ then

$$\phi\left(\left[\left(\frac{A+A^*}{2}\right)^2\right]^{1/2}\right) = \phi(A), \quad \text{and} \quad \left[\left(\frac{A+A^*}{2}\right)^2\right]^{1/2} \geqslant 0);$$

so that, as above $\|\phi(A)\|=\|A\|$, A self-adjoint in $\mathfrak{A}_1$, and $\|\phi(B)\|=\|B\|$, for each B in $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ is complete its image in $\mathfrak{A}_2$ under ϕ is complete, hence, closed, and, therefore, a B^* subalgebra of $\mathfrak{A}_2$.

If $\{\mathscr{H}_\alpha\}$ is a family of Hilbert spaces, we define the direct sum $\Sigma\oplus\mathscr{H}_\alpha$ as the subset of elements $\{x_\alpha\}$ of the Cartesian product of the family $\{\mathscr{H}_\alpha\}$ for which $\Sigma_\alpha\|x_\alpha\|^2$ is finite, this set being provided with coordinatewise addition, multiplication by scalars and the inner product $(\{x_\alpha\},\{y_\alpha\})=\Sigma_\alpha(x_\alpha,y_\alpha)$. One verifies that $\Sigma\oplus\mathscr{H}_\alpha$ with the described structure is again a Hilbert space (the essential points involve use of the special form the Schwarz inequality takes in l_2 and the completeness of $\Sigma\oplus\mathscr{H}_\alpha$ which follows closely the proof of the completeness of 1_2). If ϕ_α is a * representation (or antirepresentation) of the B^* algebra $\mathfrak{A}$ in $\mathscr{B}(\mathscr{H}_\alpha)$, we define the direct sum $\Sigma\oplus\phi_\alpha$ of $\{\phi_\alpha\}$ to be the mapping of $\mathfrak{A}$ into $\mathscr{B}(\Sigma\oplus\mathscr{H}_\alpha)$ which assigns to each element A in $\mathfrak{A}$ the operator $\Sigma\oplus\phi_\alpha(A)$ defined by: $(\Sigma\oplus\phi_\alpha(A))(\{x_\alpha\})=\{\phi_\alpha(A)x_\alpha\}$. From our preceding remarks $\Sigma_\alpha\|\phi_\alpha(A)x_\alpha\|^2\leqslant\Sigma_\alpha\|\phi_\alpha(A)\|^2\|x_\alpha\|^2\leqslant\|A\|^2\,\Sigma_\alpha\|x_\alpha\|^2$, which is finite for $\{x_\alpha\}$ in $\Sigma\oplus\mathscr{H}_\alpha$. It is routine to check that $\Sigma\oplus\phi_\alpha$ is a * representation (or antirepresentation) of $\mathfrak{A}$ (into $\mathscr{B}(\Sigma\oplus\mathscr{H}_\alpha)$).

Returning to the function algebra $C(X)$, we note that each point p of X gives rise to a * representation on the one-dimensional Hilbert $\mathscr{H}_p$ by means of the mapping which assigns to the function f in $C(X)$ the scalar operator $f(p)I$ on $\mathscr{H}_p$. The direct sum of these representations is a faithful * representation (i.e. a * isomorphism) of $C(X)$ into $\mathscr{B}(\Sigma_p\oplus\mathscr{H}_p)$ and hence onto a (commutative) C^* algebra (from preceding remarks). The associated question of which function algebras $C(X)$ are * isomorphic to the same commutative C^* algebra is simply answered by: $C(X)$ and $C(Y)$ are isomorphic if and only if X and Y are homeomorphic (by means of a homeomorphism giving rise to the given isomorphism). A homeomorphism η between X and Y gives rise to the * isomorphism $f\to f\circ\eta$ of $C(X)$ onto $C(Y)$. On the other hand, if ϕ is an isomorphism of $C(X)$ onto $C(Y)$, ϕ preserves spectrum; and, since the real functions are precisely the elements with real spectrum, ϕ is a * isomorphism. As noted more generally, ϕ is an order * isomorphism of $C(X)$ onto $C(Y)$, and therefore effects a w^* homeomorphism of the pure state space of $C(X)$ onto that of $C(Y)$. The identification of the pure states with the points of the underlying space gives rise, now, to a homeomorphism between X and Y which induces ϕ. To establish this identification, note that there is a * isomorphism ϕ of $C(X)$ with $C(X_0)$, where each multiplicative linear functional on $C(X)$ has the form $f\to\phi(f)(p_0)$, for some p_0 in X_0; since c(X) is a commutative C^* algebra. As points of X_0 separate $C(X_0)$, there is just one p_0 with this property. With p in X, let $\eta(p)$ be the point of X_0 such that $f(p)=\phi(f)[\eta(p)]$ for all f in $C(X)$. Then $f\circ\eta^{-1}=\phi(f)$. It is readily checked that η is a homeomorphism of X onto X_0, since ϕ is an isomorphism of $C(X)$ onto

$C(X_0)$ (using the complete regularity of X_0: there is a function in $C(X_0)$ which is 1 at a given point of an open subset of X_0 and vanishes outside that subset). Thus η identifies the points of X with the multiplicative linear functionals on $C(X)$, and, from an earlier discussion, with the pure states of $C(X)$.

3. *Representations from States*

The basic technique for constructing * representations of a C^* algebra is described in the proof of:

Theorem 2.3.1. To each state ρ of a C^* algebra $\mathfrak{A}$ there corresponds a * representation ϕ of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ and a unit vector x_0 in $\mathscr{H}$ such that $\{\phi(A)x_0 : A \text{ in } \mathfrak{A}\}$ $(=\{\phi(\mathfrak{A})x_0\})$ is dense in $\mathscr{H}$ and $\rho(A)=(\phi(A)x_0,x_0)$ $(=\omega_{x_0}{}^0\phi(A))$ for each A in $\mathfrak{A}$. Any other * representation for which such a vector exists is unitarily equivalent to ϕ.

The principal tools and concepts necessary for the proof of this theorem have already been developed—the commutative spectral theorem, the inner product on $\mathfrak{A}$ associated with ρ and the properties of states. Set $[A,B]=\rho(B^*A)$. Since ρ is a state $[\ ,\ ]$ is a positive semi-definite inner product on $\mathfrak{A}$. The Schwarz inequality for this inner product tells us that $[A,B]=0$ if $[A,A]=0$; so that the set $\mathscr{K}$ of A in $\mathfrak{A}$ such that $[A,A]=0$ (the *null vectors* of the inner product) form a linear space. Moreover $\mathscr{K}$ is a left ideal in $\mathfrak{A}$ (the so-called *left kernel* of ρ) since $[CA,B]=[A,C^*B]$. The quotient linear space $\mathfrak{A}/\mathscr{K}$ (i.e. the set of classes $A+\mathscr{K}$ with the quotient linear space structure) has, induced on it, an inner product $\langle A+\mathscr{K},B+\mathscr{K}\rangle=\rho(B^*A)$ which is positive definite. The completion $\mathscr{H}$ of $\mathfrak{A}/\mathscr{K}$ relative to this inner product is the Hilbert space of our representation. With $\phi_0(A)$ defined by $\phi_0(A)(B+\mathscr{K})=AB+\mathscr{K}$, one verifies that $\phi_0(\alpha A+B)=\alpha\phi_0(A)+\phi_0(B)$, $\phi_0(AB)=\phi_0(A)\phi_0(B)$ and

$$\langle\phi_0(A^*)(B+\mathscr{K}),C+\mathscr{K}\rangle=\langle B+\mathscr{K},\phi_0(A)(C+\mathscr{K})\rangle.$$

Now

$$\begin{aligned}\langle\phi_0(A)(B+\mathscr{K}),\phi_0(A)(B+\mathscr{K})\rangle &= \rho(B^*A^*AB)\\ &\leqslant \|A^*A\|\rho(B^*B)\\ &= \|A\|^2\rho(B^*B),\end{aligned}$$

since $A^*A \leqslant \|A^*A\|I$ (so that $B^*A^*AB \leqslant B^*\|A^*A\|IB$) and ρ is order preserving. Thus $\phi_0(A)$, being defined on the dense subset $\mathfrak{A}/\mathscr{K}$ of $\mathscr{H}$ and being bounded there by $\|A\|$, has a (unique) bounded linear extension $\phi(A)$ defined on $\mathscr{H}$; and $\|\phi(A)\| \leqslant \|A\|$. The properties noted for ϕ_0 and simple limiting arguments now yield that ϕ is a * representation-action of $\mathfrak{A}$. Moreover, taking x_0 to be $I+\mathscr{K}$, $\{\phi(\mathfrak{A})x_0\}$ is $\mathfrak{A}/\mathscr{K}$ which is dense in $\mathscr{H}$, and $\langle\phi(A)x_0, x_0\rangle = \rho(A)$.

Suppose that ϕ_1 is a * representation of $\mathfrak{A}$ on $\mathscr{H}_1$ and x_1 is a unit vector in $\mathscr{H}_1$ such that $\{\phi_1(\mathfrak{A})x_1\}$ is dense in $\mathscr{H}_1$ for which $\omega_{x_1}(\phi_1(A)) = \rho(A)$, for each A in $\mathfrak{A}$. Then $\|\phi(A)x_0\|^2 = \rho(A^*A)$ $= \|\phi_1(A)x_1\|^2$; and the mapping taking $\phi(A)x_0$ onto $\phi_1(A)x_1$, for each A in $\mathfrak{A}$ is an isometric linear mapping of the dense subset $\{\phi(\mathfrak{A})x_0\}$ of $\mathscr{H}$ onto the dense subset $\{\phi_1(\mathfrak{A})x_1\}$ of $\mathscr{H}_1$. It has a unique extension to a unitary transformation U of $\mathscr{H}$ onto $\mathscr{H}_1$ which, one can show, implements a unitary equivalence of ϕ and ϕ_1 (i.e. $\phi_1(A) = U\phi(A)U^{-1}$, for each A in $\mathfrak{A}$).

A vector such as x_0 for which $\{\phi(\mathfrak{A})x_0\}$ is dense in $\mathscr{H}$ is said to be *cyclic* for $\phi(\mathfrak{A})$ and ϕ is said to be a *cyclic representation* of $\mathfrak{A}$. An immediate corollary of the result just established is the fact that each cyclic * representation of a C^* algebra $\mathfrak{A}$ is unitarily equivalent to one arising from a state of $\mathfrak{A}$; for if x_0 is a cyclic unit vector for the * representation ϕ then ρ defined by $\rho(A)$ $= (\phi(A)x_0, x_0)$ gives rise to a * representation of $\mathfrak{A}$ unitarily equivalent to ϕ from the last assertion of the theorem.

Certain states give rise to representations with special properties. In particular:

Theorem 2.3.2. The representation ϕ associated with a state ρ of the C^* algebra $\mathfrak{A}$ is irreducible if and only if ρ is a pure state of $\mathfrak{A}$.

We say that a subset $\mathscr{M}$ of $\mathscr{H}$ is *stable* under the action of a family $\mathscr{F}$ of operators when Ax is in $\mathscr{M}$ for each A in $\mathscr{F}$ and x in $\mathscr{M}$. We say that $\mathscr{F}$ acts *irreducibly* on $\mathscr{H}$ when it has no stable closed linear subspaces other than (0) and $\mathscr{H}$. A representation ϕ of $\mathfrak{A}$ on $\mathscr{H}$ is said to be *irreducible* when $\phi(\mathfrak{A})$ acts irreducibly on $\mathscr{H}$. An easy computation establishes that the orthogonal projection on a closed subspace commutes with each operator in a self-adjoint family of bounded operators (i.e. A is in the family if and only if A^* is) if and only if the subspace is stable under the family. For the proof of the theorem just stated we need:

Lemma 2.3.3. The self-adjoint family $\mathscr{F}$ acts irreducibly on $\mathscr{H}$ if and only if its commutant $\mathscr{F}'$ (i.e. the set of all bounded operators on $\mathscr{H}$ commuting with each operator in $\mathscr{F}$) consists of scalars.

If $\mathscr{F}'$ consists of scalars then no orthogonal projection other than 0 and I commutes with $\mathscr{F}$ and no closed subspace other then (0) and $\mathscr{H}$ is stable under $\mathscr{F}$, i.e. $\mathscr{F}$ acts irreducibly. If we assume, now, that $\mathscr{F}$ acts irreducibly, $\mathscr{F}'$ is self-adjoint since $\mathscr{F}$ is. Thus, to show A in $\mathscr{F}'$ is a scalar, it will suffice to show that $(A+A^*)/2$ and $-i(A-A^*)/2$ are scalars; so that we may assume that A is self-adjoint. Using the commutative spectral theorem, we represent the (commutative) C^* algebra generated by A and I as $C(\sigma(A))$. If $\sigma(A)$ has two distinct points λ_1 and λ_2 we can find positive continuous functions f_1 and f_2 such that $f_1 f_2 = 0$, $f_1(\lambda_1) = 1 = f_2(\lambda_2)$ with f_1 and f_2 defined on $\mathbf{R}$. Then $f_1(A)f_2(A) = 0$ but $f_1(A) \neq 0 \neq f_2(A)$. Since A commutes with $\mathscr{F}$, limits of polynomials in A lie in $\mathscr{F}'$; so that $f_1(A)$ and $f_2(A)$ lie in $\mathscr{F}'$. It follows that the closure of the ranges of $f_1(A)$ and $f_2(A)$ are stable under $\mathscr{F}$. But neither of these ranges is (0) since neither of $f_1(A)$ nor $f_2(A)$ is 0; and neither of their closures is $\mathscr{H}$ since $f_1(A)f_2(A) = 0$. This contradicts the irreducibility of $\mathscr{F}$. Thus $\sigma(A)$ consists of a single point and A is a scalar again from the representation of $\mathfrak{A}$ as $C(\sigma(A))$.

To prove the theorem, suppose first that ϕ is irreducible. If $\rho = (\rho_1 + \rho_2)/2$ with ρ_1 and ρ_2 states of $\mathfrak{A}$ and, with x_0 a cyclic vector for $\phi(\mathfrak{A})$ such that $\rho(A) = (\phi(A)x_0, x_0)$ for each A in $\mathfrak{A}$, define $[\phi(A)x_0, \phi(B)x_0]$ to be $\rho_1(B^*A)$. Then $[\ \ ,\ \ ]$ is a bounded bilinear form on $\{\phi(\mathfrak{A})x_0\}$ and extends to a unique bounded bilinear form on $\mathscr{H}$. From the remarks at the end of I (2), there is a bounded operator T such that

$$(T\phi(A)x_0, \phi(B)x_0) = [\phi(A)x_0, \phi(B)x_0]\,(= \rho_1(B^*A))$$

for all A and B in $\mathfrak{A}$. It follows that

$$(T\phi(C)\phi(A)x_0, \phi(B)x_0) = \rho_1(B^*CA) = (\phi(C)T\phi(A)x_0, \phi(B)x_0)$$

for each C in $\mathfrak{A}$; and T is in $\phi(\mathfrak{A})'$. Since ϕ is irreducible, $T = \alpha I$

for some scalar α, from the preceding lemma. Taking A and B to be I, above, we see that $\alpha = 1$; so that $\rho_1(A) = (\phi(A)x_0, x_0) = \rho(A)$ for each A in $\mathfrak{A}$. Thus $\rho_1 = \rho$ and ρ is a pure state of $\mathfrak{A}$.

Suppose now that ρ is pure and that E is a projection commuting with $\phi(\mathfrak{A})$. Note that if η is a linear functional on $\mathfrak{U}$ such that it and $\rho - \eta$ are positive (we write $0 \leqslant \eta \leqslant \rho$ in this case) then $\eta = \eta(I)\rho$ —in fact this is equivalent to ρ being pure. To see this note that if $\eta(I) = 0$ then $\eta = 0$ (since $\eta \geqslant 0$) and $\eta = \eta(I)\rho$. If $\eta(I) = 1$ then $(\rho - \eta)(I) = 0$ and $\rho = \eta$ (since $\rho - \eta \geqslant 0$). If $0 < \eta(I) < 1$ then from $\rho = \eta(I)\eta/\eta(I) + (1 - \eta(I))(\rho - \eta)/(1 - \eta(I))$ and the fact that ρ is pure, we have $\eta = \eta(I)\rho$. On the other hand, assuming this property for the state ρ of $\mathfrak{A}$, if $\rho = a\eta + (1 - a)\tau$ with $0 < a < 1$ and n, τ states of $\mathfrak{A}$ then $0 \leqslant a\eta \leqslant \rho$ so that $a\rho = a\eta$ and $\eta = \rho$ – from which, ρ is pure. Returning to our proof, with η defined by $\eta(A) = (\phi(A)Ex_0, x_0)$, we have $0 \leqslant \eta \leqslant \rho$; so that $\eta = \eta(I)\rho$, that is $(\phi(A)Ex_0, x_0) = (Ex_0, x_0)(\phi(A)x_0, x_0)$. Since x_0 is cyclic for $\phi(\mathfrak{A})$ and $(\phi(A)Ex_0, x_0) = (\phi(A)x_0, Ex_0)$, taking A in $\mathfrak{A}$ so that $\phi(A)x_0$ is near Ex_0, we have $(Ex_0, x_0)^2 = (Ex_0, Ex_0) = (Ex_0, x_0)$. Thus (Ex_0, x_0) $(= \|Ex_0\|^2)$ is 0 or 1. In the first case $0 = Ex_0 = \phi(\mathfrak{A})(Ex_0) = E\phi(\mathfrak{A})x_0$ and $E = 0$; in the second, $((I - E)x_0, x_0) = 0$ and by the same argument $I = E$. Thus $\phi(\mathfrak{A})$ acts irreducibly.

An important technical addition to our information about constructing * representations of C^* algebras is the fact that the image of such a representation is automatically a C^* algebra, i.e. norm closed as well as being a self-adjoint algebra of operators. To demonstrate this we note first two special cases. If ϕ is a * isomorphism (or anti-isomorphism) of the C^* algebra $\mathfrak{A}$ we have noted in the proof of the commutative spectral theorem that ϕ is isometric so that $\phi(\mathfrak{A})$ is norm closed. If $\mathfrak{A}$ acts on the Hilbert space $\mathscr{H}$ and E is a projection in $\mathfrak{U}'$, $A \to AE$ is a * representation of $\mathfrak{A}$ (acting on $E(\mathscr{H})$). We show that $\mathfrak{A}E$ is norm closed in $\mathscr{B}(E(\mathscr{H}))$. Of course $A \to AE$ is norm-continuous so that $\mathscr{I}$, the two sided ideal of zeros of this mapping is closed. With η the quotient mapping of $\mathfrak{A}$ as a linear space onto $\mathfrak{A}/\mathscr{I}$, η is both continuous and open; moreover τ defined by $\tau(A + \mathscr{I}) = \phi(A)$ is an isomorphism of $\mathfrak{A}/\mathscr{I}$ onto $\mathfrak{A}E$. If $\mathscr{O}$ is an open set in $\mathfrak{A}E$, $\tau^{-1}(\mathscr{O}) = \eta(\phi^{-1}(\mathscr{O}))$, which is open in $\mathfrak{A}/\mathscr{I}$ (provided with its quotient Banach space norm). Thus τ is continuous. If we can prove that τ^{-1} is continuous then $\mathfrak{A}E$ with its operator norm is topologically linear isomorphic with the Banach space $\mathfrak{A}/\mathscr{I}$ so that $\mathfrak{A}E$ is complete—hence closed in $\mathscr{B}(E(\mathscr{H}))$. That τ^{-1} is continuous will follow if we can show that $\|AE\| \geqslant \|A+\mathscr{I}\|$ for each self-adjoint A in $\mathfrak{A}$. With $\mathfrak{A}_0$ the commutative C^* algebra generated by A and I, and $\mathscr{I}_0$ the kernel of $B \to BE$ in $\mathfrak{A}_0$, we have $\mathscr{I}_0$ contained in $\mathscr{I}$ so that $\|A+\mathscr{I}_0\| = \inf\{\|A+B\| : B \text{ in } \mathscr{I}_0\} \geqslant \inf\{\|A+B\| : B \text{ in } \mathscr{I}\} = \|A+\mathscr{I}\|$. We complete the proof that $\mathfrak{A}E$ is closed by showing that $\|A+\mathscr{I}_0\| = \|AE\|$. Now $\mathfrak{A}_0$ is isomorphic and isometric with $C(\sigma(A))$, $\mathscr{I}_0$ corresponds to the set of functions vanishing on a closed set S in $\sigma(A)$ and $\mathfrak{A}_0/\mathscr{I}_0$ is isomorphic and isometric with $C(S)$. Thus τ on $\mathfrak{A}_0$ corresponds to an isomorphism of $C(S)$ onto $\mathfrak{A}_0E$. Since * isomorphisms of C^* algebras are isometries, $\|A+\mathscr{I}_0\| = \|AE\|$.

To complete the proof, let ψ be $\iota \oplus \phi$, a * isomorphism of $\mathfrak{A}$, where ι is the identity mapping of $\mathfrak{A}$ onto $\mathfrak{A}$ (acting on $\mathscr{H}$). From the foregoing, $\psi(\mathfrak{A})$ is a C^* algebra commuting with E, the orthogonal projection of $\mathscr{H} \oplus \mathscr{H}_0$ onto $\mathscr{H}_0$, the Hilbert space on which $\phi(\mathfrak{A})$ acts; and $\psi(\mathfrak{A})E = \phi(\mathfrak{A})$ acting on $\mathscr{H}_0$ is a C^* algebra.

At the end of II(1), we noted that if we can show that the states of a C^* algebra $\mathfrak{A}$ are plentiful (form a separating family) then so are the pure states of $\mathfrak{A}$. We show that for each self-adjoint A in $\mathfrak{A}$ and each λ in $\sigma(A)$ there is a state ρ of $\mathfrak{A}$ such that $\rho(A) = \lambda$. Let $\mathfrak{A}_0$ be the (commutative) C^* algebra generated by A and I. Then $\mathfrak{A}_0$ is isomorphic with $C(\sigma(A))$ and A corresponds to the identity mapping of $\sigma(A)$ onto itself (i.e. to the "polynomial x"). Under this isomorphism the mapping which assigns to a function in $C(\sigma(A))$ its value at λ corresponds to a state ρ_0 of $\mathfrak{A}_0$ and $\rho_0(A) = \lambda$. From II(1), we know that ρ_0 has an extension ρ which is a state of $\mathfrak{A}$.

4. *Further Information about States and Representations*

In describing "irreducible" representations of C^* algebras, we imposed the condition that no *closed* subspace of the representation space other than (0) and this space itself be stable under the image.

By so doing we have singled out the *topologically* irreducible representations. There remain the *algebraically* irreducible representations—those for which no linear manifold (closed or otherwise) is stable under the image except for (0) and the representation space. For C^* algebras it is a fact that these classes of * representations coincide (that is to say, each topologically irreducible representation of a C^* algebra is an algebraically irreducible). More is true:

Theorem 2.4.1. If the C^* algebra $\mathfrak{A}$ acts irreducibly on $\mathscr{H}$ and $x_1, \ldots, x_n$ are linearly independent vectors in $\mathscr{H}$ then, with $y_1, \ldots, y_n$, n arbitrary vectors in $\mathscr{H}$, there is an operator B in $\mathfrak{A}$ such that $Bx_j = y_j$, $j = 1, \ldots, n$. If $y_j = Ax_j$, $j = 1, \ldots, n$ for some self-adjoint or unitary A in $\mathscr{B}(\mathscr{H})$, we can choose B in $\mathfrak{A}$ so that B is self-adjoint or unitary, respectively. In any case, if $y_j = Ax_j$, with A self-adjoint or not, the B chosen can be taken also satisfying $\|B\| \leqslant \|A\|$.

The proof of this result makes use of two "density" theorems the discussion of which we defer until later. The first, the von Neumann density theorem, states that a self-adjoint operator algebra $\mathfrak{A}$ is strong-operator dense in $\mathfrak{A}''$ (its second commutant). The second, the Kaplansky density theorem, states that if $\mathscr{M}$ is a self-adjoint subalgebra of the self-adjoint operator algebra $\mathscr{N}$ then $\mathscr{M}$ is strong-operator dense in $\mathscr{N}$ if and only if the unit ball of $\mathscr{M}$ is strong-operator dense in the unit ball of $\mathscr{N}$. From the first of these and lemma 2.3.3, we deduce another criterion for irreducibility of a self-adjoint operator algebra $\mathfrak{A}$. The irreducibility of $\mathfrak{A}$ acting on $\mathscr{H}$ is equivalent to $\mathfrak{A}$ being strong-operator dense in $\mathscr{B}(\mathscr{H})$—which from the Kaplansky density theorem is equivalent to the unit ball of $\mathfrak{A}$ being strong-operator dense in the unit ball of $\mathscr{B}(\mathscr{H})$.

To prove the theorem, a successive approximation scheme is employed. There is, of course, an operator A in $\mathscr{B}(\mathscr{H})$ such that $Ax_j = y_j$. Moreover, if all the y_j are suitably small, A can be chosen with small norm. Now B_1 in $\mathfrak{A}$ can be chosen with norm not exceeding that of A such that $(A - B_1)x_j$ is small for all $j = 1, \ldots, n$. Thus we can find A_1 in $\mathscr{B}(\mathscr{H})$ with small norm so that $A_1 x_j = (A - B_1)x_j$, $j = 1, \ldots, n$ and hence B_2 in $\mathfrak{A}$ with small norm such that $(A_1 - B_2)x_j$ is quite small. Continuing in this way, we construct the sequence

$B_1, B_2, B_3, \ldots$ of operators in $\mathfrak{A}$ such that $\sum_{j=1}^{\infty} B_j$ converges in norm to B in $\mathfrak{A}$, since $\mathfrak{A}$ is closed. By construction $Bx_j = Ax_j = x_j = y_j$, $j = 1, \ldots, n$. The degree of approximation is at our disposal from the first stage, the choice of B_1, so that we can arrange to find B in $\mathfrak{A}$ with $\|B\| \leqslant \|A\| + 1/n$ for any given positive integer n. With A self-adjoint, the Kaplansky density theorem allows us to make our successive approximations with self-adjoint operators B in $\mathfrak{A}$ so that the B we construct is self-adjoint. If E is the orthogonal projection on the (finite-dimensional) space generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, then EAE is a self-adjoint operator acting on $E(\mathscr{H})$ and has a diagonalizing orthonormal basis $z_1, \ldots, z_m$ in $E(\mathscr{H})$. Any operator on $\mathscr{H}$ which acts like EAE on $z_1, \ldots, z_m$ coincides with EAE on $E(\mathscr{H})$ and, therefore, maps x_j on y_j, $j = 1, \ldots, n$. Thus it suffices to consider the case where $\{x_j\}$ is an orthonormal set and $Ax_j = \lambda_j x_j$, $j = 1, \ldots, n$. Now $|\lambda_j| \leqslant \|A\|$, $j = 1, \ldots, n$. Having constructed B in $\mathfrak{A}$ with B self-adjoint and $Bx_j = \lambda_j x_j$, we have that the vector states ω_{x_j} restricted to $\mathfrak{A}_0$, the commutative C^* algebra generated by B and I are multiplicative (hence pure). Now $\mathfrak{A}_0$ is isomorphic to $C(\sigma(B))$ and ω_{x_j} transforms under this isomorphism into "evaluation at λ_j" on $C(\sigma(B))$. The function f on $\sigma(B)$ which assigns λ to λ if $-\|A\| \leqslant \lambda \leqslant \|A\|$, $-\|A\|$ to λ if $\lambda < -\|A\|$ and $\|A\|$ to λ if $\|A\| < \lambda$ is continuous and corresponds to a self-adjoint operator B_0 in $\mathfrak{A}_0$ (hence in $\mathfrak{A}$). Since $\|f\| \leqslant \|A\|$ and $f(\lambda_j) = \lambda_j$, $j = 1, \ldots, n$ (for $|\lambda_j| \leqslant \|A\|$); $\|B_0\| \leqslant \|A\|$ and $\omega_{x_j}(B_0) = \lambda_j$, $j = 1, \ldots, n$.

But x_j is an eigenvector for each operator in $\mathfrak{A}_0$; so that $B_0 x_j = \lambda_j x_j$, $j = 1, \ldots, n$. We conclude that if $Ax_j = y_j$ for some self-adjoint A in $\mathscr{B}(\mathscr{H})$ then there is a self-adjoint B in $\mathfrak{A}$ with $Bx_j = y_j$ and $\|B\| \leqslant \|A\|$.

From this same argument, if A is unitary, by choosing a diagonalizing orthonormal basis for $E(\mathscr{H})$, it suffices, in order to produce a unitary B in $\mathfrak{A}$ with $Bx_j = y_j$, to deal with the case where $x_1, \ldots, x_n$ is an orthonormal set and to find B unitary in $\mathfrak{A}$ such that $Bx_j = \lambda_j x_j$ where $\lambda_1, \ldots, \lambda_n$ is a prescribed set of complex numbers of modulus 1. With $\lambda_j = \exp(i\alpha_j)$, α_j real, there is, of course, a bounded self-adjoint A_0 such that $A_0 x_j = \alpha_j, x_j, j = 1, \ldots, n$.

From the foregoing, we can find B_0 self-adjoint in $\mathfrak{A}$ such that $B_0 x_j = \alpha_j x_j$, $j = 1, \ldots, n$. Then $B = \exp(iB_0)$ is a unitary operator in $\mathfrak{A}$ and $Bx_j = \lambda_j x_j$.

When we established, in the preceding section that an invertible operator A in a C^* algebra has its inverse in that algebra, we noted that A could be written as a product $U(A^*A)^{1/2}$, where U is a unitary operator and $(A^*A)^{1/2}$ is an (invertible) operator which is positive (both operators lying in the C^* algebra, in this case). This decomposition is a special case of the so-called *polar-decomposition* which expresses an arbitrary bounded operator (valid also for a closed, densely-defined operator) A as a product $V(A^*A)^{1/2}$, where V is a unitary transformation of the range of $(A^*A)^{1/2}$ onto that of A. To effect the decomposition, one simply defines V as mapping $(A^*A)^{1/2}x$ onto Ax for each x in $\mathscr{H}$ and notes that V is isometric hence well-defined. Of course, V extends (uniquely) to an isometric mapping of the closure of the range of $(A^*A)^{1/2}$ onto that of A; and to "complete" V to an operator defined on all of $\mathscr{H}$ in a canonical fashion we define it to be 0 on the orthogonal complement of the closure of the range of $(A^*A)^{1/2}$ (extending it linearly, now, to all of $\mathscr{H}$). We call V a *partial isometry* with *initial space* the closure of the range $(A^*A)^{1/2}$ and *final* space that of A. If E is the orthogonal projection on the first closure and F on the second, one can check, without difficulty, that $V^*V = E$, $VV^* = F$ and that these equalities characterize a partial isometry V with initial space $E(\mathscr{H})$ and final space $F(\mathscr{H})$ (either equality alone tells us that V is a partial isometry). This polar decomposition of A is unique in the sense that if $A = WH$ with $H \geqslant 0$ and W a partial isometry having the closure of the range of H as initial space, then $H = (A^*A)^{1/2}$ and $W = V$. In fact, with E the orthogonal projection on this closure, $A^*A = HW^*WH = HEH = H^2$; so that $H = (A^*A)^{1/2}$. (Note that H commutes with A^*A and is a positive square root of A^*A, so that it is $(A^*A)^{1/2}$, using the fact, evident from the isomorphism of the commutative C^* algebra generated by H with $C(\sigma(H))$, that there is just one positive square root of a positive element in this algebra). Thus $W(A^*A)^{1/2}x = Ax$; and $Wy = 0$ if y is orthogonal to the range of $(A^*A)^{1/2}$, so that $W = V$.

Returning to the proof of our theorem, with $Ax_j = y_j$ and $A = VH$, the polar decomposition of A, we can find $\mathscr{K}$ self-adjoint in $\mathfrak{A}$ such that $\mathscr{K}x_j = Hx_j, j = 1, \ldots, n$ and $\|K\| \leqslant \|H\|$. Since V is isometric on the range of H, there is a unitary operator on $\mathscr{H}$ mapping Hx_j onto $VHx_j, j = 1, \ldots, n$; hence, from the foregoing, there is a unitary operator U in $\mathfrak{A}$ such that $UHx_j = UKx_j = VHx_j = y_j$, $j = 1, \ldots, n$. But $\|B\| = \|UK\| \leqslant \|H\| = \|A\|$, where $B = UK$ is in $\mathfrak{A}$ and $Bx_j = y_j$ $j = 1, \ldots, n$.

If ρ is a pure state of the C^* algebra $\mathfrak{A}$ and ϕ is the corresponding representation of $\mathfrak{A}$ on the Hilbert space $\mathscr{H}$ with unit cyclic vector x_0 such that $\omega_{x_0} \circ \phi = \rho$ then $\phi(A)x_0 = 0$ if and only if $0 = (\phi(A^*A)x_0, x_0) = \rho(A^*A)$, that is, if and only if A is in $\mathscr{K}$, the left kernel of ρ. If $\mathscr{K}_0$ is a left ideal in $\mathfrak{A}$ containing $\mathscr{K}$ but distinct from $\mathfrak{A}$, $\{\phi(\mathscr{K}_0)x_0\}$ is a linear manifold stable under $\phi(\mathfrak{A})$. With A in $\mathfrak{A}$ not in $\mathscr{K}_0$, $\phi(A)x_0$ is not in this linear manifold (for otherwise $\phi(A)x_0 = \phi(A_0)x_0$ with A_0 in $\mathscr{K}_0$ so that $A - A_0 \in \mathscr{K}$ and $A \in \mathscr{K}_0$, contrary to assumption). Since ρ is pure, $\phi(\mathfrak{A})$ is a C^* algebra which acts topologically irreducibly and, from the foregoing, algebraically irreducibly. Thus $\{\phi(\mathscr{K}_0)x_0\}$ reduces to (0), i.e. $\mathscr{K}_0 = \mathscr{K}$, and we conclude that $\mathscr{K}$ is a maximal left ideal. In addition $\{\phi(\mathfrak{A})x_0\} = \mathfrak{A}/\mathscr{K}$ is $\mathscr{H}$.

Corollary 2.4.2. If ρ is a pure state its left kernel $\mathscr{K}$ is a maximal left ideal in $\mathfrak{A}$ and $\mathfrak{A}/\mathscr{K}$ is complete relative to the inner product induced by ρ on $\mathfrak{A}/\mathscr{K}$.

Another characterization of pure states is given by:

Corollary 2.4.3. The null space $\mathscr{N}$ of a state ρ of the C^* algebra $\mathfrak{A}$ is $\mathscr{K} + \mathscr{K}^*$, where $\mathscr{K}$ is the left kernel of ρ, if and only if ρ is a pure state.

To see this note first that if $\rho = (\rho_1 + \rho_2)/2$ and $\mathscr{N} = \mathscr{K} + \mathscr{K}^*$, with ρ_1 and ρ_2 states of $\mathfrak{A}$, then, with A in $\mathscr{K}$, $0 = \rho(A^*A) = \rho_1(A^*A) = \rho_2(A^*A)$, so that $0 = \rho_1(A) = \rho_2(A)$. Thus ρ_1 and ρ_2 annihilate $\mathscr{N}$; so that $\rho = \rho_1 = \rho_2$ and ρ is a pure state.

On the other hand, if we assume that ρ is a pure state then ϕ, the representation corresponding to ρ on $\mathscr{H}$, is irreducible. With B in $\mathscr{N}$, $B + \mathscr{K}$ and $I + \mathscr{K}$ are orthogonal vectors in $\mathscr{H}$. From our theorem we can find a self-adjoint $\phi(A)$ in $\phi(\mathfrak{A})$ (and, so, can choose

A self-adjoint in $\mathfrak{A}$) such that $\phi(A)$ annihilates $I+\mathscr{K}$ and leaves $B+\mathscr{K}$ fixed, i.e. A lies in $\mathscr{K}$ and $AB-B=-C$, with C in $\mathscr{K}$. If $B=B^*$ then $B=BA+C^*$, so that B lies in $\mathscr{K}+\mathscr{K}^*$. Since $\mathscr{N}$ is self-adjoint, $\mathscr{N}$ is contained in $\mathscr{K}+\mathscr{K}^*$. As $\mathscr{K}+\mathscr{K}^*$ is contained in $\mathscr{N}$, we have $\mathscr{N}=\mathscr{K}+\mathscr{K}^*$, on the supposition that ρ is pure.

Theorem 2.4.4. The left kernel $\mathscr{I}$ of a state ρ of the C^* algebra $\mathfrak{A}$ is a maximal left ideal if and only if ρ is pure. Each maximal left ideal in $\mathfrak{A}$ is the left kernel of a unique pure state of $\mathfrak{A}$. Each closed left ideal in $\mathfrak{A}$ is the intersection of the maximal left ideals containing it.

We show first that if $\mathscr{K}$ is a left ideal in $\mathfrak{A}$ (possibly $\mathfrak{A}$ itself) containing the closed left ideal $\mathscr{L}$ of $\mathfrak{A}$, and each state of $\mathfrak{A}$ which annihilates $\mathscr{L}$ annihilates $\mathscr{K}$, then $\mathscr{K}=\mathscr{L}$. If A in $\mathscr{K}$ is positive the set S of states at which A is not less than $1/n^2$ is w^* compact. The hypothesis, compactness and the definition of the w^* topology guarantees the existence of a finite open covering $\{U_i\}$, $i=1, \ldots, m$, of S and elements A_i in $\mathscr{L}$ such that A_i (and hence $A_i^* A_i$, by the Schwarz inequality) does not vanish on U_i. Some positive multiple T_n^2 of $A_1^* A_1+\ldots+A_m^* A_m$ exceeds A on S. Since A is less than $1/n^2$ at states of $\mathfrak{A}$ not in S and some state of $\mathfrak{A}$ takes a value less than 0 at self-adjoint operators in $\mathfrak{A}$ which are not positive, $T_n^2+I/n^2 \geqslant A$. With T_n the positive square root of T_n^2, it follows from the Weierstrass approximation theorem that T_n is a uniform limit of polynomials without constant term in T_n^2; so that T_n lies in $\mathscr{L}$. Now

$$\begin{aligned}\|A^{1/2}(T_n+I/n)^{-1}T_n-A^{1/2}\|^2 &= \|(T_n+I/n)^{-1}A(T_n+I/n)^{-1}\|/n^2\\ &\leqslant \|(T_n+I/n)^{-2}(T_n^2+I/n^2)\|/n^2\\ &\leqslant 1/n^2.\end{aligned}$$

Thus $A^{1/2}$ (hence A) lies in $\mathscr{L}$. With B an arbitrary element in $\mathscr{K}$, B^*B and $(B^*B)^{1/2}$ lie in $\mathscr{K}$, from the foregoing, while

$$\|B[(B^*B)^{1/2}+I/n]^{-1}(B^*B)^{1/2}-B\| \leqslant 1/n,$$

as above, so that B lies in $\mathscr{L}$ and $\mathscr{K}=\mathscr{L}$.

Choosing $\mathscr{K}$ to be $\mathfrak{A}$ in the preceding argument, it follows that if $\mathscr{L}$ is a proper left ideal the set $\mathscr{S}$ of states of $\mathfrak{A}$ which annihilate it is non-null. Since this set is convex and w^* compact, it is the closed convex hull of its set $\mathscr{E}$ of extreme points. It is easily seen that each point of $\mathscr{E}$ is a pure state and, from an earlier result, has left kernel a maximal left ideal in $\mathfrak{A}$. The intersection of these maximal left ideals is a left ideal of $\mathfrak{A}$ containing $\mathscr{L}$, with the property that each state in $\mathscr{E}$ hence in $\mathscr{S}$ annihilates it. Assuming $\mathscr{L}$ is norm closed, the first part of this proof assures us, now, that $\mathscr{L}$ is the intersection of these maximal left ideals. It follows that each closed left ideal in $\mathfrak{U}$ is the intersection of the maximal left ideals containing it. If $\mathscr{M}$ is a maximal left ideal, we have just noted that there is a pure state ρ which annihilates it and for which it must be the left kernel (by maximality). Given that the left kernel $\mathscr{M}$ of a state η of $\mathfrak{A}$ is maximal, there is a pure state ρ annihilating $\mathscr{M}$, and for which $\mathscr{M}+\mathscr{M}^*$ is the null space, from the preceding corollary. Since η annihilates $\mathscr{M}+\mathscr{M}^*$, $\eta=\rho$. Thus if the left kernel of a state ρ of $\mathfrak{A}$ is a maximal left ideal, ρ is pure and is the only state of $\mathfrak{A}$ annihilating that ideal.

If ρ and τ are states of the C^* algebra $\mathfrak{A}$ such that there is a unitary operator U in $\mathfrak{A}$ for which $\tau(U^*AU)=\rho(A)$, all A in $\mathfrak{A}$; then with $\mathscr{J}$ and $\mathscr{K}$ the left kernels of ρ and τ, respectively, the mapping V of $\mathfrak{A}/\mathscr{J}$ onto $\mathfrak{A}/\mathscr{K}$ defined by $V(A+\mathscr{J})=AU+\mathscr{K}$ is an isometric mapping of a dense subset of the representation space corresponding to ρ onto that for τ. Hence V has an extension to a unitary mapping of one representation space onto the other. This extension implements a unitary equivalence of the representations corresponding to ρ and τ.

The converse to this is not true; and each cyclic representation of an abelian C^* algebra on a space of dimension greater than one affords a counter-example. For two states ρ and τ of an abelian C^* algebra $\mathfrak{A}$ to be *unitarily equivalent* in the sense just discussed they must be identical (for $\tau(U^*AU)=\tau(A)=\rho(A)$, for all A in $\mathfrak{A}$). However, from theorem 2.3.1, each cyclic vector in a cyclic representation gives rise to a state whose corresponding representation is unitarily equivalent to the given cyclic representation. It remains to note that if $\mathfrak{A}$ acting on $\mathscr{H}$ with cyclic vector x_0 is such that $\mathscr{H}$ has dimension greater than one, then there are cyclic vectors which give distinct states of $\mathfrak{A}$. Note that with x_0 cyclic for a self-adjoint family $\mathfrak{A}$ of operators, x_0 is separating for its commutant $\mathfrak{A}'$, that is, if $A'x_0=0$ with A' in $\mathfrak{A}'$, then $A'=0$. In particular, if $\mathfrak{A}$ is abelian, $\mathfrak{A}$ is contained in $\mathfrak{A}'$; so that each cyclic vector for an abelian self-adjoint family of operators is separating for that family. Note also that two unit vectors x_0 and y_0 in $\mathscr{H}$ give rise to the same vector state of the self-adjoint algebra $\mathfrak{A}$ of operators on $\mathscr{H}$ if and only if there is a partially isometric operator V in $\mathfrak{A}'$ with $Vx_0=y_0$, initial space $[\mathfrak{A}x_0]$ and final space $[\mathfrak{A}y_0]$. The mapping taking Ax_0 onto Ay_0 determines V, in case the vector states coincide on $\mathfrak{A}$; and these states clearly coincide in case there is such a V, since $V^*Vx_0=x_0$. Now, with $\mathfrak{A}$ abelian, x_0 cyclic for $\mathfrak{A}$ in $\mathscr{H}$, and $\mathscr{H}$ of dimension greater than one; choose A in $\mathfrak{A}$ invertible, and, using a suitable scalar multiple of A, we arrange that $\|Ax_0\|=1$. Since $\{\mathfrak{A}Ax_0\}$ contains $\{\mathfrak{A}A^{-1}Ax_0\}$ $(=\{\mathfrak{A}x_0\})$, Ax_0 is a cyclic unit vector for $\mathfrak{A}$. If Ax_0 and x_0 give rise to the same vector state of $\mathfrak{A}$, we have, from the preceding remarks, that there is a unitary operator U in $\mathfrak{A}'$ with $Ux_0=Ax_0$. Since x_0 is separating for $\mathfrak{A}'$, $U=A$, that is A is unitary. The isomorphism of $\mathfrak{A}$ with $C(X)$ shows at once that there are invertible operators in $\mathfrak{A}$ which are not scalar multiples of unitary operators (such multiples correspond to functions on $C(X)$ with constant modulus) if X has more than one point, i.e. unless $\mathfrak{A}$ consists just of scalars. With $\mathfrak{A}$ cyclic on $\mathscr{H}$, this entails that $\mathfrak{A}$ is one-dimensional contrary to assumption. Hence there are cyclic vectors for $\mathfrak{A}$ giving distinct vector states of $\mathfrak{A}$. For a specific example, we use the diagonal 2×2 matrices acting on two-dimensional Hilbert space.

A partial converse to the fact that the representations corresponding to ρ and τ are unitarily equivalent if the states ρ and τ are unitarily equivalent is valid. If ρ and τ are pure states of $\mathfrak{A}$ with corresponding representations unitarily equivalent then ρ and τ are unitarily equivalent. To demonstrate this, we need two additional observations. If ϕ is a * representation of $\mathfrak{A}$ and U is a unitary operator in $\phi(\mathfrak{A})$ with $\sigma(U)\neq\mathbf{C}_1$ (the set of complex numbers of modulus 1), then we can find U_0 in $\mathfrak{A}$ such that $\phi(U_0)=U$, with U_0 unitary. Since $\sigma(U)\neq\mathbf{C}_1$ we can find a continuous function f on $\mathbf{C}_1$ such that $f(U)$ is self-adjoint and

$\exp[if(U)] = U$. Now $f(U)$ is in $\phi(\mathfrak{A})$ (a C^* algebra), and, since $f(U)$ is self-adjoint, there is a self-adjoint A in $\mathfrak{A}$ such that $\phi(A) = f(U)$. Then $\phi(U_0) = U$, where $U_0 = \exp(iA)$ is a unitary operator in $\mathfrak{A}$. The second fact we need is a slight modification of the last statement in Theorem 2.4.1, to the effect that the unitary operator B can be chosen with $\sigma(B) \neq \mathbf{C}_1$. Re-examining the proof of Theorem 2.4.1, we see that B arises as $\exp(iB_0)$, where B_0 is a self-adjoint operator in $\mathfrak{A}$ having a given finite set of orthogonal unit eigenvectors with specified eigenvalues in $[-\pi, \pi]$. Arranging that $\sigma(B_0)$ is disjoint from some subinterval of $[-\pi, \pi]$, we have that $\exp(iB_0)$ has spectrum different from $\mathbf{C}_1$.

Suppose, now, that the representations ϕ and ψ corresponding to the pure states ρ and τ, respectively, of $\mathfrak{A}$ on $\mathscr{H}$ and $\mathscr{K}$ are unitarily equivalent; and the unitary transformation U of $\mathscr{H}$ onto $\mathscr{K}$ implements the equivalence. With x_0 and y_0 cyclic unit vectors for $\phi(\mathfrak{A})$ and $\psi(\mathfrak{A})$, respectively, such that $(\phi(A)x_0, x_0) = \rho(A)$ and $(\psi(A)y_0, y_0) = \tau(A)$, for all A in $\mathfrak{A}$; we can find a unitary V in $\psi(\mathfrak{A})$ such that $\sigma(V) \neq \mathbf{C}_1$ and $VUx_0 = y_0$. Thus there is a unitary U_0 in $\mathfrak{A}$ such that $\psi(U_0)Ux_0 = y_0$; so that $\tau(A) = \rho(U_0^* A U_0)$ for all A in $\mathfrak{A}$.

The proof of Theorem 2.4.1 relies on the Kaplansky and von Neumann density theorems. The Kaplansky density theorem follows from the fact that a continuous function on $\mathbf{R}$ to $\mathbf{R}$ which vanishes outside a compact set is continuous on $\mathscr{B}(\mathscr{H})_*$, the self-adjoint operators in $\mathscr{B}(\mathscr{H})$, taken in the strong-operator topology. Suppose we are given this fact and that $\mathscr{N}$ is a * sub-algebra of $\mathscr{B}(\mathscr{H})$ with $\mathscr{M}$ a strong-operator dense * subalgebra of $\mathscr{N}$. To show that $\mathscr{M}_1$, the unit ball of $\mathscr{M}$, is strong-operator dense in the unit ball $\mathscr{N}_1$ of $\mathscr{N}$, it suffices to deal with the case where $\mathscr{M}$ and $\mathscr{N}$ are norm closed (i.e. are C^* algebras), for the unit ball of $\mathscr{M}$ (and of $\mathscr{N}$) is norm dense, hence strong-operator dense, in that of its norm closure. Let f be continuous on $\mathbf{R}$ with range in $[-1, 1]$, vanish outside of $[-2, 2]$ and be the identity mapping on $[-1, 1]$. Then $f(\mathscr{M}_*) = \mathscr{M}_* \cap \mathscr{M}_1 \leqslant f(\mathscr{N}_*) = \mathscr{N}_* \cap \mathscr{N}_1$, since f has range in $[-1, 1]$ and $f(A) = A$, for each self-adjoint A with spectrum in $[-1, 1]$. Since $B \to B^*$ is weak-operator continuous on $\mathscr{B}(\mathscr{H})$

and $\mathcal{M}$ is weak-operator dense in $\mathcal{N}$ (being strong-operator dense in $\mathcal{N}$), $\mathcal{M}_*$ is weak-operator dense in $\mathcal{N}_*$. Being convex, the weak and strong operator closure of $\mathcal{M}_*$ coincide (as do those of $\mathcal{N}_*$). Thus $\mathcal{M}_*$ is strong-operator dense in $\mathcal{N}_*$. Having assumed that f is strong-operator continuous on $\mathcal{B}(\mathcal{H}_*)$, $f(\mathcal{M}_*)=\mathcal{M}_*\cap\mathcal{M}_1$ is strong-operator dense in $f(\mathcal{N}_*)=\mathcal{N}_*\cap\mathcal{N}_1$. With C an operator in $\mathcal{N}_1$, we consider the 2×2 matrix C_0 with 0 entries on the diagonal and C and C^* off the diagonal in the C^* algebra of 2×2 matrices with entries in $\mathcal{N}$ acting in the usual way on $\mathcal{H}\oplus\mathcal{H}$. The subalgebra consisting of those matrices with entries in $\mathcal{M}$ is strong-operator dense in the larger one; so that, from the foregoing, there is a (self-adjoint) 2×2 matrix A_0 with norm $\leqslant 1$ strong-operator close to C_0. The appropriate entry of A_0 lies in $\mathcal{M}_1$ and is strong-operator close to C. Hence $\mathcal{M}_1$ is strong-operator dense in $\mathcal{N}_1$. We note, from this argument, that $\mathcal{M}_1\cap\mathcal{M}_*$ is strong-operator dense in $\mathcal{N}_1\cap\mathcal{N}_*$ and that $\mathcal{M}_1\cap\mathcal{M}^+$ is strong-operator dense in $\mathcal{N}_1\cap\mathcal{N}^+$, where $\mathcal{M}^+$ and $\mathcal{N}^+$ are the sets of positive operators in $\mathcal{M}$ and $\mathcal{N}$.

It remains to establish the strong continuity of a function f on $\mathbf{R}$ to $\mathbf{R}$ continuous and vanishing outside a compact set. We note first that each continuous function on $\mathbf{C}$ to $\mathbf{C}$ is strong-operator continuous on any bounded set S of normal operators. Suppose $\|A\|\leqslant K$ for each A in S, then since $\|(AB-A_0B_0)x\| \leqslant \|A\|\cdot\|(B-B_0)x\|+\|(A-A_0)B_0x\|$, AB is strong-operator close to A_0B_0 provided A, B in S are strong-operator close to A_0 and B_0, respectively, in S. Since multiplication and addition are jointly strong-operator continuous on S, polynomials are strong-operator continuous on S. If f is continuous on the disk $\mathbf{D}$ with center 0 and radius K in $\mathbf{C}$ to $\mathbf{C}$, then, from an extension of the Weierstrass polynomial approximation theorem, there is a sequence (p_n) of polynomials in z and $\bar{z}$ such that $\|f-p_n\|<1/n$, $n=1, 2, \ldots$, where $\|\ \|$ refers to the "supnorm" of functions taken over $\mathbf{D}$. Since $\sigma(A)$ is contained in $\mathbf{D}$ for each A in S, $\|f(A)-p_n(A)\|<1/n$, for all A in S. It follows that f, being the uniform limit of strong-operator continuous functions p_n on S, is strong-operator continuous on S.

With A self-adjoint, $\|(A+iI)^{-1}\| \leqslant 1$, while

$$(A-iI)(A+iI)^{-1}-(A_0-iI)(A_0+iI)^{-1} \\ = 2i(A+iI)^{-1}(A-A_0)(A_0+iI)^{-1};$$

so that the Cayley transform $A \to (A-iI)(A+iI)^{-1}$ is a strong-operator continuous mapping of $\mathscr{B}(\mathscr{H})_*$ into $\mathscr{B}(\mathscr{H})_u$, the set of unitary operators in $\mathscr{B}(\mathscr{H})$. If U is a unitary operator with 1 not in $\sigma(U)$, $-i(U+I)(U-I)^{-1}$ is a self-adjoint operator with Cayley transform U. If h is a continuous function from **R** to **R** vanishing at ∞, then defining $f(z)$ for z in $\mathbf{C}_1$ to be $h[-i(z+1)(z-1)^{-1}]$ for $z \neq 1$ and 0 for $z=1$, f is continuous. Thus f is strong-operator continuous on $\mathscr{B}(\mathscr{H})_u$; and $h(A)=f[(A-iI)(A+iI)^{-1}]$. It follows that h is strong-operator continuous on $\mathscr{B}(\mathscr{H})_*$, being a composition of strong-operator continuous mappings.

Although the strong-operator continuity of continuous functions on **R** to **R** vanishing outside compact sets (and even the broader class of continuous functions vanishing at infinity has been dealt with) is all that is needed to complete the proof of the Kaplansky density theorem; it is worth noting that each bounded continuous function h on **R** to **R** is strong-operator continuous on $\mathscr{B}(\mathscr{H})_*$, for these continuity results are important and useful in themselves. This is proved by decomposing h as $(1-h)p+hq$, where p and $q-1$ are continuous, vanish at ∞ and $p(A_0)=q(A_0)=h(A_0)$, and observing that $h(A)-h(A_0)=[1-h(A)][p(A)-p(A_0)]+h(A)[q(A)-q(A_0)]$.

For the von Neumann density theorem, we suppose that $\mathfrak{A}$ is a self-adjoint algebra of operators containing I. Its strong closure $\mathfrak{A}^-$ certainly commutes with $\mathfrak{A}'$, that is $\mathfrak{A}^-$ contained in $\mathfrak{A}''$. Suppose T lies in $\mathfrak{A}''$ and x in $\mathscr{H}$ is given. With A in $\mathfrak{A}$, A^* is in $\mathfrak{A}$ and $[\mathfrak{A}x]$ is stable under A and A^*. Thus E, the orthogonal projection with range $[\mathfrak{A}x]$, commutes with each A in $\mathfrak{A}$, lies in $\mathfrak{A}'$, hence, commutes with T. It follows that $[\mathfrak{A}x]$ is stable under T; and, since x $(=Ix)$ lies in $[\mathfrak{A}x]$, that Tx lies in $[\mathfrak{A}x]$. Hence, given $\epsilon>0$, there is an A in $\mathfrak{A}$ such that $\|(T-A)x\|<\epsilon$. Now, let $\tilde{\mathscr{H}}$ be the direct sum of $\mathscr{H}$ with itself n times, let $\mathfrak{A}_n'$ be $n \times n$ matrices with entries in $\mathfrak{A}'$ (acting on the elements of $\tilde{\mathscr{H}}$ expressed as column vectors in the usual matrix fashion), let $\tilde{B}$ be the $n \times n$ matrix with B at each diagonal entry and zeros at all others (B in $\mathscr{B}(\mathscr{H})$), and let $\tilde{\mathfrak{A}}$ be $\{\tilde{A}: A \text{ in } \mathfrak{A}\}$. Then $\tilde{\mathfrak{A}}' = \mathfrak{A}'_n$ and $\tilde{T}$ lies in $\tilde{\mathfrak{A}}''$. From the foregoing, given $\epsilon > 0$ and $\tilde{x}$ ($= (x_1, \ldots, x_n)$) in $\tilde{\mathscr{H}}$, there is an $\tilde{A}$ in $\tilde{\mathfrak{A}}$ such that $\|(\tilde{T} - \tilde{A})\tilde{x}\| < \epsilon$, from which $\|(T - A)x_j\| < \epsilon$, $j = 1, \ldots, n$. Hence T lies in $\mathfrak{A}^-$, and $\mathfrak{A}'' = \mathfrak{A}^-$.

Aside from the use to which we have already put the von Neumann density theorem, it is typically used in situations such as the following. With T in $\mathscr{B}(\mathscr{H})$ and $\mathscr{R}$ the strong-operator closed algebra generated by T and T^*, V and H lie in $\mathscr{R}$, where VH is the polar decomposition of T. Since $H = (T^*T)^{1/2}$, H is even in the C^* algebra generated by T^* and T (i.e. is a norm limit of polynomials in T^*T). With U' a unitary operator in $\mathscr{R}'$, $T = U'TU'^* = U'VU'^*U'HU'^*$, and $U'VU'^*$, $U'HU'^*$ are a partial isometry and positive operator, respectively, satisfying the same conditions that V and H do. Uniqueness of the polar decomposition now gives $V = U'VU'^*$; so that V commutes with each unitary operator in $\mathscr{R}'$. The observation $A = \frac{1}{2}(U_1 + U_2)$, where U_1 and U_2 are the unitary operators $A + i(I - A^2)^{\frac{1}{2}}$ and $A - i(I - A^2)^{\frac{1}{2}}$, respectively, $0 \leqslant A \leqslant I$, establishes that each C^* algebra is generated (linearly) by its unitary operators. Thus V lies in $\mathscr{R}''$ ($= \mathscr{R}$).

5. *Special Cases*

We apply the general theory developed in the preceding section to study $\mathscr{B}(\mathscr{H})$ and its subalgebra $\mathscr{C}$, the compact (completely continuous) operators on infinite-dimensional separable Hilbert space $\mathscr{H}$. We recall that $\mathscr{C}$ is the norm closure of $\mathscr{F}$, the operators on $\mathscr{H}$ with finite-dimensional range. Note that $\mathscr{F}$ is a proper two-sided ideal in (the Banach algebra) $\mathscr{B}(\mathscr{H})$; so that $\mathscr{C}$ is a proper, norm-closed, two-sided ideal in $\mathscr{B}(\mathscr{H})$.

Theorem 2.5.1. The ideal $\mathscr{C}$ is the only proper two-sided, norm-closed ideal in $\mathscr{B}(\mathscr{H})$.

If $\mathscr{K}$ is a proper left ideal (not necessarily closed) in $\mathscr{B}(\mathscr{H})$, then $\{\mathscr{K}x\}$ is stable under $\mathscr{B}(\mathscr{H})$ for each x in $\mathscr{H}$. Thus $\{\mathscr{K}x\}$ is either (0) or $\mathscr{H}$. Since $\mathscr{K}$ is not (0), there is a unit vector x_0 such that $\mathscr{K}x_0 = \mathscr{H}$. With T in $\mathscr{K}$ such that $Tx_0 = x_0$ and E_0 the one-dimensional projection with x_0 in its range, T^*E_0T ($=(E_0T)^*(E_0T)$) is a non-zero, self-adjoint operator in $\mathscr{K}$ with one-dimensional range. Since a self-adjoint operator annihilates the orthogonal complement of its range T^*E_0T is a (non-zero) multiple of the one-dimensional projection with the same range. Thus $\mathscr{K}$ contains some one-dimensional projection, E. If F is another one-dimensional projection, there is a partial isometry V in $\mathscr{B}(\mathscr{H})$ with initial space E and final space F. Since $V = VE$, V is in $\mathscr{K}$. Assuming, now, that $\mathscr{K}$ is a two-sided ideal, F is in $\mathscr{K}$, for $F = VV^*$. Thus $\mathscr{K}$ contains all one-dimensional projections; and, hence, contains $\mathscr{F}$.

If T is not in $\mathscr{C}$ then T^*T is not in $\mathscr{C}$; for, otherwise, $(T^*T)^{1/2}$, a norm limit of polynomials in T^*T without constant term, lies in $\mathscr{C}$ and T ($= V(T^*T)^{1/2}$) lies in $\mathscr{C}$. Thus, if $\mathscr{K}$ is a left ideal in $\mathscr{B}(\mathscr{H})$ not contained in $\mathscr{C}$, there is some positive H in $\mathscr{K}$ not in $\mathscr{C}$. Now, the identity mapping on $\sigma(H)$ is a uniform limit of positive functions in $C(\sigma(H))$ vanishing outside intervals with positive left endpoints. Some such function f vanishing outside $[a,b]$ with $a>0$ must correspond to a (positive) operator A in the C^* algebra generated by H and I which has infinite-dimensional range (otherwise H lies in $\mathscr{C}$). Since $(\lambda-a)f(\lambda)\geqslant 0$ for all λ, $(H-aI)A$ and $(H-aI)A^2$ are positive. Thus $((H-aI)Ax,Ax)\geqslant 0$ for all x in $\mathscr{H}$. With E the projection on the closure of the range of A, it follows that $0\leqslant((H-aI)Ex,Ex)=((H-aI)E^2x,x)=((H-aI)Ex,x)$; so that $HE=EH\geqslant aE$. Passing to the C^*-algebra generated by I, EH and E and to its representation by functions, we see that E is a norm limit of polynomials in EH (each of which lies in $\mathscr{K}$). Thus $\mathscr{K}$ contains the infinite-dimensional projection E. Since $\mathscr{H}$ is separable, there is a partial isometry V with initial space E, so that V ($= VE$) lies in $\mathscr{K}$, and final space I. If $\mathscr{K}$ is assumed, now, to be two-sided, I ($= VV^*$) lies in $\mathscr{K}$; so that $\mathscr{K}=\mathscr{B}(\mathscr{H})$. Thus each proper, two-sided ideal in $\mathscr{B}(\mathscr{H})$ contains $\mathscr{F}$ and is contained in $\mathscr{C}$. Our theorem follows.

Making use of this characterization of the compact operators, one can now establish the other identifications of such operators: T is compact if and only if the image of the closed unit ball in $\mathscr{H}$ under T is compact (hence the name "compact" operator); and

T is compact if and only if $Tx_n - Ty \to 0$ if $(x_n - y, z) \to 0$ as $n \to \infty$, for each z in $\mathscr{H}$ (i.e. T converts weakly convergent sequences into norm convergent sequences).

With regard to pure states of $\mathscr{B}(\mathscr{H})$:

Theorem 2.5.2. Each pure state ρ of $\mathscr{B}(\mathscr{H})$ annihilates $\mathscr{C}$ or is a vector state (i.e. $\rho(A) = (Ax, x)$ for all A in $\mathscr{B}(\mathscr{H})$).

To see this, note that if $\rho(\mathscr{C}) \neq 0$, $\mathscr{C}$ is not the kernel of ϕ, the * representation of $\mathscr{B}(\mathscr{H})$ arising from ρ. From Theorem 2.5.1, ϕ is a * isomorphism in this case. Since the one-dimensional projections E in $\mathscr{B}(\mathscr{H})$ are characterized by the property $EAE = \alpha E$ for each bounded A (and $E \neq 0$), we have $\phi(E)\phi(A)\phi(E) = \alpha\phi(E)$. Strong-operator continuity of multiplication in the unit ball yields $\phi(E)B(E) = \beta\phi(E)$ for each bounded operator B on the representation space. (Recall that $\phi(\mathscr{B}(\mathscr{H}))$ acts irreducibly since ρ is pure; so that it is strong-operator dense in all bounded operators.) Thus $\phi(E)$ is a one-dimensional projection. With x_0 a unit vector such that $\rho(A) = (\phi(A)x_0, x_0)$, for each A in $\mathscr{B}(\mathscr{H})$, strong-operator density of $\phi(\mathscr{B}(\mathscr{H}))$ permits us to choose A_n in $\mathscr{B}(\mathscr{H})$ such that $\phi(A_n)x$ are unit vectors tending to x_0 (we could use Theorem 2.4.2 to shorten the argument). Now $\phi(A_n)\phi(E)$ maps x onto $\phi(A_n)x$, where x is a unit vector in the range of $\phi(E)$; and $\phi(A_n)\phi(E)$ annihilates the subspace orthogonal to x. Thus $\phi(A_n)\phi(E)[\phi(A_n)\phi(E)]^*$ is the one-dimensional projection with $\phi(A_n)x$ in its range. These projections tend in norm to $\phi(E_0)$ (in $\phi(\mathscr{B}(\mathscr{H}))$), the one-dimensional projection with x_0 in its range. Note that E_0 is a projection, since ϕ is a * isomorphism and $\phi(E_0)$ is a projection, and that E_0 is minimal in $\mathscr{B}(\mathscr{H})$, hence, one-dimensional. Now

$$\begin{aligned}\rho(A) = (\phi(A)x_0, x_0) = (\phi(E_0)\phi(A)\phi(E_0)x_0, x_0) &= (\phi(E_0AE_0)x_0, x_0)\\ = ((Az_0, z_0)\phi(E_0)x_0, x_0) &= (Az_0, z_0),\end{aligned}$$

where z_0 is a unit vector in the range of E_0.

Of course, the pure states of $\mathscr{B}(\mathscr{H})$ corresponding to unit vectors are all unitarily equivalent to one another (since each unit vector is the image of any other under a unitary operator). Making use of special knowledge about the pure state space of the (abelian) algebra of bounded operators having some fixed orthonormal basis as eigenvectors—namely, that this space has 2^c elements, where c is the cardinal number of the continuum—together with the fact that unitarily equivalent irreducible representations of a C^* algebra correspond to unitarily equivalent pure states (proved in the preceding section) and the fact that there are only c unitary operator; we conclude (from cardinality arguments) that there are 2^c unitarily inequivalent irreducible * representations of $\mathscr{B}(\mathscr{H})$ each with kernel $\mathscr{C}$ and all on a Hilbert space of dimension c.

Turning to the C^* algebra $\mathscr{C}_0$ generated by $\mathscr{C}$ and I:

Theorem 2.5.3. There are two unitary equivalence classes of pure states of $\mathscr{C}_0$, one containing just one pure state ρ_0 defined by and corresponding to the one-dimensional irreducible representation $\lambda I + C \to \lambda$, C in $\mathscr{C}$, of $\mathscr{C}_0$, and the other consisting of all vector states and corresponding to the given (irreducible) representation of $\mathscr{C}_0$ on $\mathscr{H}$.

Note that the set of operators $\lambda I + C : \lambda$ a scalar C in $\mathscr{C}$ is a C^* algebra—obviously the smallest one containing $\mathscr{C}$ and I, since adding a closed subspace and a finite-dimensional subspace of a normed space results in a closed subspace. Moreover, the representation of an operator in $\mathscr{C}_0$ as $\lambda I + C$ with C in $\mathscr{C}$ is unique since $\mathscr{C}$ does not contain I. Thus ρ_0 is well-defined; and, being one-dimensional is irreducible and a pure state. On the other hand, if ρ is a pure state of $\mathscr{C}_0$ not equal to ρ_0 then ρ does not annihilate $\mathscr{C}$. We know that ρ has a pure state extension to $\mathscr{B}(\mathscr{H})$ which, from Theorem 2.5.2, is a vector state. Thus all pure states of $\mathscr{C}_0$ other than ρ_0 are vector states–all unitarily equivalent.