JOURNAL OF FUNCTIONAL ANALYSIS 1, 204-221 (1967)

Derivations and Automorphisms of Operator Algebras. II

RICHARD V. KADISON*

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

AND

E. CHRISTOPHER LANCE AND JOHN R. RINGROSE

Department of Mathematics, University of Newcastle, Newcastle upon Tyne, England

Communicated by Irving E. Segal

Received April 7, 1967

1. INTRODUCTION

In [7] the automorphism group of a C^*-algebra, along with several of its naturally occurring subgroups, is studied. Principally, the connected component of the identity is identified there and shown to be open as well as closed; so that the same is true for several of the other subgroups studied. The group of inner automorphisms is touched on only glancingly. In particular it is not noted there whether this group is necessarily closed or necessarily open.

The present paper is primarily concerned with this subject as well as the associated question of whether or not the space of inner derivations of the algebra is closed. We establish (Theorem 5.3) a necessary and sufficient condition for both the inner derivations and inner automorphisms to be closed sets which, in its simplest application (Corollary 5.5) shows that they are closed if the C^*-algebra possesses a faithful representation such that the image contains the center of its weak-operator closure. This is the case if the C^*-algebra has a faithful factor (primary) representation (or, more specially, if it is primitive).

A related aspect of this study is the question of the extent to which the presence of inner automorphisms on the one-parameter group generated by a derivation of a C^*-algebra forces the derivation to be inner. We show (Theorem 4.3) that, if a C^*-algebra is norm-separable

* With partial support of ONR and NSF.

(equivalently, countably generated) and an uncountable number of the automorphisms are inner, then the derivation is inner. On the other hand (Example 6.2), we produce a C^*-algebra (nonseparable, of course) acting on a separable Hilbert space for which each of the automorphisms on the one-parameter group is inner while the derivation is outer. A simple example (Example 6.1) shows that a separable C^*-algebra can have an outer derivation whose exponential is an inner automorphism. Example 6.3 describes a derivation of a separable C^*-algebra which is outer even though all the automorphisms corresponding to rational parameters on the one-parameter subgroup it generates are inner. This same C^*-algebra has its center consisting of scalars and nonclosed sets of inner automorphisms and inner derivations (so that the algebra does not admit a faithful factor representation).

Section 2 introduces our notation and basic definitions. In Section 3 we locate a canonical operator in the weak-operator closure of a (concrete) C^*-algebra inducing a given derivation (Theorem 3.1). Theorem 3.2 identifies the π-inner automorphisms of a C^*-algebra as the weakly-inner automorphisms of the universal representation and establishes that each such automorphism maps every closed two-sided ideal of the algebra onto itself.

2. Notation and Definitions

By a C^*-algebra we mean a complex Banach $*$-algebra $\mathfrak{A}$ with identity, satisfying $\| A^*A \| = \| A^* \| \| A \|$ for each A in $\mathfrak{A}$. The unit ball of $\mathfrak{A}$ is denoted by $\mathfrak{A}_1$ and its unitary group by $\mathscr{U}(\mathfrak{A})$. A *representation* of $\mathfrak{A}$ on a (complex) Hilbert space $\mathscr{H}$ is a $*$-homomorphism φ from $\mathfrak{A}$ into the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$ which carries the identity of $\mathfrak{A}$ onto the identity operator on $\mathscr{H}$. Each representation of $\mathfrak{A}$ is norm-decreasing and has norm-closed range $\varphi(\mathfrak{A})$, while faithful (i.e., one-to-one) representations are isometric. We write $\varphi(\mathfrak{A})^-$ for the weak-(equivalently, strong-, ultraweak-, ultrastrong-) operator closure of $\varphi(\mathfrak{A})$. An extension of the Gelfand-Naimark Theorem [*3*] asserts the existence of a faithful representation of every C^*-algebra.

Two representations φ and ψ of $\mathfrak{A}$, on $\mathscr{H}_\varphi$ and $\mathscr{H}_\psi$, respectively, are said to be *equivalent* if there is an isometry U from $\mathscr{H}_\varphi$ onto $\mathscr{H}_\psi$ such that $\psi(A) = U\varphi(A)\, U^*$ for each A in $\mathfrak{A}$. If there is a vector x in $\mathscr{H}_\varphi$ such that the set $\{\varphi(A)\, x : A \in \mathfrak{A}\}$ is everywhere dense in $\mathscr{H}_\varphi$, then φ is called a *cyclic representation* and x a *cyclic vector* for φ. A *state* of $\mathfrak{A}$ is a linear functional f on $\mathfrak{A}$ such that $f(A^*A) \geqslant 0$ for each A in $\mathfrak{A}$

and $f(I) = 1$, where I is the identity of $\mathfrak{A}$. With each state f is associated a cyclic representation φ_f (unique to within equivalence), and a cyclic vector x_f for φ_f, such that $f(A) = \langle \varphi_f(A) x_f, x_f \rangle$ for each A in $\mathfrak{A}$. By selecting such a representation for each state of $\mathfrak{A}$, and taking the direct sum of the resulting family of representations, we obtain the *universal representation* Ψ of $\mathfrak{A}$ (*the* universal representation, because it is determined to within equivalence). It is faithful, and if φ is any other representation of $\mathfrak{A}$ then $\varphi\Psi^{-1}$ extends to a *-homomorphism from $\Psi(\mathfrak{A})^-$ onto $\varphi(\mathfrak{A})^-$ ([*4*], Lemmas 2.3, 2.4).

A *derivation* of a C^*-algebra $\mathfrak{A}$ is a linear mapping δ from $\mathfrak{A}$ into $\mathfrak{A}$ such that $\delta(AB) = A\delta(B) + \delta(A) B$ for all A and B in $\mathfrak{A}$. Each such δ has an adjoint derivation δ^* defined by $\delta^*(A) = \delta(A^*)^*$; and we call δ a *-derivation if $\delta = \delta^*$. If $\mathfrak{A}$ is a C^*-algebra given concretely as operators on a Hilbert space $\mathscr{H}$, $B \in \mathscr{B}(\mathscr{H})$ and $BA - AB \in \mathfrak{A}$ whenever $A \in \mathfrak{A}$, then the derivation ad $iB : T \to i(BT - TB)$ of $\mathscr{B}(\mathscr{H})$ restricts to a derivation ad $iB \mid \mathfrak{A}$ of $\mathfrak{A}$. Each derivation δ of $\mathfrak{A}$ has this form, and B can be chosen in $\mathfrak{A}^-$ (with $B = B^*$ if δ is a *-derivation) [*5*], [*6*], [*11*]. It follows that, if δ is a derivation of an (abstract) C^*-algebra $\mathfrak{A}$ and φ is a faithful representation of $\mathfrak{A}$, then $\varphi\delta\varphi^{-1} = \text{ad}\, iB \mid \varphi(\mathfrak{A})$ for some B in $\varphi(\mathfrak{A})^-$. This shows, in particular, that derivations of $\mathfrak{A}$ are *bounded* linear operators, a fact that had previously been conjectured by Kaplansky [*8*] and proved by Sakai [*10*] (and which is used in proving the result that derivations of C^*-algebras have the stated form). The set $\Delta(\mathfrak{A})$ of all derivations of $\mathfrak{A}$ is a norm-closed linear subspace of the Banach space $\mathscr{B}(\mathfrak{A})$ of all bounded linear operators on $\mathfrak{A}$; so $\Delta(\mathfrak{A})$ is a Banach space under the induced norm. The inner derivations, those of the form ad $iB \mid \mathfrak{A}$ with B in $\mathfrak{A}$, form a linear subspace $\Delta_0(\mathfrak{A})$ of $\Delta(\mathfrak{A})$.

By an *automorphism* of a C^*-algebra $\mathfrak{A}$ we shall mean an automorphism for the *-algebra structure (our automorphisms are *-automorphisms, though we do *not* adopt the corresponding convention for derivations); each automorphism of $\mathfrak{A}$ is an isometric linear mapping from $\mathfrak{A}$ onto $\mathfrak{A}$, and the set $\alpha(\mathfrak{A})$ of all automorphisms of $\mathfrak{A}$ is a complete metric group with the metric it inherits from $\mathscr{B}(\mathfrak{A})$. Its unit will be denoted by ι. The inner automorphisms of $\mathfrak{A}$—those of the form $A \to UAU^*$ for some fixed U in $\mathscr{U}(\mathfrak{A})$—form a subgroup $\iota_0(\mathfrak{A})$ of $\alpha(\mathfrak{A})$.

If δ is a *-derivation of $\mathfrak{A}$, then exp δ is an automorphism of $\mathfrak{A}$. If $\mathfrak{A}$ is given concretely as a C^*-algebra of operators on $\mathscr{H}$, and $\delta = adiB \mid \mathfrak{A}$ for some self-adjoint B in $\mathfrak{A}^-$, then by comparing series coefficients (cf. the proof of Lemma 2 in [*7*]) it follows that exp δ is the automorphism $A \to UAU^*$ of $\mathfrak{A}$, where U is the unitary operator exp iB. If δ is inner we may choose B in $\mathfrak{A}$; then $U \in \mathscr{U}(\mathfrak{A})$ and exp δ is an inner automorphism.

3. Derivations and π-Inner Automorphisms

It follows from the results of [5], [6], [11] that each derivation δ of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ is ultraweakly continuous and extends (uniquely) to an inner derivation $\bar{\delta} = \text{ad}\, iH \mid \mathfrak{A}^-$ of $\mathfrak{A}^-$. If Q is a projection in the center $\mathscr{Z}$ of $\mathfrak{A}^-$, then the restriction $\bar{\delta} \mid \mathfrak{A}^- Q$ is a derivation of $\mathfrak{A}^- Q$. We shall show that, if δ is a $*$-derivation then the operator H in $\mathfrak{A}^-$ can be chosen, uniquely, to satisfy certain rather stringent conditions.

With T a self-adjoint element of $\mathfrak{A}^-$ and Q a projection in $\mathscr{Z}$, we define

$$M_Q(T) = \inf\{a : TQ \leqslant aQ\},$$
$$m_Q(T) = \sup\{b : TQ \geqslant bQ\}.$$

Theorem 3.1. *Let δ be a $*$-derivation of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$, $\bar{\delta}$ its extension to a derivation of $\mathfrak{A}^-$, and $\mathscr{Z}$ the center of $\mathfrak{A}^-$. Then there is a unique self-adjoint operator H in $\mathfrak{A}^-$ such that $\bar{\delta} = \text{ad}\, iH \mid \mathfrak{A}^-$ and, for each projection Q in $\mathscr{Z}$, $\| HQ \| = \frac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^- Q \|$. Furthermore,*

$$M_Q(H) = - m_Q(H) = \tfrac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^- Q \| .$$

Proof. The argument is divided into several stages.

(a) We prove that $\bar{\delta} = \text{ad}\, iB \mid \mathfrak{A}^-$ for some self-adjoint operator B in $\mathfrak{A}^-$ with $\| B \| \leqslant \frac{1}{2} \| \bar{\delta} \|$ $(= \frac{1}{2} \| \delta \|)$. With $\alpha_t = \exp t\bar{\delta}$ and t small, $\| \alpha_t - \iota \| < 2$; so that α_t is implemented by some unitary operator U_t in $\mathfrak{A}^-$ with spectrum $\sigma(U_t)$ in $\{z : \mathscr{R}e\, z \geqslant \frac{1}{2}(4 - \| \alpha_t - \iota \|^2)^{1/2}\}$ (from [7], Lemma 5). Then $U_t = \exp iB_t$, with B_t a self-adjoint operator in $\mathfrak{A}^-$ such that

$$\| B_t \| \leqslant \arctan \| \alpha_t - \iota \| (4 - \| \alpha_t - \iota \|^2)^{-1/2}.$$

Now $\bar{\delta} = \text{ad}\, t^{-1} iB_t$, since $t\bar{\delta}$ and ad iB_t are both equal to $\log \alpha_t$, where "log" is the principal value of the logarithm on the plane slit along the negative axis. As

$$\| t^{-1} B_t \| \leqslant t^{-1} \arctan \frac{\| \alpha_t - \iota \|}{(4 - \| \alpha_t - \iota \|^2)^{1/2}} \leqslant \frac{t^{-1} \| \alpha_t - \iota \|}{(4 - \| \alpha_t - \iota \|^2)^{1/2}}$$
$$\leqslant \frac{\| \bar{\delta} \| + t \exp \| \bar{\delta} \|}{(4 - \| \alpha_t - \iota \|^2)^{1/2}},$$

it follows that, given $\epsilon > 0$, $\| t^{-1}B_t \| < \frac{1}{2} \| \delta \| + \epsilon$ for t small and positive.

With $\epsilon = 1/n$, it follows that the set

$$\mathscr{S}_n = \left\{ B : B = B^* \in \mathfrak{A}^-, \delta = \text{ad}\, iB \mid \mathfrak{A}^-, \| B \| \leqslant \tfrac{1}{2} \| \delta \| + \frac{1}{n} \right\}$$

is not empty ($n = 1, 2, 3,\ldots$). Since $\mathscr{S}_{n+1} \subseteq \mathscr{S}_n$ and each $\mathscr{S}_n$ is weak-operator compact, there is an element B in the intersection of all the $\mathscr{S}_n$'s; and $B = B^* \in \mathfrak{A}^-$, $\delta = \text{ad}\, iB \mid \mathfrak{A}^-$, $\| B \| \leqslant \frac{1}{2} \| \delta \|$. (The foregoing can be proved directly, without appeal to [7], Lemma 5, by a somewhat longer argument.)

(b) Let $\mathscr{F}$ denote the class of all finite orthogonal families $\{Q_1 ,\ldots, Q_n\}$ of projections in $\mathscr{Z}$ with sum I. Then $\mathscr{F}$ is directed by refinement ($\{P_1 ,\ldots, P_m\} \leqslant \{Q_1 ,\ldots, Q_n\}$ means that each P_j is the sum of a subset of the Q's). If $\{Q_1 ,\ldots, Q_n\} \in \mathscr{F}$, the result of (a), applied to the derivation $\delta \mid \mathfrak{A}^-Q_k$ of $\mathfrak{A}^-Q_k$, shows that there is a self-adjoint operator B_kQ_k in $\mathfrak{A}^-Q_k$ such that $\delta \mid \mathfrak{A}^-Q_k = \text{ad}\, iB_kQ_k \mid \mathfrak{A}^-Q_k$ and $\| B_kQ_k \| \leqslant \frac{1}{2} \| \delta \mid \mathfrak{A}^-Q_k \|$. With $B_{\{Q_1,\ldots,Q_n\}}$ the self-adjoint operator $\sum_{k=1}^{n} B_kQ_k$ in $\mathfrak{A}^-$, we have $\delta = \text{ad}\, iB_{\{Q_1,\ldots,Q_n\}} \mid \mathfrak{A}^-$ and

$$\| Q_kB_{\{Q_1,\ldots,Q_n\}} \| \leqslant \tfrac{1}{2} \| \delta \mid \mathfrak{A}^-Q_k \| \qquad (k = 1,\ldots, n).$$

The net $(B_{\{Q_1,\ldots,Q_n\}} : \{Q_1 ,\ldots, Q_n\} \in \mathscr{F})$ is contained in the weak-operator-compact ball $\frac{1}{2} \| \delta \| \mathfrak{A}_1^-$, and so there is a cofinal subnet which is weak-operator convergent to an element H of that ball. Clearly $H = H^* \in \mathfrak{A}^-$ and $\delta = \text{ad}\, iH \mid \mathfrak{A}^-$. If Q is a projection in $\mathscr{Z}$ then $\{Q, I - Q\} \in \mathscr{F}$ and $\| QB_{\{Q_1,\ldots,Q_n\}} \| \leqslant \frac{1}{2} \| \delta \mid \mathfrak{A}^-Q \|$ whenever $\{Q_1 ,\ldots, Q_n\} \geqslant \{Q, I - Q\}$. For by renumbering we may suppose that $Q = Q_1 + \cdots + Q_k$ for some $k \leqslant n$, and

$$\begin{aligned} \| QB_{\{Q_1,\ldots,Q_n\}} \| &= \max_{1 \leqslant r \leqslant k} \| Q_rB_{\{Q_1,\ldots,Q_n\}} \| \\ &\leqslant \tfrac{1}{2} \max_{1 \leqslant r \leqslant k} \| \delta \mid \mathfrak{A}^-Q_r \| \leqslant \tfrac{1}{2} \| \delta \mid \mathfrak{A}^-Q \| . \end{aligned}$$

Since the mapping $B \to \| QB \|$ is upper semicontinuous in the weak-operator topology, $\| QH \| \leqslant \frac{1}{2} \| \delta \mid \mathfrak{A}^-Q \|$ for each projection Q in $\mathscr{Z}$.

(c) If C is any self-adjoint element of $\mathfrak{A}^-$ such that $\delta = \text{ad}\, iC \mid \mathfrak{A}^-$ and Q is a projection in $\mathscr{Z}$, then

$$\| \delta(AQ) \| = \| CQ \cdot AQ - AQ \cdot CQ \| \leqslant 2 \| CQ \| \| AQ \|$$

for A in $\mathfrak{A}^-$; so $\frac{1}{2} \| \delta \mid \mathfrak{A}^-Q \| \leqslant \| CQ \|$. If $\| CQ \| = \frac{1}{2} \| \delta \mid \mathfrak{A}^-Q \|$ then $M_Q(C) = - m_Q(C) = \frac{1}{2} \| \delta \mid \mathfrak{A}^-Q \|$; for otherwise, with

$a = \frac{1}{2}[M_Q(C) + m_Q(C)]$ and $C_0 = C - aQ$, $\bar{\delta} = \operatorname{ad} iC_0 \mid \mathfrak{A}^-$ and $\| C_0Q \| < \| CQ \| = \frac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^-Q \|$, a contradiction.

By applying these results when C is the operator H constructed in (b), it follows that $\frac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^-Q \| = \| HQ \| = M_Q(H) = - m_Q(H)$ for every projection Q in $\mathscr{Z}$. This proves the existence of an operator H with the required properties.

(d) Suppose that $H_j = H_j^* \in \mathfrak{A}^-$, $\bar{\delta} = \operatorname{ad} iH_j \mid \mathfrak{A}^-$ and $\frac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^-Q \| = \| H_jQ \|$ for every projection Q in $\mathscr{Z}$ ($j = 1, 2$). Then $H_1 - H_2 \in \mathscr{Z}$, and the results of (c) show that

$$M_Q(H_1) = M_Q(H_2) = \tfrac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^-Q \| .$$

If $H_1 - H_2 \neq 0$ there is a nonzero projection P in $\mathscr{Z}$ and a positive scalar a such that either $(H_1 - H_2) P \geqslant aP$ or $(H_2 - H_1) P \geqslant aP$: we may suppose the former. Then

$$\tfrac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^-P \| = M_P(H_1) \geqslant M_P(H_2 + aP) > M_P(H_2) = \tfrac{1}{2} \| \bar{\delta} \mid \mathfrak{A}^-P \| ,$$

a contradiction. Hence $H_1 = H_2$, and the uniqueness of H is proved.

Let φ be a faithful representation of a C^*-algebra $\mathfrak{A}$. Following [7], we denote by $\iota_\varphi(\mathfrak{A})$ the group of all automorphisms α of $\mathfrak{A}$ with the property that the automorphism $\varphi\alpha\varphi^{-1}$ of $\varphi(\mathfrak{A})$ is implemented by a unitary operator in $\varphi(\mathfrak{A})^-$; and we describe such an α as "weakly-inner in the representation φ". With $\pi(\mathfrak{A})$ the intersection of all the groups $\iota_\varphi(\mathfrak{A})$, the elements of $\pi(\mathfrak{A})$ are described as *permanently weakly-inner*, or *π-inner*. The following theorem shows that π-inner automorphisms carry each closed ideal J of $\mathfrak{A}$ onto itself, and induce π-inner automorphisms on each homomorphic image of $\mathfrak{A}$. This, together with the fact that derivations of $\mathfrak{A}$ are weakly-inner in every faithful representation and leave each closed ideal invariant ([2] 1.9. 11d, p. 20), shows that many of the results of [7] can be extended, in an obvious sense, so as to apply to representations of $\mathfrak{A}$ other than faithful ones.

THEOREM 3.2. *Let $\mathfrak{A}$ be a C^*-algebra, Ψ its universal representation, J a closed ideal in $\mathfrak{A}$. Then*

(i) $\pi(\mathfrak{A}) = \iota_\Psi(\mathfrak{A})$,

(ii) *if $\alpha \in \pi(\mathfrak{A})$ then $\alpha(J) = J$, and the mapping*

$$\alpha_J : A + J \to \alpha(A) + J$$

is a π-inner automorphism of $\mathfrak{A}/J$.

Proof. We may suppose that $\mathfrak{A}$ acting on $\mathscr{H}$ is the universal representation Ψ of $\mathfrak{A}$. Let φ be any other representation of $\mathfrak{A}$, and denote by $\bar{\varphi}$ the extension of φ to a *-homomorphism from $\mathfrak{A}^-$ onto $\varphi(\mathfrak{A})^-$ ([4], Lemmas 2.3, 2.4). If $\alpha \in \iota_\Psi(\mathfrak{A})$, there is a unitary operator U in $\mathfrak{A}^-$ such that $\alpha(A) = UAU^*$ for each A in $\mathfrak{A}$. Thus

$$\varphi(\alpha(A)) = \bar{\varphi}(\alpha(A)) = \bar{\varphi}(UAU^*) = U_\varphi \varphi(A) U_\varphi^*, \tag{3.1}$$

for each A in $\mathfrak{A}$, where U_φ is the unitary operator $\bar{\varphi}(U)$ in $\varphi(\mathfrak{A})^-$.

If φ is a faithful representation then (3.1) asserts that $(\varphi\alpha\varphi^{-1})(B) = U_\varphi B U_\varphi^*$ for each B in $\varphi(\mathfrak{A})$. Hence $\alpha \in \iota_\varphi(\mathfrak{A})$ for each faithful representation φ; that is $\alpha \in \pi(\mathfrak{A})$. This shows that $\iota_\Psi(\mathfrak{A}) \subseteq \pi(\mathfrak{A})$; and since the reverse inclusion is obvious, $\iota_\Psi(\mathfrak{A}) = \pi(\mathfrak{A})$.

Next, let φ be a representation for which $\varphi^{-1}(0)$ is J. By (3.1), $\varphi(\alpha(A)) = 0$ if and only if $\varphi(A) = 0$; that is, $\alpha(A) \in J$ if and only if $A \in J$. Hence $\alpha(J) = J$, and the mapping α_J is an automorphism of $\mathfrak{A}/J$. Each faithful representation θ of $\mathfrak{A}/J$ has the form $A + J \to \varphi(A)$, where φ is a representation of $\mathfrak{A}$ such that $\varphi^{-1}(0) = J$. Then $\theta\alpha_J\theta^{-1}$ is the automorphism $\varphi(A) \to \varphi(\alpha(A)) = U_\varphi\varphi(A) U_\varphi^*$ of $\theta(\mathfrak{A}/J)$ $(= \varphi(\mathfrak{A}))$. Since $U_\varphi \in \theta(\mathfrak{A}/J)^-$, it follows that $\alpha_J \in \iota_\theta(\mathfrak{A}/J)$ for each such θ; hence $\alpha_J \in \pi(\mathfrak{A}/J)$.

Remark 3.3. Let $\mathfrak{A}$ be the factor of type II_1 generated by the left regular representation of the free group on two generators, α the outer automorphism of $\mathfrak{A}$ corresponding to the automorphism of the group which exchanges these generators ([1], Exercise 15, p. 308). The only ideals in $\mathfrak{A}$ are (0) and $\mathfrak{A}$; so α carries each ideal onto itself, but is not π-inner.

It is shown in [9] that, if $\mathfrak{A}$ is a separable GCR C^*-algebra, and α is an automorphism of $\mathfrak{A}$ which maps each closed ideal into itself, then α is π-inner.

4. Derivations which Generate Inner Automorphisms

Suppose that $\mathfrak{A}$ is a C^*-algebra, H is a self-adjoint operator in $\mathfrak{A}$ and δ is the inner derivation ad iH of $\mathfrak{A}$. Then, for each real t, exp $t\delta$ is the inner automorphism implemented by the unitary operator exp itH. Our purpose in this section is to study the following question, together with some of its refinements: if δ is a *-derivation of a C^*-algebra and the automorphism exp $t\delta$ is inner for all real t, does it follow that δ is an inner derivation?

Lemma 4.1. *If the automorphism α of the C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$ is implemented by a unitary operator V in $\mathfrak{A}$ with* *spectrum in* $\mathbf{C}^+$, *the open right half-plane, then there is an inner derivation* δ *of* $\mathfrak{A}$ *with spectrum in* $\{it : -\pi < t < \pi\}$ *such that* $\exp \delta = \alpha$. *If* δ_0 *is another derivation of* $\mathfrak{A}$ *such that* $\alpha = \exp \delta_0$ *and* δ_0 *has spectrum in the strip* $S = \{z : |\operatorname{Im} z| < \pi\}$, *then* $\delta_0 = \delta$.

Proof. With "log" defined as the principal value of logarithm on the plane slit along the negative axis, $\log V = iH$ for some self-adjoint H in $\mathfrak{A}$ whose spectrum lies in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Moreover, $\exp \operatorname{ad} iH = \alpha$; and $\operatorname{ad} iH\ (= \delta)$ has spectrum in $\{it : -\pi < t < \pi\}$ (cf. [7] proofs of Lemma 6, p. 46 and Lemma 2, p. 40). If δ_0 is another derivation of $\mathfrak{A}$ such that $\alpha = \exp \delta_0$ and $\sigma(\delta_0)$ lies in S, then, since $z = \log \exp z$ on S and both δ and δ_0 have their spectra in S,

$$\delta = \operatorname{ad} iH = \log(\exp \operatorname{ad} iH) = \log \alpha = \log(\exp \delta_0) = \delta_0 .$$

COROLLARY 4.2. *If* δ *is a *-derivation of the* C^**-algebra* $\mathfrak{A}$ *such that, for some* $t \neq 0$ *with* $\| t\delta \| < \pi$, $\alpha_t = \exp t\delta$ *is implemented by a unitary operator* V *in* $\mathfrak{A}$ *with* $\sigma(V)$ *in* $\mathbf{C}^+$, *then* δ *is inner.*

Proof. From Lemma 4.1, $\alpha_t = \exp \delta_0$ for some inner δ_0. Since $\| t\delta \| < \pi$, $\sigma(t\delta)$ lies in S; so that $t\delta = \delta_0$, from the uniqueness assertion in Lemma 4.1. Thus $\delta = t^{-1}\delta_0$, and δ is inner.

THEOREM 4.3. *If* δ *is a *-derivation of the separable* C^**-algebra* $\mathfrak{A}$, *and* $\alpha_t = \exp t\delta$ *is inner for an uncountable set of real* t, *then* δ *is inner.*

Proof. With α_t inner, let U_t in $\mathfrak{A}$ be a unitary operator which implements α_t; and let S_t be the open ball of radius $\frac{1}{2}\sqrt{2}$ and center U_t. Since there are an uncountable number of S_t and $\mathfrak{A}$ is (norm-) separable, there are distinct s and t as close as we please such that S_s and S_t have nonvoid intersection. With $0 \leqslant s < t$ and $r = t - s$, we may assume that $r \| \delta \| < \pi$. Letting

$$V = U_t U_s^*, \qquad \| V - I \| = \| U_t - U_s \| < \sqrt{2};$$

so that $\sigma(V)$ lies in $\mathbf{C}^+$. In addition, V implements $\alpha_r = \exp r\delta$. From Corollary 4.2, δ is inner.

Remark. 4.4. In Example 6.2 we exhibit a C^*-algebra $\mathfrak{A}$ acting on a separable Hilbert space, and an outer derivation δ of $\mathfrak{A}$ such that $\exp t\delta$ is an inner automorphism of $\mathfrak{A}$ for all real t. This shows that some condition such as the restriction to *separable* C^*-algebras must be included in Theorem 4.3. In Example 6.3 we construct a separable C^*-algebra $\mathfrak{A}$ and an outer derivation δ of $\mathfrak{A}$ such that $\exp t\delta$ is an inner automorphism of $\mathfrak{A}$ for all rational t.

Remark. 4.5. The first few lines of the proof of Lemma 4.1 establish that an automorphism of a C^*-algebra implemented by a unitary operator in the algebra whose spectrum is not the entire unit circle is the exponential of an inner derivation of the algebra.

THEOREM 4.6. *Let $\mathfrak{A}$ be a C^*-algebra which has a faithful representation φ in which $\varphi(\mathfrak{A})$ contains the center of $\varphi(\mathfrak{A})^-$. Suppose that δ is a *-derivation of $\mathfrak{A}$, $\| \delta \| < 2\pi$, and* exp δ *is an inner automorphism of $\mathfrak{A}$. Then δ is an inner derivation.*

Proof. We may suppose that $\mathfrak{A}$, acting on a Hilbert space $\mathscr{H}$, is the given representation φ. By Theorem 3.1 there exists H in $\mathfrak{A}^-$ such that

$$H = H^*, \qquad \| H \| = \tfrac{1}{2} \| \delta \| < \pi, \qquad \delta = \operatorname{ad} iH \mid \mathfrak{A}.$$

The automorphism exp δ is implemented both by exp iH and by some U in $\mathscr{U}(\mathfrak{A})$. Thus

$$U^* \exp iH \in \mathscr{U}(\mathfrak{A}^- \cap \mathfrak{A}') \subseteq \mathscr{U}(\mathfrak{A}),$$

and so exp $iH \in \mathfrak{A}$. Since $\| H \| < \pi$, the principal value log z is well-defined and continuous on $\sigma(\exp iH) = \{\exp it : t \in \sigma(H)\}$, and $\log (\exp it) = it$ for each t in $\sigma(H)$. By means of the (continuous) functional calculus for normal operators we have $iH = \log (\exp iH)$, and since exp $iH \in \mathfrak{A}$ it follows that $H \in \mathfrak{A}$. Thus δ ($= \operatorname{ad} iH \mid \mathfrak{A}$) is an inner derivation of $\mathfrak{A}$.

Remark. 4.7. The class of C^*-algebras considered in Theorem 4.6 includes primitive C^*-algebras and, more generally, any C^*-algebra which has a faithful factor representation. We shall see in Example 6.1. that Theorem 4.6 fails for *-derivations δ such that $\| \delta \| = 2\pi$.

5. CONDITIONS UNDER WHICH $\iota_0(\mathfrak{A})$ AND $\Delta_0(\mathfrak{A})$ ARE NORM-CLOSED

If $\mathfrak{A}$ is a C^*-algebra then various natural subgroups of $\alpha(\mathfrak{A})$—in particular $\pi(\mathfrak{A})$ nad the connected component $\gamma(\mathfrak{A})$ of ι—are open, hence closed, in $\alpha(\mathfrak{A})$ ([7], Theorem 7 and Remark G). For the class of C^*-algebras studied in [7], Example d, $\gamma(\mathfrak{A}) \subseteq \iota_0(\mathfrak{A})$ and so $\iota_0(\mathfrak{A})$ is open. In Example 6.1. we exhibit a C^*-algebra $\mathfrak{A}$ for which $\iota_0(\mathfrak{A})$ is closed, but not open, in $\alpha(\mathfrak{A})$; and in Example 6.3 we produce a C^*-algebra for which $\iota_0(\mathfrak{A})$ is not closed in $\alpha(\mathfrak{A})$. In the present section we investigate the conditions under which $\iota_0(\mathfrak{A})$ is closed in $\alpha(\mathfrak{A})$, and relate this problem to the corresponding question for derivations.

With A in $\mathfrak{A}$ and $\mathscr{B}$ a subset of $\mathfrak{A}$ we write $d(A, \mathscr{B})$ for the distance from A to $\mathscr{B}$.

LEMMA 5.1. *Suppose that $\mathfrak{A}$ is a C^*-algebra with center $\mathscr{C}$, and $U \in \mathscr{U}(\mathfrak{A})$. Then*

$$d(U, \mathscr{C}) \leqslant d(U, \mathscr{U}(\mathscr{C})) \leqslant 2d(U, \mathscr{C}).$$

Proof. The first inequality is obvious since $\mathscr{U}(\mathscr{C}) \subseteq \mathscr{C}$; in proving the second we shall assume that $d(U, \mathscr{C}) < 1$, since each U in $\mathscr{U}(\mathfrak{A})$ satisfies $d(U, \mathscr{U}(\mathscr{C})) \leqslant \| U - I \| \leqslant 2$. With Z in $\mathscr{C}$ such that $\| U - Z \| (= \| I - U^{-1}Z \|) < 1$, $U^{-1}Z$ and hence Z are invertible. Letting V be $Z \mid Z \mid^{-1}$ and passing to the function algebra representing the commutative C^*-algebra generated by U and Z, we see that $d(U, \mathscr{U}(\mathscr{C})) \leqslant \| U - V \| \leqslant 2 \| U - Z \|$; since, for complex a and b with $|a| = 1$ and $b \neq 0$,

$$| a - b | b |^{-1} | \leqslant | a - b | + | b - b | b |^{-1} | \leqslant 2 | a - b | .$$

Thus

$$d(U, \mathscr{U}(\mathscr{C})) \leqslant 2 \inf\{\| U - Z \| : Z \in \mathscr{C}\} = 2d(U, \mathscr{C}).$$

LEMMA 5.2. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\mathscr{H}$, $\mathscr{C} = \mathfrak{A} \cap \mathfrak{A}'$, $\mathscr{Z} = \mathfrak{A}^- \cap \mathfrak{A}'$. Then the following conditions are equivalent.*

(i) *There is a positive constant k such that $d(A, \mathscr{C}) \leqslant k\, d(A, \mathscr{Z})$ for each A in $\mathfrak{A}$.*

(ii) *There is a positive constant m such that*

$$d(U, \mathscr{U}(\mathscr{C})) \leqslant m\, d(U, \mathscr{U}(\mathscr{Z}))$$

for each U in $\mathscr{U}(\mathfrak{A})$.

Proof. Suppose that (i) is satisfied, and let U be in $\mathscr{U}(\mathfrak{A})$. Since $\mathscr{U}(\mathscr{Z}) \subseteq \mathscr{Z}$ it follows from Lemma 5.1 that

$$d(U, \mathscr{U}(\mathscr{C})) \leqslant 2d(U, \mathscr{C}) \leqslant 2kd(U, \mathscr{Z}) \leqslant 2kd(U, \mathscr{U}(\mathscr{Z}));$$

so (ii) is satisfied, with $m = 2k$.

Suppose, conversely, that (ii) is satisfied. If $T = T^* \in \mathfrak{A}$ then, for each positive t, $U_t = \exp(itT)$ is a unitary operator in $\mathfrak{A}$. By Lemma 5.1, with $\mathfrak{A}^-$ in place of $\mathfrak{A}$,

$$\begin{aligned} d\left(\frac{U_t - I}{it}, \mathscr{C}\right) &= t^{-1} d(U_t, \mathscr{C}) \\ &\leqslant t^{-1} d(U_t, \mathscr{U}(\mathscr{C})) \\ &\leqslant m t^{-1} d(U_t, \mathscr{U}(\mathscr{Z})) \\ &\leqslant 2m t^{-1} d(U_t, \mathscr{Z})) \\ &= 2md\left(\frac{U_t - I}{it}, \mathscr{Z}\right). \end{aligned}$$

By taking limits as $t \to 0+$ we obtain $d(T, \mathscr{C}) \leqslant 2md(T, \mathscr{Z})$, for each self-adjoint T in $\mathfrak{A}$. For general T in $\mathfrak{A}$, $d(T^*, \mathscr{Z}) = d(T, \mathscr{Z})$, thus

$$d(\tfrac{1}{2}(T + T^*), \mathscr{Z}) \leqslant d(T, \mathscr{Z})$$

and

$$d(\tfrac{1}{2}(T + T^*), \mathscr{C}) \leqslant 2md(\tfrac{1}{2}(T + T^*), \mathscr{Z}) \leqslant 2md(T, \mathscr{Z}).$$

With iT in place of T, this gives $d(\frac{1}{2}(T - T^*), \mathscr{C}) \leqslant 2md(T, \mathscr{Z})$, so

$$d(T, \mathscr{C}) \leqslant d(\tfrac{1}{2}(T + T^*), \mathscr{C}) + d(\tfrac{1}{2}(T - T^*), \mathscr{C}) \leqslant 4md(T, \mathscr{Z}).$$

Thus (i) is satisfied, with $k = 4m$.

THEOREM. 5.3. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\mathscr{H}$, $\mathscr{C} = \mathfrak{A} \cap \mathfrak{A}'$, $\mathscr{Z} = \mathfrak{A}^- \cap \mathfrak{A}'$. Then the following three conditions are equivalent.*

(i) *$\Delta_0(\mathfrak{A})$ is a norm-closed subspace of $\Delta(\mathfrak{A})$.*

(ii) *There is a positive constant k such that*

$$d(A, \mathscr{C}) \leqslant kd(A, \mathscr{Z}) \quad \textit{for each} \quad A \quad \textit{in} \quad \mathfrak{A}.$$

(iii) *There is a positive constant m such that*

$$d(U, \mathscr{U}(\mathscr{C})) \leqslant md(U, \mathscr{U}(\mathscr{Z})) \quad \textit{for each} \quad U \quad \textit{in} \quad \mathscr{U}(\mathfrak{A}).$$

Each of these conditions implies

(iv) *$\iota_0(\mathfrak{A})$ is a norm-closed subgroup of $\alpha(\mathfrak{A})$.*

If $\mathfrak{A}$ is a separable C^*-algebra then conditions (i),..., (iv) *are all equivalent.*

Proof. Let $\mathscr{D} = \{D \in \mathfrak{A}^- : DA - AD \in \mathfrak{A}$ whenever $A \in \mathfrak{A}\}$, and for each D in $\mathscr{D}$ define δ_D in $\Delta(\mathfrak{A})$ by $\delta_D = \operatorname{ad} D \mid \mathfrak{A}$. Then $\mathscr{D}$ is a norm-closed linear subspace of $\mathfrak{A}^-$, $\mathfrak{A} + \mathscr{L} \subseteq \mathscr{D}$ and, by [*11*], Theorem 2, the linear mapping $D \to \delta_D$ carries $\mathscr{D}$ *onto* $\Delta(\mathfrak{A})$. This mapping has kernel $\mathscr{L}$ and so induces a one-to-one linear mapping $\varphi : D + \mathscr{L} \to \delta_D$ of the quotient space $\mathscr{D}/\mathscr{L}$ onto $\Delta(\mathfrak{A})$. We have $\| \delta_D \| \leqslant 2 \| D \|$, hence $\| \delta_D \| \leqslant 2d(D, \mathscr{L})$, for each D in $\mathscr{D}$; so φ is continuous. By the closed-graph theorem, φ^{-1} is also continuous. Hence there is a constant K such that

$$\tfrac{1}{2} \| \delta_D \| \leqslant d(D, \mathscr{L}) \leqslant K \| \delta_D \| \qquad (D \in \mathscr{D}). \tag{5.1}$$

It follows easily from Theorem 3.1 that $\| \delta_D \| = 2d(D, \mathscr{L})$ when $D = D^* \in \mathscr{D}$, and hence that (5.1) is satisfied, with $K = 1$, for all D in $\mathscr{D}$. However, the particular value of K has no bearing on the proof of the present theorem.

The linear mapping $A \to \delta_A$ from $\mathfrak{A}$ into $\Delta(\mathfrak{A})$ has range $\Delta_0(\mathfrak{A})$ and kernel $\mathscr{C}$, and so induces a one-to-one linear mapping $\psi : A + \mathscr{C} \to \delta_A$ from the quotient space $\mathfrak{A}/\mathscr{C}$ into $\Delta(\mathfrak{A})$, with range $\Delta_0(\mathfrak{A})$. Since $\mathscr{C} \subseteq \mathscr{L}$, it follows from (5.1) that $\| \delta_A \| \leqslant 2d(A, \mathscr{C})$, so ψ is continuous. If $\Delta_0(\mathfrak{A})$ is a norm-closed subspace of $\Delta(\mathfrak{A})$ (and hence complete) the closed-graph theorem asserts that ψ^{-1} is continuous, so there is a constant M such that $d(A, \mathscr{C}) \leqslant M \| \delta_A \|$ for each A in $\mathfrak{A}$. By (5.1), $d(A, \mathscr{C}) \leqslant 2M\, d(A, \mathscr{L})$, so condition (ii) is satisfied with $k = 2M$.

Suppose conversely that $d(A, \mathscr{C}) \leqslant k\, d(A, \mathscr{L})$ for each A in $\mathfrak{A}$. By (5.1),

$$\begin{aligned}\tfrac{1}{2} \| \psi(A + \mathscr{C}) \| = \tfrac{1}{2} \| \delta_A \| \leqslant d(A, \mathscr{L}) \leqslant d(A, \mathscr{C}) \leqslant kd(A, \mathscr{L}) \leqslant kK \| \delta_A \| \\ = kK \| \psi(A + \mathscr{C}) \|\end{aligned}$$

for each A in $\mathfrak{A}$. It follows that the norm inherited by $\Delta_0(\mathfrak{A})$ from $\Delta(\mathfrak{A})$ is equivalent to the (complete) norm it acquires, through ψ, from the Banach space, $\mathfrak{A}/\mathscr{C}$. Thus $\Delta_0(\mathfrak{A})$ is complete in both these norms, and is therefore a norm-closed subspace of $\Delta(\mathfrak{A})$.

We have now proved the equivalence of (i) and (ii), and Lemma 5.2 asserts that (ii) is equivalent to (iii). We now assume that (iii) is satisfied, and show that this implies (iv). Suppose that $\alpha \in \alpha(\mathfrak{A})$, $\alpha_n \in \iota_0(\mathfrak{A})$ $(n = 1, 2,...)$ and $\lim_{n\to\infty} \| \alpha - \alpha_n \| = 0$. By passing to a subsequence we may suppose that, if

$$\epsilon_n = \| \alpha_{n+1} - \alpha_n \|, \qquad r_n = [2 - (4 - \epsilon_n^{\,2})^{1/2}]^{1/2},$$

then

$$\epsilon_n < 2 \qquad (n = 1, 2,...), \qquad \sum_{n=1}^{\infty} r_n < \infty. \tag{5.2}$$

Let U_n be an element of $\mathscr{U}(\mathfrak{A})$ which implements the inner automorphism $\alpha_{n+1}\alpha_n^{-1}$. Since

$$\| \alpha_{n+1}\alpha_n^{-1} - \iota \| = \| \alpha_{n+1} - \alpha_n \| = \epsilon_n < 2,$$

it follows, from [7], Remark E, p. 48, that $\alpha_{n+1}\alpha_n^{-1}$ is also implemented by an element V_n of $\mathscr{U}(\mathfrak{A}^-)$ with spectrum $\sigma(V_n)$ contained in the arc of the unit circle which lies in the half-plane $\{z : \mathscr{R}e\, z \geqslant \frac{1}{2}(4 - \epsilon_n{}^2)^{1/2}\}$. Since this arc is contained in the disc $\{z : | z - 1 | \leqslant r_n\}$, while the norm of $V_n - I$ is its spectral radius, it follows that $\| V_n - I \| \leqslant r_n$. Since U_n, $V_n \in \mathscr{U}(\mathfrak{A}^-)$ and both implement $\alpha_{n+1}\alpha_n^{-1}$, $U_nV_n^* \in \mathscr{U}(\mathscr{Z})$. Thus

$$d(U_n, \mathscr{U}(\mathscr{C})) \leqslant md(U_n, \mathscr{U}(\mathscr{Z})) \leqslant m \| U_n - U_nV_n^* \|$$

$$= m \| I - V_n^* \| = m \| I - V_n \| \leqslant mr_n .$$

Hence there exists Z_n in $\mathscr{U}(\mathscr{C})$ such that $\| U_n - Z_n \| < 2mr_n$. With $W_n = U_nZ_n^*$ we have $W_n \in \mathscr{U}(\mathfrak{A})$, $\| W_n - I \| < 2mr_n$ and W_n implements $\alpha_{n+1}\alpha_n^{-1}$. Thus

$$X_n = W_nW_{n-1} \cdots W_1 \in \mathscr{U}(\mathfrak{A}), \qquad \| X_n - X_{n-1} \| = \| W_n - I \| < 2mr_n ,$$

and

$$X_nAX_n^* = \alpha_{n+1}\alpha_1^{-1}(A) \qquad (A \in \mathfrak{A}). \tag{5.3}$$

Since $\sum \| X_n - X_{n+1} \| \leqslant \sum 2mr_n < \infty$, (X_n) converges in norm to some element X of $\mathscr{U}(\mathfrak{A})$. By (5.3),

$$\alpha(A) = \lim_{n\to\infty} \alpha_{n+1}\alpha_1^{-1}(\alpha_1(A))$$

$$= \lim_{n\to\infty} X_n\alpha_1(A)\, X_n^*$$

$$= X\alpha_1(A)\, X^* \qquad (A \in \mathfrak{A});$$

and since α_1 is an inner automorphism of $\mathfrak{A}$, so is α. This completes the proof that $\iota_0(\mathfrak{A})$ is closed when $\mathfrak{A}$ satisfies (iii).

Finally, we suppose that $\mathfrak{A}$ is a separable C^*-algebra that satisfies (iv), and deduce that $\Delta_0(\mathfrak{A})$ is a norm-closed subspace of $\Delta(\mathfrak{A})$. Suppose $\delta \in \Delta(\mathfrak{A})$, $\delta_n \in \Delta_0(\mathfrak{A})$ $(n = 1, 2,\ldots)$ and $\lim_{n\to\infty} \| \delta - \delta_n \| = 0$. We shall prove that $\delta \in \Delta_0(\mathfrak{A})$.

Since

$$\delta = \frac{1}{2}(\delta + \delta^*) + i\frac{1}{2i}(\delta - \delta^*)$$
$$= \lim_{n\to\infty} \frac{1}{2}(\delta_n + \delta_n^*) + i \lim_{n\to\infty} \frac{1}{2i}(\delta_n - \delta_n^*),$$

while $\frac{1}{2}(\delta_n + \delta_n^*)$, $(1/2i)(\delta_n - \delta_n^*) \in \Delta_0(\mathfrak{A})$, it is sufficient to consider the case in which each δ_n is a *-derivation in $\Delta_0(\mathfrak{A})$. In this case, $\exp t\delta_n \in \iota_0(\mathfrak{A})$ for all real t; and since $\iota_0(\mathfrak{A})$ is norm-closed,

$$\exp t\delta = \lim_{n\to\infty} \exp t\delta_n \in \iota_0(\mathfrak{A}).$$

By Theorem 4.3, $\delta \in \Delta_0(\mathfrak{A})$.

Remark. 5.4. While conditions (ii) and (iii) in Theorem 5.3 involve spatial concepts, condition (i) is concerned with just the algebraic structure of $\mathfrak{A}$. It follows that (ii) and (iii) are in fact independent of the faithful representation in which $\mathfrak{A}$ is considered.

In Example 6.2 we construct a C^*-algebra $\mathfrak{A}$, acting on a separable Hilbert space, with the property that $\iota_0(\mathfrak{A})$ is closed while $\Delta_0(\mathfrak{A})$ is not. This shows that some condition such as the restriction to *separable* C^*-algebras must be included in the final sentence of Theorem 5.3. The proof given is not applicable when $\mathfrak{A}$ is not separable since Theorem 4.3 fails for general C^*-algebras.

COROLLARY 5.5. *Suppose that $\mathfrak{A}$ is a C^*-algebra with a faithful representation φ in which $\varphi(\mathfrak{A})$ contains the center of $\varphi(\mathfrak{A})^-$. Then $\Delta_0(\mathfrak{A})$ is a norm-closed subspace of $\Delta(\mathfrak{A})$ and $\iota_0(\mathfrak{A})$ is a norm-closed subgroup of $\alpha(\mathfrak{A})$.*

Proof. We may apply Theorem 5.3, with $\varphi(\mathfrak{A})$ in place of $\mathfrak{A}$; since $\mathscr{C} = \mathscr{Z}$, condition (ii) of that theorem is satisfied, with $k = 1$.

Remark. 5.6. Corollary 5.5 shows that $\iota_0(\mathfrak{A})$ and $\Delta_0(\mathfrak{A})$ are closed if $\mathfrak{A}$ is a primitive C^*-algebra or, more generally, a C^*-algebra with a faithful factor representation.

6. EXAMPLES

In this section we exploit the properties of the ideal $\mathscr{C}(\mathscr{H})$ of all compact linear operators on a separable Hilbert space $\mathscr{H}$ to construct

C^*-algebras which exhibit various automorphism and derivation phenomena related to, but not included in, the results of Sections 4 and 5. We shall need the fact that, if $C \in \mathscr{C}(\mathscr{H})$, a and b are scalars and E, F are projections with infinite-dimensional ranges such that $E + F = I$, then $\| aE + bF + C \| \geqslant \| aE + bF \|$. For this, if (x_n) is an orthonormal sequence in the range of E then $Ex_n = x_n$, $Fx_n = 0$, $\lim \| Cx_n \| = 0$ and thus

$$| a | = \lim \| (aE + bF + C) x_n \| \leqslant \| aE + bF + C \| .$$

This, and the analogous inequality for b, give

$$\| aE + bF + C \| \geqslant \max (| a | , | b |) = \| aE + bF \| .$$

6.1. Let $\mathscr{H}$ be a separable Hilbert space, $\mathfrak{A}$ the C^*-algebra $\{aI + C : a \text{ scalar}, \ C \in \mathscr{C}(\mathscr{H})\}$. Each T in $\mathscr{B}(\mathscr{H})$ gives rise to a derivation $\delta = \operatorname{ad} T \mid \mathfrak{A}$ of $\mathfrak{A}$; since $\mathfrak{A}' = \{aI\} \subseteq \mathfrak{A}$, δ is inner if and only if $T \in \mathfrak{A}$. Similarly each unitary operator U on $\mathscr{H}$ implements an automorphism α of $\mathfrak{A}$, and α is inner if and only if $U \in \mathfrak{A}$. Let E and F be projections with infinite-dimensional ranges and with sum I, and define $\delta = \operatorname{ad} i\pi(E - F) \mid \mathfrak{A}$. Then δ is an outer derivation of $\mathfrak{A}$ and $\| \delta \| \leqslant 2\pi \| E - F \| = 2\pi$. If V is a partial isometry with V^*V and VV^* 1-dimensional subprojections of E and F, respectively, then $V \in \mathfrak{A}$ and $\delta(V) = -2\pi i V$, so $\| \delta \| = 2\pi$. Since $\exp i\pi(E - F) = -I$, $\exp \delta = \iota$ (an inner automorphism!). This shows that the results of Theorem 4.6 can fail for derivations of norm 2π, since $\mathfrak{A}$ is clearly a C^*-algebra of the type considered in that theorem.

By Corollary 5.5, $\Delta_0(\mathfrak{A})$ is a norm-closed subspace of $\Delta(\mathfrak{A})$ and $\iota_0(\mathfrak{A})$ is a norm-closed subgroup of $\alpha(\mathfrak{A})$. However, $\iota_0(\mathfrak{A})$ is not an open subgroup of $\alpha(\mathfrak{A})$: for with t real and $U_t = E + \exp(it) F$, U_t implements an automorphism α_t of $\mathfrak{A}$ and $\| \alpha_t - \iota \| \leqslant 2 \| U_t - I \| \to 0$ as $t \to 0+$, although α_t is an outer automorphism when $0 < t < \pi$.

6.2. We construct a C^*-algebra $\mathfrak{A}$, acting on a separable Hilbert space, with the property that $\iota_0(\mathfrak{A})$ is a norm-closed subgroup of $\alpha(\mathfrak{A})$ although $\Delta_0(\mathfrak{A})$ is not a norm-closed subspace of $\Delta(\mathfrak{A})$. Furthermore, $\mathfrak{A}$ has an outer $*$-derivation δ such that, for all real t, $\exp t\delta$ is an inner automorphism of $\mathfrak{A}$.

For each integer n ($\geqslant 0$) let $\mathscr{H}_n$ be a separable Hilbert space, and E_n, F_n be projections on $\mathscr{H}_n$ with infinite-dimensional ranges and sum I. With $\mathscr{H} = \oplus_n \mathscr{H}_n$, $\mathscr{A}$ the Abelian C^*-algebra consisting of all bounded operators A on $\mathscr{H}$ of the form $\oplus_n (a_n E_n + a_{n+1} F_n)$, $\mathscr{C}_0$ the algebra $\oplus_n \mathscr{C}(\mathscr{H}_n)$, and $C = \oplus_n C_n$ in $\mathscr{C}_0$,

$$\| A + C \| = \sup_n \| a_n E_n + a_{n+1} F_n + C_n \|$$

$$\geqslant \sup_n \| a_n E_n + a_{n+1} F_n \| = \| A \| .$$

Since the "angle" between the closed subspaces $\mathscr{A}$ and $\mathscr{C}_0$ of $\mathscr{B}(\mathscr{H})$ is not 0, their linear span $\mathscr{A} + \mathscr{C}_0$ is closed and, hence, a C^*-algebra $\mathfrak{A}$. Moreover, $\mathfrak{A}^- = \oplus_n \mathscr{B}(\mathscr{H}_n)$, and $\mathfrak{A}'$ (which is also the center $\mathscr{Z}$ of $\mathfrak{A}^-$) consists of all bounded operators of the form $\oplus_n c_n I$, where c_n is a scalar.

We show that the center $\mathscr{C}$ of $\mathfrak{A}$ consists of scalars. For this, suppose that $Z \in \mathscr{C} = \mathfrak{A} \cap \mathfrak{A}'$. Then $Z = A + C = \oplus_n c_n I$, with $A = \oplus_n (a_n E_n + a_{n+1} F_n)$ in $\mathscr{A}$ and $C = \oplus_n C_n$ in $\mathscr{C}_0$. Since C_n is compact and $a_n E_n + a_{n+1} F_n + C_n = c_n I$, it follows that $C_n = 0$ and $a_n = a_{n+1} = c_n$. Thus $c_n = a_n = a_0$ for all n and $Z = \oplus_n c_n I = a_0 I$.

For $k = 1, 2,\ldots$ let A_k be the unitary operator

$$\bigoplus_n \left\{ \exp \frac{i\pi n}{k} E_n + \exp \frac{i\pi(n+1)}{k} F_n \right\}$$

in $\mathfrak{A}$. Since both 1 and -1 lie in the spectrum of A_k, $d(A_k, \mathscr{C}) = 1$. With $Z_k = \oplus_n \exp(i\pi n/k)\, I$ in $\mathscr{Z}$, it follows easily that $d(A_k, \mathscr{Z}) \leqslant \| A_k - Z_k \| \to 0$ as $k \to \infty$. Hence $\mathfrak{A}$ does not satisfy (ii) of Theorem 5.3; and, from that theorem, $\Delta_0(\mathfrak{A})$ is not a norm-closed subspace of $\Delta(\mathfrak{A})$.

Next, we show that $\iota_0(\mathfrak{A})$ is a norm-closed subgroup of $\alpha(\mathfrak{A})$. The restriction of $\mathfrak{A}$ to the subspace $\mathscr{H}_n$ of $\mathscr{H}$ is the C^*-algebra $\mathfrak{A}_n$ consisting of all operators of the form $aE_n + bF_n + C$, with a, b scalars and C compact. By Corollary 5.5, $\iota_0(\mathfrak{A}_n)$ is a norm-closed subgroup of $\alpha(\mathfrak{A}_n)$. Each inner automorphism of $\mathfrak{A}$ has the form $\oplus_n \alpha_n$, with α_n in $\iota_0(\mathfrak{A}_n)$. Suppose conversely that $\alpha_n \in \iota_0(\mathfrak{A}_n)$ $(n = 0, 1, 2,\ldots)$ and let V_n be a unitary operator in $\mathfrak{A}_n$ which implements α_n. Then V_n has the form $a_n E_n + b_n F_n + C_n$ with C_n compact and a_n, b_n scalars of modulus 1. With $c_0 = 1$ and $c_n = \prod_{r=1}^{n} b_{r-1} a_r^{-1}$ when $n > 0$, we have $c_{n+1} a_{n+1} = c_n b_n$ $(n \geqslant 0)$, and $\oplus_n c_n V_n$ is a unitary operator in $\mathfrak{A}$. The corresponding inner automorphism of $\mathfrak{A}$ is $\oplus_n \alpha_n$. We have now shown that $\iota_0(\mathfrak{A})$ consists of all mappings $\oplus_n \alpha_n$, with α_n an arbitrary element of $\iota_0(\mathfrak{A}_n)$. Since each $\iota_0(\mathfrak{A}_n)$ is norm-closed, so is $\iota_0(\mathfrak{A})$.

We show, finally, that $\mathfrak{A}$ has an outer *-derivation which gives rise to a one-parameter group of inner automorphisms. The operator $H = \oplus_n F_n$ commutes with each A in $\mathscr{A}$; so that, with $C = \oplus_n C_n$ in $\mathscr{C}_0$, $H(A + C) - (A + C) H = \oplus_n (F_n C_n - C_n F_n) \in \mathscr{C}_0$. Thus $\delta = \text{ad}\, iH \mid \mathfrak{A}$ is a *-derivation of $\mathfrak{A}$. The automorphism $\exp t\delta$ is implemented by the unitary operator

$$W_t = \exp itH = \bigoplus_n (E_n + \exp (it) F_n).$$

Since $W_t = V_t Z_t$, where

$$V_t = \bigoplus_n (\exp (int) E_n + \exp (i(n+1)\, t) F_n)$$

is in $\mathfrak{A}$ and

$$Z_t = \bigoplus_n \exp (-\, int) I$$

is in $\mathscr{Z}$, $\exp t\delta$ is the inner automorphism implemented by V_t.

We claim that δ is an outer derivation of $\mathfrak{A}$. Suppose the contrary. Then there exist operators $A = \oplus_n (a_n E_n + a_{n+1} F_n)$ in $\mathscr{A}$, $C = \oplus_n C_n$ in $\mathscr{C}_0$ and $Z = \oplus_n c_n I$ in $\mathscr{Z}$, such that $H = A + C + Z$. Since C_n is compact and $C_n + (a_n + c_n) E_n + (a_{n+1} + c_n - 1) F_n = 0$ it follows that $C_n = 0$, $a_n + c_n = a_{n+1} + c_n - 1 = 0$. Thus $a_{n+1} = a_n + 1$, $a_n = a_0 + n$, a contradiction since the sequence (a_n) is bounded. Thus δ is outer.

6.3. By making slight modifications in the preceding example, we exhibit a separable C^*-algebra $\mathfrak{A}_0$ with center the scalars (acting on a separable Hilbert space), and an outer *-derivation δ_0 of $\mathfrak{A}_0$ with the property that $\exp t\delta_0$ is an inner automorphism for each rational t. The group $\mathscr{G} = \{t : \exp t\delta_0 \text{ is inner}\}$ therefore contains the rationals and, by Theorem 4.3, is countable; so both $\mathscr{G}$ and its complement are everywhere dense in R. It follows that $\iota_0(\mathfrak{A}_0)$ is not a norm-closed subgroup of $\alpha(\mathfrak{A}_0)$, and Theorem 5.3 implies that $\Delta_0(\mathfrak{A}_0)$ is not a closed subspace of $\Delta(\mathfrak{A}_0)$. By Remark 5.6, $\mathfrak{A}_0$ has no faithful factor representation and so, in particular, is not primitive.

With $\mathscr{H}$, $\mathscr{A}$, $\mathfrak{A}$, H, and V_t constructed as in Example 6.2, let $\mathscr{A}_0$ be the C^*-subalgebra of $\mathfrak{A}$ which is generated by $\{V_t : t \text{ rational}\}$; and let $\mathfrak{A}_0$ be $\mathscr{A}_0 + \mathscr{C}^0$, where $\mathscr{C}^0 = \{\oplus_n C_n : C_n \in \mathscr{C}(\mathscr{H}), \| C_n \| \to 0\}$. Just as in the previous case, $\mathfrak{A}_0$ is a C^*-algebra with center the scalars; and $\mathfrak{A}_0^- = \mathfrak{A}^-$, $\mathfrak{A}_0' = \mathfrak{A}'$. Since $\mathscr{A}_0$ and $\mathscr{C}^0$ are separable, so is $\mathfrak{A}_0$. Since $HA - AH \in \mathscr{C}^0$ for each A in $\mathfrak{A}$, $\delta_0 = \text{ad}\, iH \mid \mathfrak{A}_0$ is a

-derivation of $\mathfrak{A}_0$. If δ_0 is inner, then $H \in \mathfrak{A}_0 + \mathfrak{A}_0' \subseteq \mathfrak{A} + \mathfrak{A}_0' = \mathfrak{A} + \mathfrak{A}'$, and $\delta = \text{ad } iH \mid \mathfrak{A}$ is inner, which we have shown to be false. Hence δ_0 is an outer -derivation of $\mathfrak{A}_0$: the automorphism $\exp t\delta_0$ is implemented by V_t and is therefore inner for each rational t.

Acknowledgments

We wish to record our gratitude to S. Doplicher and D. Kastler, for a discussion in which Theorem 3.2 was conjectured.

References

1. Dixmier, J., "Les algèbres d'opérateurs dans l'espace hilbertien." Gauthiers-Villars, Paris, 1957.
2. Dixmier, J., "Les C^*-algèbres et leurs représentations." Gauthiers-Villars, Paris, 1964.
3. Gelfand, I. and Naimark, M., On the imbedding of normed rings into the ring of operators in Hilbert space, *Rec. Math. (mat. Sb.)* **12** (1943), 197–213.
4. Kadison, R., Transformation of states in operator theory and dynamics. *Topology Suppl.* **2** (1965), 177–198.
5. Kadison, R., Derivations of operator algebras. *Ann. Math.* **83** (1966), 280–293.
6. Kadison, R. and Ringrose, J., Derivations of operator group algebras. *Am. J. Math.* **88** (1966), 562–576.
7. Kadison, R. and Ringrose, J., Derivations and automorphisms of operator algebras. *Commun. Math. Phys.* **4** (1967), 32–63.
8. Kaplansky, I., Modules over operator algebras. *Am. J. Math.* **75** (1953), 839–859.
9. Lance, C., Automorphisms of postliminal C^*-algebras. (to appear in *Pacific J. Math.*).
10. Sakai, S., On a conjecture of Kaplansky. *Tôhoku Math. J.* **12** (1960), 31–33.
11. Sakai, S., Derivations of W^*-algebras. *Ann. Math.* **83** (1966), 273–279.