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TRANSFORMATIONS OF STATES IN OPERATOR THEORY AND DYNAMICS

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§1. INTRODUCTION

TWO BASIC constituents of a physical system are its family $\mathfrak{A}$ of observable attributes and the family S of states in which the system can be found. In classical (particle) mechanics, the observables are algebraic combinations of the (canonical) coordinates and (conjugate) momenta. Each state is described by an assignment of numbers to these observables—the values certain to be found by measuring these observables in the given state. The totality of numbers associated with a given observable is its *spectrum*. In this view of classical statics, the observables $\mathfrak{A}$ are represented as functions on the space S of states—they form an algebra (necessarily commutative) relative to pointwise operations. The dynamics (or law of motion) of this system describes the way the states evolve in time (i.e. specifies trajectories through states in S).

The experiments involving atomic and sub-atomic phenomena made it clear that this Newtonian view of mechanics would not suffice for their basic theory. Speculation on the meaning of these experimental results eventually led to the conclusion that the only physically meaningful description of a state was in terms of an assignment of probability measures to the spectra of the observables (a measurement of the observable with the system in a given state will produce a value in a given portion of the spectrum with a specific probability). Moreover, it was necessary to assume, in this physical realm, that a state which assigns a "definite" value to one observable (position) assigns a dispersed measure to the spectrum of some other observable (momentum)—the amount of dispersion involving the experimentally reappearing Planck's constant (The Uncertainty Principle). Further analysis shows that this entails the non-commutativity of the algebra of observables.

The search for a mathematical model which could mirror the structural features of this system and in which computations in accord with experimental results could be made produced the (possibly unbounded) self-adjoint operators on a Hilbert space as the observables and the unit vectors (up to a complex multiple of modulus 1) as corresponding to the states. This correspondence between vectors and states is made as follows: if A is a (bounded) self-adjoint operator and $\sigma(A)$ is its spectrum, the state corresponding to the unit vector $\mathbf{x}$

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assigns the measure to $\sigma(A)$ which has as "nth moment" $(A^n\mathbf{x}, \mathbf{x})$. The dynamics of such a system is assumed to be given by a (strongly-continuous) one-parameter unitary group $t \to U_t$ (the integrated form of the Schrödinger Wave Equation—the system initially in the state corresponding to $\mathbf{x}$ will evolve at time t into the state $U_t\mathbf{x}$). From the mathematical viewpoint, the description of the trajectories $t \to U_{-t}AU_t$ of observables also determines the dynamics, for the probability measure assigned to the spectrum of $U_{-t}AU_t$ by the state corresponding to $\mathbf{x}$ is the same as that assigned by $U_t\mathbf{x}$ to the spectrum of A. The view of dynamics as states transforming in time is sometimes called "the Schrödinger Picture" and that of the "moving observables", "the Heisenberg Picture".

The structure discernible in the physics of early quantum mechanics did not "force" this mathematical model (i.e. no representation theorem involving operators on Hilbert space was proved)—and later investigation showed that no such theorem is possible on the basis of this general structure (cf. [16, 22] for a discussion of this). This model is the simplest compatible with the additional structure needed. More recent studies have indicated that all self-adjoint operators may not be adequate as the model for the algebra of observables of every physical system. The C^* algebras are a long step from this special model, but still not into the chaos of abstract structures consistent with the general features of physical systems. (See §2 for definitions.)

Proceeding from the (*ad hoc* but considerably weakened) assumption that the (bounded) observables of a physical system are the self-adjoint elements of a C^* algebra and some plausible (general) physical assumptions about the way states evolve in time, we shall derive as much of the usual formulation of (quantum) dynamics as seems possible. Under general assumptions, we derive something close to the Heisenberg Picture from the Schrödinger picture (Theorem 3.3). (Even if the dynamics is given by a one-parameter unitary group $t \to U_t$ it is not *a priori* clear that the automorphism of all bounded operators induced by U_t will map the algebra of bounded observables into itself, if this algebra is not all bounded operators.) Adding conditions (on the algebra of observables and on the dynamical group), by steps, we derive a full analogue of the Schrödinger Picture (Theorem 3.4) and, then, the description of the dynamics in terms of a one-parameter unitary group (or Hamiltonian—Theorem 3.8). We note that deductive treatments of quantum dynamics (with all self-adjoint operators as model for the observables) are to be found in lecture notes of E. P. Wigner (we are told by A. S. Wightman) and in [16]. The dynamics (motions) of systems (with a C^* algebra as model) is considered in [23; Section 5] from the point of view of the Heisenberg Picture, for the purpose of analyzing their stationary states.

The (physical) scope of this paper is broader than the deductive derivation of (non-relativistic) quantum dynamics which we have chosen as unifying descriptive theme; since the groups whose representations by transformations of families of states we consider include such classes as the simply-connected, semi-simple Lie groups (cf. Theorem 4.13). In §2, we list some preliminary definitions and results. A review of the *universal representation* of a C^* algebra, its relation to the second dual of the algebra, and its use in completing the description of (Jordan) C^* homomorphisms of C^* algebras given in [8;

Theorem 10] are included in this section. Section 3 contains the statements of the main results in terms of dynamical systems along with accompanying discussion and definitions. Their proofs as well as more general auxiliary results are contained in §4.

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§2. NOTATION AND PRELIMINARY RESULTS

We deal with complex Hilbert spaces $\mathscr{H}$, denoting by $(\mathbf{x}, \mathbf{y})$ the inner product of two vectors $\mathbf{x}, \mathbf{y}$ in $\mathscr{H}$, and by $\|\mathbf{x}\|$ the norm (or length) of $\mathbf{x}$. Our operators are bounded unless otherwise noted; and the norm (or bound) of an operator A is denoted by $\|A\|$. We denote by $\mathscr{B}(\mathscr{H})$ the set of all operators on $\mathscr{H}$, and refer to the metric topology induced on it by the norm as the *norm topology*. A self-adjoint family of operators is one which contains A^*, the adjoint of A, if it contains A; and a $*$ algebra of operators is a self-adjoint family which is an algebra relative to the usual operations on operators. A C^* algebra is a Banach algebra with a distinguished, conjugate-linear, anti-automorphic involution ($*$ operation) which is $*$ isomorphic (and isometric) with a norm-closed $*$ algebra of operators (a C^* algebra of operators). In essence, the result of [6] says that a Banach algebra with such an involution which satisfies $\|A^*A\| = \|A^*\| \cdot \|A\|$ is a C^* algebra. For the most part, our algebras contain an identity.

We shall also be concerned with the weak and strong operator topologies on $\mathscr{B}(\mathscr{H})$ (the weakest topologies on $\mathscr{B}(\mathscr{H})$ such that the mappings $A \to (A\mathbf{x}, \mathbf{x})$ and $A \to A\mathbf{x}$ are continuous, respectively, for each $\mathbf{x}$ in $\mathscr{H}$), and with the $*$ algebras of operators called *von Neumann algebras*, closed in these topologies (closure in either implies closure in the other). For their theory, we make general reference to [3]. The case of von Neumann algebras whose centers consist of scalar multiples of the identity operator I, called *factors* [18] and abelian von Neumann algebras are of special interest. Those abelian von Neumann algebras which are generated by their minimal projections (equivalently, whose identity is the sum of their minimal projections), we call *totally atomic*. We denote the weak closure of a family $\mathscr{F}$ of operators by $\mathscr{F}^-$, and the set of operators in $\mathscr{B}(\mathscr{H})$ which commute with it (its commutant) by $\mathscr{F}'$.

A $*$ homomorphism ϕ of a C^* algebra $\mathfrak{A}$ into $\mathscr{B}(\mathscr{H})$ is called a representation of $\mathfrak{A}$ (on $\mathscr{H}$). The image $\phi(\mathfrak{A})$ of such a representation is norm closed (a C^* algebra of operators) [20]. The representation obtained by composing ϕ with restriction of the operators in $\phi(\mathfrak{A})$ to a (closed) invariant subspace is called a *subrepresentation* of ϕ. We use *subspace* to mean *closed* linear manifold; and adopt the convention of identifying terminology and notation for a subspace and the orthogonal projection operator having it as range (where no confusion can arise). With $\mathscr{F}$ a family of operators and V a set of vectors, we denote by $[\mathscr{F}V]$ the subspace spanned by $\{A\mathbf{x} : A \text{ in } \mathscr{F} \text{ and } \mathbf{x} \text{ in } V\}$. A separating projection E for a $*$ algebra of operators is one such that $AE = 0$ implies $A = 0$, if A is in the algebra. A C^* homomorphism (or C^* representation) ϕ of a C^* algebra $\mathfrak{A}$ is a linear $*$ preserving mapping of $\mathfrak{A}$ into $\mathscr{B}(\mathscr{H})$ such that $\phi(A^2) = \phi(A)^2$, for each A in $\mathfrak{A}$ (equivalently, $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$).

A state ρ of a C^* algebra $\mathfrak{A}$ is a linear functional such that $\rho(I) = 1$ and $\rho(A) \geq 0$ if $A \geq 0$ (i.e. ρ is a normalized, positive, linear functional on $\mathfrak{A}$). The value, $\rho(A)$, is the expectation of the observable A in the state ρ when these mathematical entities are assigned their physical interpretation. This view of states (as functionals) makes use of the familiar identification of a measure with its associated integration process. We define the linear functional $\omega_{\mathbf{x},\mathbf{y}}$ on $\mathscr{B}(\mathscr{H})$ by $\omega_{\mathbf{x},\mathbf{y}}(A) = (A\mathbf{x}, \mathbf{y})$, and denote those functionals for which $\mathbf{y} = \mathbf{x}$ by $\omega_{\mathbf{x}}$. With $\mathbf{x}$ a unit vector, $\omega_{\mathbf{x}}$ is a state of $\mathscr{B}(\mathscr{H})$. Its restriction, $\omega_{\mathbf{x}}|\mathfrak{A}$, to $\mathfrak{A}$ (in general, we denote the restriction of a mapping ϕ to K by $\phi|K$) is called a *vector state* of $\mathfrak{A}$. Each state of $\mathfrak{A}$ has norm 1; and, so, lies in the continuous dual $\overline{\mathfrak{A}}$ of $\mathfrak{A}$. We shall make use of the w^* topology on $\overline{\mathfrak{A}}$, the weakest topology relative to which the mappings $\rho \to \rho(A)$ are continuous for each A in $\mathfrak{A}$, and its associated w^* uniform structure whose neighborhood entourages are given by a positive ε and a finite set of elements $A_1, \ldots, A_n$ of $\mathfrak{A}$ as $N_{\varepsilon,A_1,\ldots,A_n}(\rho_0) = \{\rho : |\rho(A_j) - \rho_0(A_j)| < \varepsilon, j = 1, \ldots, n, \rho \text{ in } \overline{\mathfrak{A}}\}$. The set of all states of $\mathfrak{A}$ will be denoted by $S(\mathfrak{A})$. It is convex and compact in the w^* topology. From the Krein–Milman theorem, $S(\mathfrak{A})$ is the closed convex hull of its extreme points—the *pure states* of $\mathfrak{A}$. In general, a mapping ϕ of a convex subset K of a vector space into another vector space will be said to be an affine mapping when $\phi(ak + (1-a)k') = a\phi(k) + (1-a)\phi(k')$, where $0 \leq a \leq 1$.

For the purpose of distinguishing the spectra of the elements of a C^* algebra $\mathfrak{A}$, we shall deal with special convex subsets of $S(\mathfrak{A})$.

DEFINITION (2.1). *A full family of states S_0 of $\mathfrak{A}$ is a convex subset of $S(\mathfrak{A})$ such that $A \geq 0$ if $\rho(A) \geq 0$ for all ρ in S_0.*

THEOREM (2.2). *A convex subset S_0 of the state space $S(\mathfrak{A})$ of a C^* algebra $\mathfrak{A}$ is full if and only if it is w^* dense in $S(\mathfrak{A})$.*

Proof. If S_0 is w^* dense in $S(\mathfrak{A})$, and $\rho(A) \geq 0$ for each ρ in S_0; then, since $\rho \to \rho(A)$ is w^* continuous on $S(\mathfrak{A})$, $\rho(A) \geq 0$ for each ρ in $S(\mathfrak{A})$. Thus $A \geq 0$, and S_0 is full.

If S_0 is full and L is the representing function system of $\mathfrak{A}$ on $S(\mathfrak{A})$ [9; p. 312], the restriction mapping of L into functions on S_0 is a linear isomorphism; for, if $\rho(A) = 0$ for all ρ in S_0, then A and $-A$ are positive, so that $A = 0$. It is also an order isomorphism by virtue of the assumption that S_0 is full. The argument of [9; p. 328] now yields that the w^* closure of S_0, a convex set, contains all pure states of $\mathfrak{A}$; and the Krein–Milman theorem shows that this w^* closure is then $S(\mathfrak{A})$.

If ϕ is a representation of the C^* algebra $\mathfrak{A}$ on the Hilbert space $\mathscr{H}$, we denote by S_ϕ the convex hull (not its closure) of the set of states $\{\omega_{\mathbf{x}}\phi : \mathbf{x} \text{ a unit vector in } \mathscr{H}\}$ of $\mathfrak{A}$. If ϕ is faithful, $\phi(A) \geq 0$ if and only if $A \geq 0$; so that S_ϕ is full. The weakly continuous states of $\phi(\mathfrak{A})$ have the form $(a_1\omega_{\mathbf{x}_1} + \cdots + a_n\omega_{\mathbf{x}})|\phi(\mathfrak{A})$, where $0 \leq a_j \leq 1$, $\mathbf{x}_j$ is a unit vector in $\mathscr{H}$, and $a_1 + \cdots + a_n = 1$. The normal states, those weakly continuous on the unit ball of $\phi(\mathfrak{A})$, have the form

$$\left(\sum_{j=i}^{\infty} a_j \omega_{\mathbf{x}_j}\right)\Big|\phi(\mathfrak{A}), \qquad \text{where} \qquad a_j \geq 0, \quad \sum_{j=i}^{\infty} a_j = 1$$

and each $\mathbf{x}_j$ is a unit vector in $\mathscr{H}$ [3, Théorème 1, p. 54]. Thus, each normal state ω of $\phi(\mathfrak{A})$ is a norm limit of weakly continuous states. (In the presence of a separating vector for $\phi(\mathfrak{A})^-$ each normal state is a vector state.) We refer to the state $\omega\phi$ of $\mathfrak{A}$ as a normal state of ϕ.

We shall make frequent use of a certain canonical faithful representation of a C^* algebra $\mathfrak{A}$ which we refer to as the *universal representation* of $\mathfrak{A}$. It has a "universal" property for extension of cyclic representations of $\mathfrak{A}$ which we describe in the following outline along with its other main properties. Let ϕ be the direct sum of all representations of $\mathfrak{A}$ corresponding to states of $\mathfrak{A}$ [21] (i.e. of all cyclic representations of $\mathfrak{A}$); and let $\mathscr{H}$ be the Hilbert space on which $\phi(\mathfrak{A})$ acts. In effect, ϕ is the representation used in [6] to prove that an abstract C^* algebra has a concrete representation. Since each cyclic representation ψ of $\mathfrak{A}$ is (unitarily equivalent to) a direct summand of ϕ, there is a cyclic projection E' in $\phi(\mathfrak{A})'$ such that ψ is (unitarily equivalent to) the representation $A \to \phi(A)E'$ of $\mathfrak{A}$ on $E'(\mathscr{H})$. Of course, $B \to BE'$ is a strongly (and weakly)-continuous representation of $\phi(\mathfrak{A})^-$, mapping the weak (and strong) closure of $\phi(\mathfrak{A})$ onto $\phi(\mathfrak{A})^-E'$. Thus, if we identify $\mathfrak{A}$ with $\phi(\mathfrak{A})$, we may say that each cyclic representation ψ of $\mathfrak{A}$ has a strongly (and weakly)-continuous extension to a representation of $\mathfrak{A}^-$ onto $\psi(\mathfrak{A})^-$. (Note that $\phi(\mathfrak{A})^-E'$ is a von Neumann algebra [3; Prop. 1, p. 18].) Moreover, each state ρ of $\mathfrak{A}$ has a weakly continuous extension to $\mathfrak{A}^-$ (unique, since $\mathfrak{A}$ is weakly dense in $\mathfrak{A}^-$) which, in fact, corresponds to a unit vector. (If ϕ_ρ is the representation arising from ρ, and $\mathbf{x}_\rho$ its generating unit vector, $(\phi_\rho(A)\mathbf{x}_\rho, \mathbf{x}_\rho) = \rho(A)$. But ϕ_ρ is unitarily equivalent to $A \to AE'$ for some cyclic projection E' in $\mathfrak{A}'$; so that there is a unit vector $\mathbf{x}$ in $E'(\mathscr{H})$ such that $\rho = \omega_{\mathbf{x}}|\mathfrak{A}$.) With η a bounded linear functional on $\mathfrak{A}$, let $\eta^*(A)$ be $\overline{\eta(A^*)}$. Then $\eta = \eta_1 + i\eta_2$, where $\eta_1 = (\eta + \eta^*)/2 = \eta_1^*$ and $\eta_2 = (\eta - \eta^*)/2i = \eta_2^*$; and $\eta_1 = \eta_1^+ - \eta_1^-$, $\eta_2 = \eta_2^+ - \eta_2^-$, with η_1^+, η_1^-, η_2^+, η_2^- positive linear functionals on $\mathfrak{A}$. Thus η has a weakly continuous extension to $\mathfrak{A}^-$. If $\eta = a_1\rho_1 + \cdots + a_n\rho_n$, with $\rho_1, \ldots, \rho_n$ distinct states of $\mathfrak{A}$ and $a_1, \cdots, a_n$ complex scalars; then $\rho_j = \omega_{\mathbf{x}_j}$, with $\mathbf{x}_j$ a unit vector in $\mathscr{H}$ and $[\mathfrak{A}\mathbf{x}_j]$, $j = 1, \ldots, n$, orthogonal subspaces of $\mathscr{H}$. Thus, with $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$ and $\mathbf{y} = \mathbf{x}_1 + \cdots + \mathbf{x}_n$, $\eta(A) = (A\mathbf{x}, \mathbf{y}) = \omega_{\mathbf{x},\mathbf{y}}(A)$, for each A in $\mathfrak{A}$. If τ is a positive linear mapping of $\mathfrak{A}$ into the algebra of bounded operators on some Hilbert space $\mathscr{K}$ then τ is norm continuous on $\mathfrak{A}$, since $-\|A\|\tau(I) \leq \tau(A) \leq \|A\|\tau(I)$, for each self-adjoint operator A in $\mathfrak{A}$. Thus with $\mathbf{y}$, $\mathbf{z}$ in $\mathscr{K}$, $A \to (\tau(A)\mathbf{y}, \mathbf{z})$ has a weakly continuous extension to $\mathfrak{A}^-$. It follows [9; Remark 2.2.3] that τ is weakly continuous on $\mathfrak{A}$ and has a weakly-continuous extension to $\mathfrak{A}^-$ which is, again, positive linear. By weak continuity of this extension (which we denote, again, by τ), $\tau(\mathfrak{A}^-) \subseteq \tau(\mathfrak{A})^-$. If $\tau(\mathfrak{A})$ is again a C^* algebra, then $\tau(\mathfrak{A})$ is norm closed; so that τ is an open mapping (Closed Graph Theorem); and, with $\mathfrak{S}$ the (norm) closed unit ball in $\mathfrak{A}$, $\tau(\mathfrak{S})$ contains $\mathfrak{S}_0$, the (norm) closed ball of radius $r > 0$ about 0 in $\tau(\mathfrak{A})$. Now $\mathfrak{S}^-$ is weakly compact, so that $\tau(\mathfrak{S}^-)$ is weakly compact, weakly closed, and contains $\mathfrak{S}_0^-$. But with $\tau(\mathfrak{A})$ a self-adjoint algebra, $\mathfrak{S}_0^-$ is the closed ball of radius r in $\tau(\mathfrak{A})^-$, by Kaplansky's Density Theorem [12, and 3; p. 46]. Thus $\tau(\mathfrak{A}^-)$ contains $(\tau\mathfrak{A})^-$; and $\tau(\mathfrak{A}^-) = \tau(\mathfrak{A})^-$. We have shown:

LEMMA (2.3). *Each linear mapping τ of a linear space of operators $\mathfrak{A}$ onto a C^* algebra which is norm continuous and which has a weakly (or ultra weakly)-continuous extension to $\mathfrak{A}^-$ maps $\mathfrak{A}^-$ onto $\tau(\mathfrak{A})^-$.*

With regard to the universal representation, we shall also need:

LEMMA (2.4). *If $\mathfrak{A}$ acting on $\mathscr{H}$ is the universal representation of the C^* algebra $\mathfrak{A}$ and ϕ is a C^* homomorphism of $\mathfrak{A}$, the weakly continuous extension of ϕ to $\mathfrak{A}^-$ is a C^* homomorphism. If ϕ is a representation of $\mathfrak{A}$ so is its extension.*

Proof. Employing the decomposition of an operator as the sum of a self-adjoint and skew-adjoint operator, it will suffice to show that $\phi(A^2) = \phi(A)^2$, with $A = A^*$, $\|A\| \leq 1$ and A in $\mathfrak{A}^-$, in order to prove that the extension of ϕ to $\mathfrak{A}^-$ (which we denote, again, by ϕ) is a C^* homomorphism; and that $\phi(AB) = \phi(A)\phi(B)$, with $A = A^*, B = B^*, \|A\| \leq 1, \|B\| \leq 1$, A and B in $\mathfrak{A}^-$, in order to show that ϕ is a representation. By virtue of Kaplansky's Density Theorem, the joint continuity of multiplication on the unit ball relative to the strong topology, and the fact that ϕ satisfies the corresponding identity on $\mathfrak{A}$, it suffices to establish the strong continuity of ϕ on the set of self-adjoint operators in the closed unit ball of $\mathfrak{A}^-$. Since the strong topology induces a topological linear, and hence, uniform structure on $\mathfrak{A}^-$ and ϕ is linear, it suffices to prove the strong continuity of ϕ at 0 on the closed unit ball $\mathfrak{S}$ in the space $\mathfrak{A}_*$ of self-adjoint operators in $\mathfrak{A}$ in order to establish the existence of a strongly continuous extension ϕ_0 of ϕ from $\mathfrak{S}$ to $\mathfrak{S}^-$, the closed unit ball in the space of self-adjoint elements in $\mathfrak{A}^-$ [12]. Since strong convergence implies weak convergence and the weak topology on $\phi(\mathfrak{A})^-$ is Hausdorff, ϕ_0 and ϕ agree on $\mathfrak{S}^-$; and ϕ is strongly continuous on $\mathfrak{S}^-$.

To prove the strong continuity of ϕ at 0 on $\mathfrak{S}$, let a vector $\mathbf{x}$ in the Hilbert space on which $\phi(\mathfrak{A})$ acts be given. If A in $\mathfrak{S}$ is sufficiently close to 0 in the strong topology, A^2 is sufficiently close to 0 in the strong and, hence, weak topology; so that $(\phi(A^2)\mathbf{x}, \mathbf{x}) = (\phi(A)^2\mathbf{x}, \mathbf{x}) = \|\phi(A)\mathbf{x}\|^2 \leq 1$, by weak continuity of ϕ—from which the strong continuity of ϕ at 0 on $\mathfrak{S}$ follows.

For completeness, we include a proof of the fact that the second dual of $\mathfrak{A}$ is canonically isometric with $\mathfrak{A}^-$. This was first noted in [25]. A discussion along the lines indicated [25] is to be found in [28] and an independent proof in [3; p. 49 prob. 6a, p. 64 prob. 5]. Let f be a bounded linear functional on the dual $\bar{\mathfrak{A}}$ of $\mathfrak{A}$. For each pair of vectors $\mathbf{x}, \mathbf{y}$ in $\mathscr{H}$, $\omega_{\mathbf{x},\mathbf{y}}|\mathfrak{A}$ is in $\bar{\mathfrak{A}}$; and $\{\mathbf{x}, \mathbf{y}\} \to f(\omega_{\mathbf{x},\mathbf{y}}|\mathfrak{A})$ is a conjugate bilinear functional on $\mathscr{H}$ with bound not exceeding $\|f\|$. From the Riesz representation of such functionals, there is a (unique) bounded operator B on $\mathscr{H}$ such that $f(\omega_{\mathbf{x},\mathbf{y}}|\mathfrak{A}) = (B\mathbf{x}, \mathbf{y})$, for all $\mathbf{x}, \mathbf{y}$ in $\mathscr{H}$; and $\|B\| \leq \|f\|$. Moreover, the mapping $f \to B$ is linear. If $A' \in \mathfrak{A}'$, then $(BA'\mathbf{x}, \mathbf{y}) = f(\omega_{A'\mathbf{x},\mathbf{y}}|\mathfrak{A}) = f(\omega_{\mathbf{x},A'^*\mathbf{y}}|\mathfrak{A}) = (A'B\mathbf{x}, \mathbf{y})$. Thus, B is in $\mathfrak{A}''$ $(= \mathfrak{A}^-)$. As noted, each element η of $\bar{\mathfrak{A}}$ has a unique weakly-continuous extension to $\mathfrak{A}^-$; and this extension has the same norm as η, since the unit ball of $\mathfrak{A}$ is weakly dense in that of $\mathfrak{A}^-$ [12]. Thus, each B in $\mathfrak{A}^-$ induces a linear functional f on $\bar{\mathfrak{A}}$ with $\|f\| \leq \|B\|$, and $\bar{\mathfrak{A}}$ is isometric with $\mathfrak{A}^-$ via the mapping described.

The closed ideals of $\mathfrak{A}$ (left, right and two-sided) have a description in terms of the universal representation (cf. [29]) as the intersection with $\mathfrak{A}$ of principal ideals in $\mathfrak{A}^-$ generated by (self-adjoint) idempotents.

THEOREM (2.5). *With $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$ the universal representation of the C^* algebra $\mathfrak{A}$ and $\mathscr{I}$ a (norm) closed left (right) ideal in $\mathfrak{A}$, we have $\mathscr{I} = \mathfrak{A} \cap \mathfrak{A}^-E$ ($\mathfrak{A} \cap E\mathfrak{A}^-$), with E a projection in $\mathfrak{A}^-$. If $\mathscr{I}$ is two-sided, E may be chosen to be a central projection in $\mathfrak{A}^-$. In either case, $\mathscr{I}^- = \mathfrak{A}^-E$ (or $E\mathfrak{A}^-$, when $\mathscr{I}$ is a right ideal).*

Proof. In [10; Theorem 2], it is shown that $\mathscr{I}$ is the intersection of the left kernels of the (pure) states of $\mathfrak{A}$ which annihilate it (for states, alone, the essence of this fact is contained in [20]). Each state of $\mathfrak{A}$ has the form $\omega_{\mathbf{x}}|\mathfrak{A}$, with $\mathbf{x}$ a unit vector in $\mathscr{H}$. Now, A in $\mathfrak{A}$ is in the left kernel of $\omega_{\mathbf{x}}|\mathfrak{A}$, if and only if $0 = \omega_{\mathbf{x}}(A^*A) = \|A\mathbf{x}\|^2$. Thus, $\mathscr{I}$ is the annihilator in $\mathfrak{A}$ of the subspace of $\mathscr{H}$ it annihilates. Let $I - E$ be the orthogonal projection on this subspace. Then $\mathscr{I} = \mathfrak{A} \cap \mathfrak{A}^-E$. Since E is the intersection of projections on null spaces of operators in $\mathfrak{A}$, E is in $\mathfrak{A}^-$ (as defined, E is clearly invariant under $\mathfrak{A}'$, hence in $\mathfrak{A}'' = \mathfrak{A}^-$). With $\mathscr{I}$ a closed right ideal in $\mathfrak{A}$, $\mathscr{I}^*$ is a closed left ideal; whence $\mathscr{I} = \mathfrak{A} \cap E\mathfrak{A}^-$, for some projection E in $\mathfrak{A}^-$, in this case. If $\mathscr{I}$ is a closed two-sided ideal in $\mathfrak{A}$, the subspace it annihilates is invariant under both $\mathfrak{A}$ and $\mathfrak{A}'$. Thus $\mathscr{I} = \mathfrak{A} \cap \mathfrak{A}^-E$, with E in the center of $\mathfrak{A}^-$.

Since $\mathscr{I}$ annihilates $(I - E)\mathscr{H}$, $\mathscr{I}^-$ does. Moreover, since $(I - E)\mathscr{H}$ is the null space of $\mathscr{I}$, the closure of the range of $\mathscr{I}^*$ is $E\mathscr{H}$ (that is, $[\mathscr{I}^*\mathscr{H}] = E\mathscr{H}$, from the general fact that $[\mathscr{F}\mathscr{H}]$ is the orthogonal complement of the null space of $\mathscr{F}^*$, for an arbitrary family of operators $\mathscr{F}$). Now, the closure of the ranges of A and AA^* are the same (A^* and AA^* have the same null space); and $\mathscr{I}^*$ is a right ideal. Thus $E\mathscr{H}$ is the closure of the span of ranges of the positive operators in $\mathscr{I}^*$—each of which lies in $\mathscr{I}$. The projection on the closure of the range of a self-adjoint operator is, by spectral theory, the strong limit of polynomials without constant terms in the operator. Thus, the range projection of each positive operator in $\mathscr{I}$ lies in $\mathscr{I}^-$, as does their union, E. From strong continuity of left multiplication by an operator, $A\mathscr{I}^- \subseteq \mathscr{I}^-$, with A in $\mathfrak{A}$, since $A\mathscr{I} \subseteq \mathscr{I}$. By strong continuity of right multiplication by an operator, $\mathfrak{A}^-B \subseteq \mathscr{I}^-$, since $\mathfrak{A}B \subseteq \mathscr{I}^-$, with B in $\mathscr{I}^-$. Thus, $\mathscr{I}^-$ is a left ideal in $\mathfrak{A}^-$ containing E and annihilating $(I - E)\mathscr{H}$. It follows that $\mathscr{I}^- = \mathfrak{A}^-E$.

The following application of the universal representation extends [8; Theorem 10] to a complete description of the C^* homomorphisms of one C^* algebra onto another.

THEOREM (2.6). *A linear, adjoint-preserving mapping α of a C^* algebra $\mathfrak{A}$ onto a C^* algebra of operators $\mathscr{B}$ acting on the Hilbert space $\mathscr{H}$ is a C^* homomorphism if and only if there is a projection P in the center of $\mathscr{B}^-$ such that $\alpha(A)\alpha(B)P = \alpha(AB)P$ and $\alpha(AB)(I - P) = \alpha(B)\alpha(A)(I - P)$, for all A and B in $\mathfrak{A}$.*

Proof. If such a P exists, then, even without the assumption that α is onto, we can conclude that it is a C^* homomorphism.

Suppose, now, that α is a C^* homomorphism, that $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$ is the universal representation of $\mathfrak{A}$, and that the weakly-continuous, C^* homomorphic extension of

α mapping $\mathfrak{A}^-$ onto $\mathcal{B}^-$ (whose existence is guaranteed by Lemmas (2.3), (2.4) and the discussion preceding them) is denoted, again, by α. If A is self-adjoint and $\alpha(A) = 0$, then $\alpha(A^2) = 0$; so that $0 = (\alpha(AB)\mathbf{x}, \mathbf{x}) = (\alpha(BA)\mathbf{x}, \mathbf{x})$ for each $\mathbf{x}$ in $\mathcal{H}$ and B in $\mathfrak{A}^-$ (by applying the Cauchy–Schwarz Inequality to the positive semi-definite inner product $[C, D] = (\alpha(D^*C)\mathbf{x}, \mathbf{x})$). Thus $0 = \alpha(AB) = \alpha(BA)$; and the kernel $\mathcal{I}$ of α is a weakly-closed, two-sided ideal in $\mathfrak{A}^-$. From Theorem (2.5), $\mathcal{I} = \mathfrak{A}^-(I - R)$, with R a central projection in $\mathfrak{A}^-$. Thus α is a C^* isomorphism of $\mathfrak{A}^-R$ onto $\mathcal{B}^-$. According to [8; Theorem 10], there is a central projection Q in $\mathfrak{A}^-R$ (hence, in $\mathfrak{A}^-$) such that α is a $*$ isomorphism on $\mathfrak{A}^-Q$ and a $*$ anti-isomorphism on $\mathfrak{A}^-(R - Q)$ (hence, a $*$ anti-homomorphism on $\mathfrak{A}^-(I - Q)$). Taking P to be $\alpha(Q)$, the proof is complete.

Remark (2.7). The union of a family of central projections each of whose members has the same property as P in Theorem (2.6) has this property; so that there is a maximal such projection in $\mathcal{B}^-$. We call this maximal projection the *homomorphic carrier* of α.

Remark (2.8). With the notation of the statement of Theorem (2.6), we note that the subspace E of $\mathcal{H}$ defined by $\{\mathbf{x} : (CD - DC)\mathbf{x} = 0 \text{ for all } C, D \text{ in } \mathcal{B}\}$ is invariant under $\mathcal{B}'$ and $\mathcal{B}$; for $(CD - DC)B\mathbf{x} = B(CD - DC)\mathbf{x} = 0$, with B in $\mathcal{B}$. Thus, E is a projection in the center of $\mathcal{B}^-$. Now, $\alpha(AB)E = \alpha(A)\alpha(B)PE + \alpha(B)\alpha(A)(I - P)E = \alpha(A)\alpha(B)EP + \alpha(A)\alpha(B)E(I - P) = \alpha(A)\alpha(B)E$. By maximality, E is contained in the homomorphic carrier of α.

§3. THE BASIC FORMULATION

We assume that the bounded observables of the physical system under study are (identified with) the self-adjoint operators in a C^* algebra $\mathfrak{A}$ and that the physically meaningful states of this system form a full family S_0.

DEFINITION (3.1). *A physical system is a pair* $(\mathfrak{A}, S_0)$ *consisting of a* C^* *algebra* $\mathfrak{A}$ *and a full family* S_0 *of states of* $\mathfrak{A}$.

Despite the fact that causality in the fully classical sense does not hold for a general physical system (a phenomenon referred to as "indeterminacy" in quantum mechanics), that aspect of causality which relates to the evolution of (undisturbed) systems in time remains valid. Namely, at time t units after a given time, a system initially in the state ρ will be in some specific state $\nu_t(\rho)$; so that ν_t is a mapping of the family of (physically meaningful) states into itself. This aspect of causality (which may also be expressed by saying that the system obeys a "law of motion") entails also $\nu_t\nu_{t'} = \nu_{t+t'}$. We assume, further, about the dynamics of our system a type of (theoretical) reversibility—specifically, for each interval of time t and each η in S_0, there is precisely one state ρ from which the system will evolve into the state η after the time interval t (i.e. $\nu_t(\rho) = \eta$). This assumption that each ν_t is a 1–1 mapping of S_0 onto itself amounts, roughly, to our considering systems whose laws of motion satisfy a certain non-singularity condition. If we denote the mapping inverse to ν_t by ν_{-t}, it is easily verified that $t \to \nu_t$ is a one-parameter group of 1–1 transformations of S_0 onto itself (e.g. with $t' > t \geq 0$, $\nu_t\nu_{t'-t} = \nu_{t'} = \nu_t\nu_{-t}\nu_{t'}$, so that $\nu_{t'-t} = \nu_{-t}\nu_{t'}$).

We shall also need some plausible (physical) continuity assumptions about the way the states of the system evolve in time. The first of these requires that, for a given interval of time t, if two states in S_0 are suitably close, the states into which they evolve after time t are close—where closeness of two states is measured by the closeness of the expectations of a given observable in these states. In precise mathematical form, we assume that each v_t is a unimorphism of S_0 onto itself relative to the w^* uniform structure (formally, given B in $\mathfrak{A}$ and t, there are $A_1, \ldots, A_n$ in $\mathfrak{A}$ such that if $|\rho(A_j) - \tau(A_j)| < 1, j = 1, \ldots, n$, with ρ and τ in S_0, then $|[v_t(\rho)](B) - [v_t(\tau)](B)| < 1$). The second continuity assumption involves the trajectory of a given state and requires that in a short enough interval of time the state into which the given state evolves will be close to the given state (again, measured by the expectation of an assigned observable). More precisely, we assume that $t \to [v_t(\rho)](A)$ is continuous for each ρ in S_0 and A in $\mathfrak{A}$; and we refer to this as *weak continuity* of the mapping $t \to v_t$.

Our final assumption is that mixtures of states are preserved by the dynamics of a physical system—formally, $v_t[a\rho + (1 - a)\tau] = av_t(\rho) + (1 - a)v_t(\tau)$, with ρ and τ in S_0 and $0 \leq a \leq 1$. In particular, states which cannot be expressed as non-trivial mixtures of other states (pure states) evolve as such states under the action of the "dynamical group". This is the quantum theory analogue of the deterministic evolution of classical mechanical systems (states of such systems being described by an assignment of definite numerical values to the canonical coordinates and conjugate momenta are pure and evolve, according to their laws of motion, into such states). The formalism of mixing states in quantum mechanics replaces the deterministic description of states in classical mechanics. The assumption just made is that this formalism remains intact under the dynamical evolution of the system (by analogy with the preservation of the deterministic nature of classical mechanical systems under dynamical evolution).

Our assumptions about the dynamics of a physical system may be summed up by saying that they conform to a "dynamical group" as in:

DEFINITION (3.2). *A dynamical group of a physical system* $(\mathfrak{A}, S_0)$ *is a (weakly) continuous, one-parameter group* $t \to v_t$ *of affine* w^* *unimorphisms* v_t *of* S_0. *The triple* $(\mathfrak{A}, S_0, t \to v_t)$ *will be called a dynamical system.*

The description of the dynamics in terms of a dynamical group corresponds to the Schrödinger Picture. Our first main result deduces the possibility of describing the dynamics in terms of a (modified) Heisenberg Picture.

THEOREM (3.3). *If* $(\mathfrak{A}, S_0)$ *is a physical system,* $t \to v_t$ *is a dynamical group of it if and only if there is a weakly-continuous, one-parameter group* $t \to \alpha_t$ *of* C^* *automorphisms of* $\mathfrak{A}$ *such that* $\rho(\alpha_t(A)) = [v_t(\rho)](A)$, *for each* ρ *in* S_0, A *in* $\mathfrak{A}$ *and real* t.

While the Jordan algebraic structure of the self-adjoint operators in $\mathfrak{A}$ is all we should expect to have preserved by the mappings α_t, in certain circumstances we can assert that each α_t is a $*$ automorphism.

THEOREM (3.4). *If* $(\mathfrak{A}, S_0, t \to v_t)$ *is a dynamical system, there is a weakly-continuous, one-parameter group* $t \to \alpha_t$ *of* $*$ *automorphisms of* $\mathfrak{A}$ *such that* $\rho(\alpha_t(A)) = [v_t(\rho)](A)$, *for each* ρ *in* S_0, A *in* $\mathfrak{A}$ *and real* t, *provided* S_0 *satisfies any one of the following*:

(a) $S_0 = S(\mathfrak{A})$.

(b) S_0 *contains the vector states of some separating family of irreducible representations of* $\mathfrak{A}$.

(c) S_0 *contains the vector states of some separating family of factor representations of* $\mathfrak{A}$.

This result and those to follow indicate the desirability of formalizing a concept of representation of a physical system and of a dynamical system.

DEFINITION (3.5). *If* $(\mathfrak{A}, S_0)$ *and* $(\mathfrak{A}, S_0, t \to v_t)$ *are a physical system and associated dynamical system, respectively, a representation* ϕ *of* $\mathfrak{A}$ *by operators on the Hilbert space* $\mathscr{H}$ *is said to be a representation of* $(\mathfrak{A}, S_0)$ *when* $\omega_x\phi$ *lies in* S_0 *for each unit vector* $\mathbf{x}$ *in* $\mathscr{H}$. *If* $\omega\phi$ *lies in* S_0, *for each normal state* ω *of* $\phi(\mathfrak{A})$, *we say that* ϕ *is a complete representation of* $(\mathfrak{A}, S_0)$. *If, for each* t *and unit vector* $\mathbf{x}$, $v_t(\omega_x\phi)$ *is* $\omega_y\phi$ *for some unit vector* $\mathbf{y}$ *in* $\mathscr{H}$, *we say that* ϕ *is a representation of* $(\mathfrak{A}, S_0, t \to v_t)$.

We shall make free use of all the standard terminology appertaining to representations of operator algebras in the context of representations of physical and dynamical systems without further explanation—for example, we will speak of faithful or factor representations of physical systems when the corresponding representation of the associated operator algebra is a faithful or factor representation, respectively.

Since we are emphasizing considering an abstract physical system as independent of its specific representations, it seems appropriate to comment on the physical significance we ascribe to a representation of the system. Mathematically, a representation of $(\mathfrak{A}, S_0)$ selects a certain "coherent" family of states from among the states of S_0 and, at the same time, "coalesces" some of the algebraic structure of $\mathfrak{A}$. This is the effect of introducing the system into an inhibiting physical environment (compatible with it)—e.g. placing interferometers, spectrometers, polarimeters about the system, or enclosing the system within reflecting walls (placing such a wall between two rigidly linked particles would violate the algebraic relations between the position observables of these particles and not be compatible with the system). A representation corresponding to enclosing a system in a box has in its kernel each position observable with spectrum "outside the box". At the same time, this representation selects states which assign a 0 probability distribution to the spectra of such observables (though, annihilating the kernel, alone, does not characterize the vector states or even the normal states of the representation). For such an enclosure to yield a representation of a dynamical system associated with this physical system, no such state must evolve under the given dynamical group into one which is not compatible with the representation. Passing from a representation of $(\mathfrak{A}, S_0)$ to a subrepresentation corresponds to introducing further (or more restrictive) compatible constraints—e.g. making the enclosure smaller. These more restrictive constraints make themselves evident in a higher concentration of probability distribution corresponding to the states of this subrepresentation over the spectra of the observables—contracting the enclosure, for example, "concentrates position". So to speak, passing to a subrepresentation makes the states "purer".

No attempt is made to list here physical interpretations for all the mathematical constructs which play a rôle in this theory. The remarks above deal with some of the basic constructs. It may be worth adding to these the comment that if the physical system is of the form $(\mathfrak{A}, S(\mathfrak{A}))$, the universal representation of $\mathfrak{A}$ is a representation of this system and corresponds physically to choosing an environment for the system which imposes no restriction on it.

The following theorem is a step toward describing the dynamics of a system in terms of a one-parameter, unitary group (or, in differential form, by a Hamiltonian and Schrödinger Wave Equation).

THEOREM (3.6). *If ϕ is a complete (separable) faithful representation of a dynamical system $(\mathfrak{A}, S_0, t \to v_t)$ such that $(\mathfrak{A}, S_0)$ has a separating family of factor representations, then there is a complete (separable) faithful representation ψ of $(\mathfrak{A}, S_0, t \to v_t)$ with the same normal states as ϕ such that $[v_t(\rho)](A) = \rho[\psi^{-1}(U_t\psi(A)U_t^*)]$, for each ρ in S_0, A in $\mathfrak{A}$ and all real t, where U_t is a unitary operator on the representation space of ψ.*

Each U_t of Theorem (3.6) may be multiplied by an arbitrary unitary operator in $\psi(\mathfrak{A})'$. The question of whether the U_t can be chosen so that $t \to U_t$ is a group representation becomes, then, a problem in the cohomology of the additive group of reals with coefficients in the (non-commutative) group of unitary operators in $\psi(\mathfrak{A})'$. To arrive at a (strongly) continuous group representation involves cross section problems for such unitary groups modulo closed normal subgroups and the restriction to *topological* cohomology. In case the automorphisms induced by the dynamical group correspond to unitary operators in $\psi(\mathfrak{A})^-$ (are "weakly" inner), the coefficients lie in the unitary group of the center of $\psi(\mathfrak{A})^-$; so that the cohomology considerations become commutative.

DEFINITION (3.7). *If $(\mathfrak{A}, S_0, t \to v_t)$ and ϕ satisfy the hypothesis of Theorem (3.6), and each U_t of the conclusion of Theorem (3.6) can be chosen in $\psi(\mathfrak{A})^-$; we say that $t \to v_t$ is an inner dynamical group relative to ϕ.*

The representations with inner groups are those for which there is an "observable energy". For the dynamics to be generated by a Hamiltonian, it is also necessary to introduce strengthened continuity conditions. In fact, if ϕ is a faithful representation of the dynamical system $(\mathfrak{A}, S_0, t \to v_t)$ by operators on $\mathscr{H}$; and $[v_t(\omega_{\mathbf{x}}\phi)](A) = (\phi(A)U_t\mathbf{x}, U_t\mathbf{x})$, for each unit vector $\mathbf{x}$ in $\mathscr{H}$, A in $\mathfrak{A}$ and all t, where $t \to U_t$ is a strongly-continuous, one-parameter, unitary group on $\mathscr{H}$; then $t \to v_t(\omega_{\mathbf{x}}\phi)$ is continuous, where S_0 has its norm topology. This follows from the strong continuity assumption and the fact that $\|\omega_{\mathbf{x}} - \omega_{\mathbf{y}}\| \leq 2\|\mathbf{x} - \mathbf{y}\|$. If the dynamical group satisfies this type of continuity condition, we say that it is *norm continuous* relative to ϕ.

Since we cannot expect this type of continuity to follow from weak continuity in the case of more general C^* algebras of observables (though it does when $\mathfrak{A}$ is assumed to be all bounded operators on some Hilbert space [16; p. 131]), we must assume it to conclude that the dynamics is given by a strongly-continuous, one-parameter, unitary group. The physical interpretation of this assumption is that the dynamics of the system is such that each observable with spectrum in a given interval has expectation in a state into which the given state evolves, after a suitably short time interval, close to its expectation in the given state.

THEOREM (3.8). *If $t \to v_t$ is a norm-continuous, inner dynamical group relative to the complete, faithful representation ϕ of the dynamical system $(\mathfrak{A}, S_0, t \to v_t)$ by operators on a separable Hilbert space $\mathscr{H}$, and the associated representation of $\mathfrak{A}$ is a direct sum of factor representations; then there is a (complete) faithful, separable, representation ψ of $(\mathfrak{A}, S_0, t \to v_t)$ with the same normal states as ϕ and a strongly-continuous, one-parameter, unitary group $t \to U_t$, with U_t in $\psi(\mathfrak{A})^-$, such that $[v_t(\omega\psi)](A) = \omega(U_t\psi(A)U_t^*)$, for each A in $\mathfrak{A}$, each normal state ω of $\psi(\mathfrak{A})$, and all real t.*

§4. THE PROOFS AND RELATED RESULTS

The three lemmas which follow establish that an affine unimorphism of a full family of states has a w^* continuous linear extension to the continuous dual. It then follows that such a mapping is induced by a C^* automorphism.

LEMMA (4.1). *If ϕ_0 is an affine mapping of a convex subset K of a vector space E which lies in no hyperplane into another vector space F, then there is a unique linear transformation ϕ of E into F and a unique vector $\mathbf{x}_0$ in F such that $(T_{\mathbf{x}_0}\phi)|K = \phi_0$, where $T_{\mathbf{x}_0}(\mathbf{y}) = \mathbf{y} + \mathbf{x}_0$ for $\mathbf{y}$ in F.*

Proof. With k in K, $T_{-\phi_0(k)}\phi_0 T_k$ $(= \phi_1)$ is an affine mapping of $K - k$ onto $\phi_0(K) - \phi_0(k)$, and $\phi_1(0) = 0$. Since K lies in no hyperplane, $K - k$ generates E. If we establish the existence of a linear transformation ϕ of E into F such that $\phi|(K - k) = \phi_1$, then $\phi_0 = T_{\phi_0(k)}\phi T_{-k}|K = T_{\phi_0(k)-\phi(k)}\phi|K$. To prove the existence of the asserted decomposition, it suffices to deal with the case in which K contains 0 with $\phi_0(0) = 0$, and to show, in this case, that ϕ_0 has a linear extension to E. For uniqueness, note that if $T_{\mathbf{y}}\psi|K = T_{\mathbf{x}}\phi|K$, then $T_{\mathbf{y}-\mathbf{x}}\psi|K = \phi|K$. Since ϕ is linear and K spans E, $T_{\mathbf{y}-\mathbf{x}}\psi = \phi$. In particular, $T_{\mathbf{y}-\mathbf{x}}\psi(0) = \phi(0)$, $\mathbf{y} - \mathbf{x} = 0$, and $\psi = \phi$.

Assuming K contains 0 and $\phi_0(0) = 0$, define $\phi(b_1k_1 + \cdots + b_nk_n)$ to be $b_1\phi_0(k_1) + \cdots + b_n\phi_0(k_n)$ for arbitrary real $b_1, \ldots, b_n$ and $k_1, \ldots, k_n$ in K. The proof that ϕ is well-defined and linear (clearly $\phi|K = \phi_0$) rests on showing that if $c_1k_1 + \cdots + c_nk_n = 0$, with $c_1, \ldots, c_n$ real, $k_1, \ldots, k_n$ in K, then $c_1\phi_0(k_1) + \cdots + c_n\phi_0(k_n) = 0$. Note, first, that, if $a_1, \ldots, a_n$ are in $[0, 1]$, $\sum a_j = 1$, and $k_1, \ldots, k_n$ are in K; then $\phi_0(a_1k_1 + \cdots + a_nk_n) = a_1\phi_0(k_1) + \cdots + a_n\phi_0(k_n)$. This holds for just two terms k_1, k_2, by hypothesis on ϕ_0; and knowing it for $n - 1$, it follows that

$$\phi_0\left[a_1k_1 + (1 - a_1)\sum_{j=2}^{n} a_jk_j/(1 - a_1)\right] = a_1\phi_0(k_1) + (1 - a_1)\phi_0\left[\sum_{j=2}^{n} a_jk_j/(1 - a_1)\right]$$
$$= \sum_{l=2}^{n} a_j\phi_0(k_j).$$

Suppose, now, that $a_1k_1 + \cdots + a_nk_n - (b_1k_1' + \cdots + b_mk_m') = 0$, with $a_1, \ldots, a_n$; $b_1, \ldots, b_m$ non-negative and $k_1, \ldots, k_n$; $k_1', \ldots, k_m'$ in K. By convexity of K, we may write this equality as $ak - bk' = 0$, with a, b non-negative and k, k' in K. From the preceding remarks, it suffices to show that $a\phi_0(k) - b\phi_0(k') = 0$ in order to show that $a_1\phi_0(k_1) + \cdots + a_n\phi_0(k_n) - (b_1\phi_0(k_1') + \cdots + b_m\phi_0(k_m')) = 0$. If at least one of a, b is 0 the result is immediate, knowing that $\phi_0(0) = 0$. If $ab \neq 0$, then $ck - k' = 0$ and $k - c'k' = 0$ where one of c, c' is not less than 1, say, $c \geq 1$. Then $ck, k, 0$ are in K; and $\phi_0(k) = \phi_0[c^{-1}ck + (c-1)c^{-1}0] = c^{-1}\phi_0(ck) + (c-1)c^{-1}\phi_0(0) = c^{-1}\phi_0(ck)$. Thus $\phi_0(k') = c\phi_0(k)$.

LEMMA (4.2). *If K is a compact subset of the topological linear space E and ϕ is a linear transformation of E into the topological linear space F such that $\phi|K$ is continuous then $\phi|C$ is continuous, where $C = \{ak : 0 \leq a \leq 1,\ k \text{ in } K\}$, and $\phi|D$ is continuous at 0, where $D = \{\mathbf{y} - \mathbf{y}' : \mathbf{y} \text{ and } \mathbf{y}' \text{ in } C\}$.*

Proof. Let $\mathbf{y}$ in C be given along with a neighborhood M of 0 in F. By compactness of $[0, 1]$ and joint continuity of scalar multiplication and addition in F, there is a neighborhood M' of 0 in F such that $aM' + M' \subseteq M$, for each a in $[0, 1]$. Since K is compact and $\phi|K$ is continuous, it is uniformly continuous. Thus, there is a neighborhood N of 0 in E such that if $k - k' \in N$ and k, k' are in K then $\phi(k) - \phi(k') \in M'$. Moreover, $\phi(K)$ is compact, whence by continuity of multiplication by scalars, there is an $\varepsilon > 0$ such that $c\phi(K) \subseteq M'$, if $|c| < \varepsilon$.

We assert the existence of a neighborhood N' of 0 in E such that if $\mathbf{y}' \in (\mathbf{y} + N') \cap C$ then there exist a, a' in $[0, 1]$ with $|a - a'| < \varepsilon$ and k, k' in K with $k - k'$ in N such that $\mathbf{y} = ak$ and $\mathbf{y}' = a'k'$. If no such neighborhood exists, then there is a net $\{\mathbf{y}_j\}$ converging to $\mathbf{y}$ of elements $\mathbf{y}_j$ in C such that if $\mathbf{y}_j = a'k'$ and $\mathbf{y} = ak$ with k, k' in K and a, a' in $[0, 1]$ then either $|a' - a| \geq \varepsilon$ or $k - k' \notin N$. But, since $\mathbf{y}_j \in C$, $\mathbf{y}_j = a_j k_j$ with k_j in K and $0 \leq a_j \leq 1$. Now, K and $[0, 1]$ are compact; so that there exists a subnet $\{\mathbf{y}_{j_r}\}$ with $\{a_{j_r}\}$ convergent to a and $\{k_{j_r}\}$ to k. Hence $\{\mathbf{y}_{j_r}\}$ converges to ak $(= \mathbf{y})$. But, for appropriate r, $|a - a_{j_r}| < \varepsilon$ and $k - k_{j_r} \in N$; contrary to the choice of $\mathbf{y}_{j_r}$. Thus N' with the properties described exists.

Suppose, now, that $\mathbf{y}' \in (\mathbf{y} + N') \cap C$. Then, there are scalars a, a' in $[0, 1]$ with $|a - a'| < \varepsilon$ and elements k, k' in K with $k - k'$ in N, such that $\mathbf{y} = ak$ and $\mathbf{y}' = a'k'$. We have $\phi(\mathbf{y}) - \phi(\mathbf{y}') = a\phi(k - k') + (a - a')\phi(k') \subseteq aM' + M' \subseteq M$. Thus $\phi|C$ is continuous. By compactness of C, $\phi|C$ is uniformly continuous. Let a neighborhood M of 0 in F be assigned. Choose a neighborhood N of 0 in E such that $\phi(\mathbf{y}) - \phi(\mathbf{y}') \in M$ if $\mathbf{y}, \mathbf{y}'$ are in C and $\mathbf{y} - \mathbf{y}' \in N$. Then with $\mathbf{z}$ in $N \cap D$, $\mathbf{z} = \mathbf{y} - \mathbf{y}'$ where $\mathbf{y}, \mathbf{y}'$ are in C; so that $\phi(\mathbf{z}) \in M$, and $\phi|D$ is continuous at 0.

LEMMA (4.3). *If E is a Banach space, $\bar{E}$ its continuous dual, K a convex compact subset of $\bar{E}$, $C = \{ak : a \text{ in } [0, 1], k \text{ in } K\}$ is contained in no hyperplane, $D = \{\mathbf{y} - \mathbf{y}' : \mathbf{y}, \mathbf{y}' \text{ in } C\}$ contains the unit ball $\bar{S}$ of $\bar{E}$, and ϕ_0 is an affine mapping of C into the scalars which is w^* continuous on K; then there is a vector $\mathbf{x}_0$ in E and a scalar a_0 such that $\phi_0(\bar{v}) = \bar{v}(\mathbf{x}_0) + a_0$, for each $\bar{v}$ in K.*

Proof. From Lemma (4.1), there is a (unique) linear functional ϕ on $\bar{E}$ and a (unique) scalar a_0 such that $\phi_0(\bar{v}) = \phi(\bar{v}) + a_0$, for each $\bar{v}$ in K. Using Lemma (4.2), ϕ_0 and hence $\phi|D$ are w^* continuous at 0, as is $\phi|(2\bar{S})$. With $\bar{v}_0$ in $\bar{S}$, $\bar{v} \to \bar{v} - \bar{v}_0 \to [\phi|(2\bar{S})](\bar{v} - \bar{v}_0) + \phi(\bar{v}_0) = \phi(\bar{v})$ is w^* continuous at $\bar{v}_0$ on $\bar{S}$. Thus $\phi|\bar{S}$ is w^* continuous (on $\bar{S}$). If $\bar{E}_0$ is the null space of ϕ, then $\bar{E}_0 \cap \bar{S}$ is closed in $\bar{S}$, by continuity of $\phi|\bar{S}$. Since $\bar{S}$ is w^* compact (Alaoglu–Bourbaki), hence closed in $\bar{E}$, $\bar{E}_0 \cap \bar{S}$ is closed in $\bar{E}$. It follows from [2; p. 129] that ϕ is w^* continuous on $\bar{E}$ and from [19; p. 116] that there exists a vector $\mathbf{x}_0$ in E such that $\phi(\bar{v}) = \bar{v}(\mathbf{x}_0)$, for each $\bar{v}$ in $\bar{E}$. The uniqueness of $\mathbf{x}_0$ is a consequence of the Hahn-Banach theorem.

Remark (4.4). We apply Lemma (4.3) in Theorem (4.5) with E the (real) Banach space of self-adjoint elements in a C^* algebra $\mathfrak{A}$ and K as $S(\mathfrak{A})$. For this application, let us note that each element of the unit ball of the dual of $\mathfrak{A}$ is in D by [28].

THEOREM (4.5). *If v is an affine mapping of the family $S_w(\mathfrak{A})$ of weakly-continuous states of one $*$ algebra $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$ into the corresponding family $S_w(\mathscr{B})$ of another $*$ algebra $\mathscr{B}$ acting on the Hilbert space $\mathscr{K}$, then there is a weakly-continuous positive linear mapping α of $\mathscr{B}^-$ into $\mathfrak{A}^-$ such that $\omega'(\alpha(B)) = (v(\omega)')(B)$ for each B in $\mathscr{B}^-$ and each ω in $S_w(\mathfrak{A})$, where ω' is the (unique) weakly-continuous (state) extension of ω to $\mathfrak{A}^-$. If v is an affine isomorphism of $S_w(\mathfrak{A})$ onto $S_w(\mathscr{B})$ then α is a C^* isomorphism of $\mathscr{B}^-$ onto $\mathfrak{A}^-$. If v is uniformly continuous relative to the w^* uniform structures on $S_w(\mathfrak{A})$ and $S_w(\mathscr{B})$, then α carries $\mathscr{B}$ into $\mathfrak{A}$.*

Proof. If $a\rho = b\tau$ for states ρ and τ, then $a\rho(I) = a = b$; so that v defined by $v'(a\omega) = av(\omega)$, for $a \geq 0$, is an affine extension of v from $S_w(\mathfrak{A})$ to the (convex) cone of all weakly-continuous positive linear functionals on $\mathfrak{A}$. According to Lemma (4.1), v has a linear extension, which we denote again by v, to the set of all self-adjoint weakly-continuous functionals on $\mathfrak{A}$, and from this real linear space to its complexification, the set of all weakly-continuous functionals on $\mathfrak{A}$. (Note that $\omega_{\mathbf{x},\mathbf{y}} = (i\omega_{\mathbf{x}+i\mathbf{y}} - i\omega_{\mathbf{x}-i\mathbf{y}} + \omega_{\mathbf{x}+\mathbf{y}} - \omega_{\mathbf{x}-\mathbf{y}})/4$; and each weakly-continuous linear functional is the sum of linear combinations of the special ones $\omega_{\mathbf{x},\mathbf{y}}$.) For each B in $\mathscr{B}^-$ and each pair of vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathscr{H}$, we define $\alpha(B)(\{\mathbf{x}, \mathbf{y}\})$ to be $(v(\omega_{\mathbf{x},\mathbf{y}}|\mathfrak{A}))(B)$. Since $\{\mathbf{x}, \mathbf{y}\} \to \omega_{\mathbf{x},\mathbf{y}}$ is a bounded, conjugate-bilinear mapping, v is linear, and $\|v(\omega_{\mathbf{x},\mathbf{y}})\| \leq 4$ when $\|\mathbf{x}\|$ and $\|y\|$ do not exceed 1 (compare the polarization formula just noted and observe that v preserves the norms of positive linear functionals); $\alpha(B)$ is a bounded conjugate-bilinear functional on $\mathscr{H}$. From the Riesz representation of such functionals, $\alpha(B)$ corresponds to an operator on $\mathscr{H}$, which we denote again by $\alpha(B)$. From this representation, we have the formula $\omega_{\mathbf{x},\mathbf{y}}(\alpha(B)) = (v(\omega_{\mathbf{x},\mathbf{y}}))(B)$. It follows that α is a positive linear mapping on $\mathscr{B}^-$. With A' in $\mathfrak{A}'$, $\omega_{A'\mathbf{x},\mathbf{y}}|\mathfrak{A} = \omega_{\mathbf{x},A'^*\mathbf{y}}|\mathfrak{A}$; whence $(\alpha(B)A'\mathbf{x}, \mathbf{y}) = (A'\alpha(B)\mathbf{x}, \mathbf{y})$, for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathscr{H}$. Hence $\alpha(B) \in \mathfrak{A}''$ ($= \mathfrak{A}^-$) and $\omega'(\alpha(B)) = (v(\omega)')(B)$, for each ω in $S_w(\mathfrak{A})$. Since the weakly-continuous linear functionals define the weak operator topologies on $\mathfrak{A}^-$ and $\mathscr{B}^-$, α is weakly continuous.

If v is an affine isomorphism of $S_w(\mathfrak{A})$ onto $S_w(\mathscr{B})$, then v^{-1} induces a mapping β of $\mathfrak{A}^-$ into $\mathscr{B}^-$ such that $(v^{-1}(\omega)')(A) = \omega'(\beta(A))$, for each ω in $S_w(\mathscr{B})$ and A in $\mathfrak{A}^-$. Combining this with the formula for α and the fact that $v^{-1}v$ and vv^{-1} are the identity transformations on $S_w(\mathfrak{A})$ and $S_w(\mathscr{B})$, respectively, we conclude that $\omega'(A) = \omega'(\alpha(\beta(A)))$ and $\omega'(B) = \omega'(\beta(\alpha(B)))$ for all ω' in $S_w(\mathfrak{A})$ and A in $\mathfrak{A}^-$ and all ω' in $S_w(\mathscr{B})$ and B in $\mathscr{B}^-$. Thus $\alpha\beta$ and $\beta\alpha$ are the identity transformations on $\mathfrak{A}^-$ and $\mathscr{B}^-$, respectively; so that α is an order isomorphism of $\mathscr{B}^-$ onto $\mathfrak{A}^-$. Now [11; Corollary 5] (or the alternative ending to the proof of [8; Theorem 7, p. 332] for the case of von Neumann algebras) establishes that α is a C^* isomorphism of $\mathscr{B}^-$ onto $\mathfrak{A}^-$.

Suppose, now, that v is uniformly continuous, where $S_w(\mathfrak{A})$ and $S_w(\mathscr{B})$ are taken in their w^* uniform structures. From Theorem (2.2) and the discussion following it, $S_w(\mathfrak{A})$ is full and w^* dense in $S(\mathfrak{A})$. Thus, the w^* compact $S(\mathfrak{A})$ is the completion of $S_w(\mathfrak{A})$ relative to the w^* uniform structure; and, by assumption of uniform continuity, v has a w^* continuous extension from $S_w(\mathfrak{A})$ to $S(\mathfrak{A})$ which is again affine. As at the beginning of this proof, the affine extension of v to $S(\mathfrak{A})$ has an affine extension, which we denote again by v, to $C = \{a\rho : a \text{ in } [0, 1], \rho \text{ in } S(\mathfrak{A})\}$. Applying Lemma (4.3) to the affine mapping $\tau \to [v(\tau)](B)$ of C into the scalars, for a fixed self-adjoint B in $\mathscr{B}$, we conclude the existence of an operator $\alpha'(B)$ in $\mathfrak{A}$ such that $\tau(\alpha'(B)) = [v(\tau)](B)$, for each τ in $S(\mathfrak{A})$. In particular, for ω in $S_w(\mathfrak{A})$, $[v(\omega)](B) = \omega(\alpha'(B))$. But $[v(\omega)](B) = \omega'(\alpha(B))$, from the first part of this proof. Thus, $\omega'(\alpha'(B)) = \omega(\alpha'(B)) = \omega'(\alpha(B))$, for each ω in $S_w(\mathfrak{A})$, $\alpha(B) = \alpha'(B) \in \mathfrak{A}$; and α maps $\mathscr{B}$ into $\mathfrak{A}$.

In the first assertion of Theorem (4.5), the weakly-continuous positive linear mapping α of $\mathscr{B}^-$ into $\mathfrak{A}^-$ (taking I onto I) need not carry $\mathscr{B}$ into $\mathfrak{A}$. In fact, any such mapping will induce an affine mapping of $S_w(\mathfrak{A})$ into $S_w(\mathscr{B})$. The set $S_w(\mathfrak{A})$ is tied to $\mathfrak{A}^-$ and not $\mathfrak{A}$; while the uniform space $S_w(\mathfrak{A})$ is bound to $\mathfrak{A}$, as emphasized by the last statement of Theorem (4.4).

COROLLARY (4.6). *A mapping v of the (convex) set $S_w(\mathfrak{A})$ of weakly-continuous states of a $*$ algebra $\mathfrak{A}$, acting on a Hilbert space, onto itself is an affine w^* unimorphism if and only if there exists a (weakly-continuous) C^* automorphism α of the weak closure $\mathfrak{A}^-$ of $\mathfrak{A}$ onto itself mapping $\mathfrak{A}$ onto $\mathfrak{A}$ and such that $(v(\omega)')(A) = \omega'(\alpha(A))$, for each ω in $S_w(\mathfrak{A})$ and each A in $\mathfrak{A}^-$.*

COROLLARY (4.7). *A mapping v of a full family of states S_1 of a C^* algebra $\mathfrak{A}_1$ onto those S_2 of another C^* algebra $\mathfrak{A}_2$ is an affine w^* unimorphism if and only if there is a C^* isomorphism α of $\mathfrak{A}_2$ onto $\mathfrak{A}_1$ such that $\rho(\alpha(B)) = (v(\rho))(B)$, for each ρ in S_1 and B in $\mathfrak{A}_2$. In particular, if $\mathfrak{A}_1 = \mathfrak{A}_2$, v is an affine w^* unimorphism if and only if α is a C^* automorphism.*

Proof. The states of a C^* algebra are w^* compact, hence complete, relative to the w^* uniform structure. From Theorem (2.2), S_1 and S_2 are w^* dense in their respective state spaces. Since v is a unimorphism it has a unique unimorphic (affine) extension mapping $S(\mathfrak{A}_1)$ onto $S(\mathfrak{A}_2)$. Now each state of a C^* algebra is weakly continuous in its universal representation; so that Theorem (4.5) applies to the extension of v and $\mathfrak{A}_1, \mathfrak{A}_2$ in their universal representations. The existence of the C^* isomorphism α follows.

We note that Corollary (4.7) provides a means for establishing [11; Corollary 5] without using the Generalized Schwarz Inequality; and as a consequence, provides an alternate approach to proving [11; Theorem 2 and Corollaries 3 and 4] as well as Theorem 7 of [8]. The key to this argument lies in proving that a linear order isomorphism mapping I onto I of one C^* algebra onto another is a C^* isomorphism [11; Corollary 5]. But such a mapping induces an affine w^* homeomorphism of the state space of one onto that of the other. Both state spaces are full families, of course, and being compact, the induced w^* homeomorphism is a unimorphism. Thus Corollary (4.7) applies to show that the given mapping is a C^* isomorphism.

We take this occasion to note and close a gap in the proof of Theorem 2 of [11] pointed out to us by L. T. Gardner. The gap occurs in attempting to prove that the (self-adjoint) unitary operator $\rho(I)$ $(=U)$ lies in the center of $\mathfrak{A}$ by using a strengthened Lemma 8 of [8]. One cannot conclude immediately that the mapping is isometric on operators of the form $A + inI$, with A self-adjoint, until it is known that the image of A is self-adjoint. The part of the argument "Now the map $\cdots$ into a self-adjoint element" should be replaced by: "One of $\|I+T\|$, $\|I-T\|$ is $1+\|T\|$, for each self-adjoint T in $\mathfrak{A}$. Thus $\|U+B\|$ or $\|U-B\|$ is $1+\|B,\|$ for each self-adjoint B in $\mathfrak{A}_1$. With $U=E-(I-E)$, E a projection in $\mathfrak{A}_1$, let $B=EA(I-E)+(I-E)AE$. Since $UB+BU=0$, $\|U+B\|^2=\|(U+B)^2\|=\|I+B^2\|=1+\|B\|^2=\|U-B\|^2=(1+\|B\|)^2$; and $\|B\|=0$."

Proof of Theorem (3.3). From Corollary (4.7), each ν_t is induced by a C^* automorphism α_t of $\mathfrak{A}$. Since $t\to\rho(\alpha_t(A))=[\nu_t(\rho)](A)$ is continuous for each ρ in S_0 and A in $\mathfrak{A}$, $t\to\alpha_t$ is a weakly-continuous one-parameter family of C^* automorphisms of $\mathfrak{A}$. In addition, $\rho(\alpha_{t+t'}(A))=[\nu_{t+t'}(\rho)](A)=[\nu_t(\nu_{t'}(\rho))](A)=\rho(\alpha_{t'}(\alpha_t(A)))$, so that $t\to\alpha_t$ is a one-parameter group.

Using the analysis of C^* homomorphisms given in§ 2, the special properties of the physical system assumed in Theorem (3.4), and the following lemmas, we can conclude that the C^* automorphisms associated with a dynamical group of this system are $*$ automorphisms.

LEMMA (4.8). *If $\mathfrak{A}$ is a C^* algebra and $\{\alpha_t\}$ is a family of C^* homomorphisms of $\mathfrak{A}$ onto a C^* algebra $\mathfrak{A}_0$ acting on the Hilbert space $\mathscr{H}$, t in a topological space X, such that $t\to\alpha_t(A)\mathbf{x}$ is a continuous mapping of X into $\mathscr{H}$ (in the norm topology) for each A in $\mathfrak{A}$ and $\mathbf{x}$ in $\mathscr{H}$; then $t\to P_t\mathbf{x}$ is a continuous mapping of X into $\mathscr{H}$, where P_t (a central projection in $\mathfrak{A}_0^-$) is the homomorphic carrier of α_t.*

Proof. Suppose $\mathbf{x}$ is a unit vector in $\mathscr{H}$ such that $P_{t'}\mathbf{x}=\mathbf{x}$. If $t\to P_t\mathbf{x}$ is not continuous at t', there is a net $\{t_j\}$ in X tending to t' such that $\|(I-P_{t_j})\mathbf{x}\|^2=((I-P_{t_j})\mathbf{x},\mathbf{x})\to\delta>0$. Let $\mathbf{x}_j=(I-P_{t_j})\mathbf{x}$, so that $\mathbf{x}-\mathbf{x}_j$ lies in $P_{t_j}(\mathscr{H})$ and $\alpha_{t_j}(AB)(\mathbf{x}-\mathbf{x}_j)=\alpha_{t_j}(A)\alpha_{t_j}(B)(\mathbf{x}-\mathbf{x}_j)$; $\alpha_{t_j}(AB)\mathbf{x}_j=\alpha_{t_j}(B)\alpha_{t_j}(A)\mathbf{x}_j$. Then

$$\begin{aligned}([\alpha_{t_j}(AB)-\alpha_{t_j}(A)\alpha_{t_j}(B)]\mathbf{x},\mathbf{y})&=([\alpha_{t_j}(B)\alpha_{t_j}(A)-\alpha_{t_j}(A)\alpha_{t_j}(B)]\mathbf{x}_j,\mathbf{y})\\&=(\mathbf{x}_j,[\alpha_{t_j}(A)\alpha_{t_j}(B)-\alpha_{t_j}(B)\alpha_{t_j}(A)]\mathbf{y}).\end{aligned}\qquad(*)$$

with A and B self-adjoint operators in $\mathfrak{A}$. By weak compactness, there is an $\mathbf{x}'$ in the closed unit ball of $\mathscr{H}$ and a subnet (which we denote again by $\{\mathbf{x}_j\}$) of $\{\mathbf{x}_j\}$ tending weakly to $\mathbf{x}'$. Since $P_{t'}\mathbf{x}=\mathbf{x}$, and by our continuity assumption; we have $[\alpha_{t_j}(AB)-\alpha_{t_j}(A)\alpha_{t_j}(B)]\mathbf{x}\to 0$, and $[\alpha_{t_j}(A)\alpha_{t_j}(B)-\alpha_{t_j}(B)\alpha_{t_j}(A)]\mathbf{y}\to[\alpha_{t'}(A)\alpha_{t'}(B)-\alpha_{t'}(B)\alpha_{t'}(A)]\mathbf{y}$. (Note that $\|\alpha_{t_j}(A)\|\le\|A\|$, so that we may avail ourselves of the joint strong continuity of multiplication on bounded sets.) If $\mathscr{F}$ is a bounded subset of $\mathscr{H}$, the mapping $\{u,v\}\to(u,v)$ of $\mathscr{F}\times\mathscr{H}$ provided with the product of the weak and norm topologies into the complex numbers is continuous. In fact,

$$|(u',v')-(u,v)|\le|(u',v')-(u',v)|+|(u',v)-(u,v)|\le\|u'\|\cdot\|v'-v\|+|(u-u',v)|.$$

It follows from this, $(*)$, and the convergences just noted, that

$$0=(\mathbf{x}',[\alpha_{t'}(A)\alpha_{t'}(B)-\alpha_{t'}(B)\alpha_{t'}(A)]\mathbf{y})=([\alpha_{t'}(B)\alpha_{t'}(A)-\alpha_{t'}(A)\alpha_{t'}(B)]\mathbf{x}',\mathbf{y}),$$

for each $\mathbf{y}$ in $\mathscr{H}$ and all self-adjoint operators A, B in $\mathfrak{A}$. Since $\alpha_{t'}$ maps onto, $(CD-DC)\mathbf{x}'=0$, for all C, D in $\mathfrak{A}_0$. Thus, from Remark (2.8), $P_t\mathbf{x}'=\mathbf{x}'$, for each t in X; and $(\mathbf{x}_j,\mathbf{x}')=((I-P_{t_j})\mathbf{x},\mathbf{x}')=0$. But $(\mathbf{x}_j,\mathbf{x}')\to(\mathbf{x}',\mathbf{x}')$; so that $\mathbf{x}'=0$. Hence $(\mathbf{x}_j,\mathbf{x})\to 0$, contradicting

$(\mathbf{x}_j, \mathbf{x}) = ((I - P_{t_j})\mathbf{x}, \mathbf{x}) \to \delta > 0$. It follows that $t \to P_t\mathbf{x}$ is continuous at t' for $\mathbf{x}$ such that $P_{t'}\mathbf{x} = \mathbf{x}$. This same argument with P_{t_j} in place of $I - P_{t_j}$ and $\alpha_{t_j}(BA)$ in place of $\alpha_{t_j}(AB)$ proves the continuity of $t \to P_t\mathbf{x}$ at t' for $\mathbf{x}$ such that $P_{t'}\mathbf{x} = 0$. (Note that $(P_{t_j}\mathbf{x}, \mathbf{x}') = (\mathbf{x}, \mathbf{x}') = (\mathbf{x}, P_{t'}\mathbf{x}') = 0$, in this case.) Thus, $t \to P_t\mathbf{x}$ is continuous for arbitrary $\mathbf{x}$ in $\mathscr{H}$.

LEMMA (4.9). *With the notation and hypotheses of Lemma (4.8), if X is connected, $P_0 = I$ for some 0 in X, and $\omega_z|\mathscr{C}$ is a pure state of the center $\mathscr{C}$ of $\mathfrak{A}_0^-$; then $P_t z = z$ for all t in X.*

Proof. From Lemma (4.8), $t \to (P_t z, z) = \omega_z(P_t)$ is continuous on X. Since $\omega_z|\mathscr{C}$ is pure, $\omega_z(P_t^2) = \omega_z(P_t)^2 = \omega_z(P_t)$; so that $\omega_z(P_t)$ is either 0 or 1. Now, X is connected and $\omega_z(P_0) = \omega_z(I) = 1$; so that $\omega_z(P_t) = 1$, for all t in X. Thus $P_t z = z$, for all t in X.

LEMMA (4.10). *If $t \to \alpha_t$ is a weakly-continuous family of C^* endomorphisms of the physical system $(\mathfrak{A}, S_0)$ and ϕ is a C^* representation of $(\mathfrak{A}, S_0)$ by operators on the Hilbert space $\mathscr{H}$, then $t \to \phi\alpha_t$ is strongly continuous.*

Proof. With $\mathbf{x}$ a unit vector in $\mathscr{H}$ and A a self-adjoint operator, we have, by assumption on $\{\alpha_t\}$, $\omega_{\mathbf{x}}([\phi\alpha_t](A)) \to \omega_{\mathbf{x}}([\phi\alpha_{t'}](A))$ as $t \to t'$. It follows that $(([\phi\alpha_t](A))\mathbf{x}, \mathbf{y}) \to (([\phi\alpha_{t'}](A))\mathbf{x}, \mathbf{y})$, for each $\mathbf{x}$, $\mathbf{y}$ in $\mathscr{H}$ as $t \to t'$, by polarization. Thus $\|([\phi\alpha_t](A))\mathbf{x} - ([\phi\alpha_{t'}](A))\mathbf{x}\|^2 = (([\phi\alpha_t](A^2))\mathbf{x}, \mathbf{x}) - (([\phi\alpha_t](A))\mathbf{x}, ([\phi\alpha_{t'}](A))\mathbf{x}) - (([\phi\alpha_{t'}](A))\mathbf{x}, ([\phi\alpha_t](A))\mathbf{x}) + (([\phi\alpha_{t'}](A))\mathbf{x}, ([\phi\alpha_{t'}](A))\mathbf{x}) \to 0$ as $t \to t'$. Since α_t and ϕ are linear, the same holds for arbitrary A in $\mathfrak{A}$.

Proof of Theorem (3.4). Let $\{\phi_j\}$ be a separating family of factor representations of $(\mathfrak{A}, S_0)$. Theorem (3.3) tells us that the dynamical group is induced by a weakly-continuous, one-parameter group $t \to \alpha_t$ of C^* automorphisms. From the preceding lemma, $t \to \phi_j\alpha_t$ is strongly continuous; so that Lemma (4.9) applies, and $\phi_j\alpha_t$ is a $*$ homomorphism, for each j and t (since $\phi_j(\mathfrak{A})^-$ is a factor, each vector state is pure on its center, the scalars). With A and B in $\mathfrak{A}$, $0 = \phi_j(\alpha_t(AB)) - \phi_j(\alpha_t(A))\phi_j(\alpha_t(B)) = \phi_j[\alpha_t(AB) - \alpha_t(A)\alpha_t(B)]$ for each j and t. Since $\{\phi_j\}$ is separating, each α_t is a $*$ automorphism. Both (a) and (b) are special cases of (c).

Remark (4.11). Lemma (4.10) is valid if we assume that $t \to \alpha_t$ is a weakly-continuous, *group* of linear order-endomorphisms of $(\mathfrak{A}, S_0)$. In this case, $0 \le \|([\phi\alpha_t](A))\mathbf{x} - \phi(A)\mathbf{x}\|^2 = (([\phi\alpha_t](A))^2\mathbf{x}, \mathbf{x}) - (([\phi\alpha_t](A))\mathbf{x}, \phi(A)\mathbf{x}) - (\phi(A)\mathbf{x}, ([\phi\alpha_t](A))\mathbf{x}) + (\phi(A)\mathbf{x}, \phi(A)\mathbf{x}) \le (([\phi\alpha_t](A^2))\mathbf{x}, \mathbf{x}) - (([\phi\alpha_t](A))\mathbf{x}, \phi(A)\mathbf{x}) - (\phi(A)\mathbf{x}, ([\phi\alpha_t](A))\mathbf{x}) + (\phi(A)\mathbf{x}, \phi(A)\mathbf{x}) \to 0$, as $t \to 0$. Thus $\|([\phi\alpha_t](A))\mathbf{x} - ([\phi\alpha_{t'}](A))\mathbf{x}\| = \|([\phi\alpha_{t-t'}](\alpha_{t'}(A)))\mathbf{x} - \phi(\alpha_{t'}(A))\mathbf{x}\| \to 0$ as $t \to t'$.

The next phase of our work is concerned with showing that the dynamical transformations of a physical system are induced by a (strongly-continuous) one-parameter, unitary group in a suitable representation (with certain restrictions on the system) and is therefore described by a Hamiltonian [26]. As a first step, we show that slight modification of a given faithful representation of the system guarantees that each α_t is unitarily induced. For this, we shall want:

LEMMA (4.12). *If $\mathscr{R}$ is a von Neumann algebra acting on a d-dimensional Hilbert space $\mathscr{H}$ and α is a $*$ automorphism of $\mathscr{R}$, then there is a unitary operator U on $\mathscr{H} \otimes \mathscr{H}'$ $(= \mathscr{K})$ such that $U^*(A \otimes I)U = \alpha(A) \otimes I$, for each A in $\mathscr{R}$, where $\mathscr{H}'$ has dimension $d' \ge$* $\max\{\aleph_1, d\}$ *or $\ge \aleph_0$ in case $d \le \aleph_0$.*

Proof. We show that $(\mathscr{R} \otimes I)'(= \mathscr{R}' \otimes \mathscr{B}(\mathscr{H}'))$ has coupling character d' [9; Definition 4.1.1]. According to [9; Lemma 4.1.3] there is a countably-decomposable central projection P in $(\mathscr{R} \otimes I)'$ which has coupling character a. Since $\mathscr{K}$ has dimension $d'(= dd')$, $a \leq d'$. Suppose that $\{F_k\}$ is an orthogonal family of cyclic projections in $(\mathscr{R} \otimes I)'$ with sum P and cardinality a. Let $\mathbf{x}_k$ be a generating vector for F_k, $\{E_j'\}$ be a maximal orthogonal family of one-dimensional projections in $\mathscr{H}'$, E_j be $P(I \otimes E_j')$ and $\mathscr{T}_k$ be the set of j such that $E_j\mathbf{x}_k \neq 0$. Since $\{E_j\}$ is an orthogonal family, $\mathscr{T}_k$ is denumerable for each k. If $E_j\mathbf{x}_k = 0$, then $0 = (\mathscr{R} \otimes I)E_j\mathbf{x}_k = E_j[(\mathscr{R} \otimes I)\mathbf{x}_k] = E_jF_k$. Since $E_j \neq 0$, $E_j \leq P$, and $\sum F_k = P$; $E_j\mathbf{x}_k \neq 0$, for some k. Thus, each j lies in some $\mathscr{T}_k$. Now, the cardinality of $\{E_j\}$ is d', so that $d' \leq a\aleph_0$. Thus $d' \leq a$, since $d' \geq \aleph_1$; and $d' = a$. It follows that $(\mathscr{R} \otimes I)'$ has coupling character d'; and from [9; Lemma 4.1.7], that the automorphism $A \otimes I \to \alpha(A) \otimes I$ of $\mathscr{R} \otimes I$ is unitarily induced.

If $d \leq \aleph_0$, let P be the maximal finite central projection in $\mathscr{R}$, and take d' to be $\aleph_0$. Then $\mathscr{K}$ is separable and $(\mathscr{R} \otimes I)'$ is purely infinite (it contains $I \otimes \mathscr{B}(\mathscr{H}')$). If $P \neq 0$, $(\mathscr{R} \otimes I)'(P \otimes I)$ (with finite commutant, $(\mathscr{R}P) \otimes I$) has coupling character $\aleph_0$ (each projection cyclic under $(\mathscr{R}P) \otimes I$ is finite [9; Lemma 3.3.3], while $(\mathscr{R} \otimes I)'(P \otimes I)$ is purely infinite). Since $\alpha(P) = P$, $(AP) \otimes I \to (\alpha(A)P) \otimes I$ is unitarily induced [9; Lemma 4.1.7].

If $P \neq I$, then $\mathscr{R}(I - P)$ and $(\mathscr{R}(I - P)) \otimes I$ are purely infinite. Since $\mathscr{K}$ is separable, $(\mathscr{R}(I - P)) \otimes I$ has a cyclic vector [9; Lemmas 3.3.6 and 3.3.3]; so that $((\mathscr{R}(I - P)) \otimes I)'$, which is purely infinite, has coupling character 1. Again, $(A(I - P)) \otimes I \to (\alpha(A)(I - P)) \otimes I$ is unitarily induced; and $A \otimes I \to \alpha(A) \otimes I$ is unitarily induced.

Proof of Theorem (3.6). From Theorem (3.4), the dynamical group $t \to v_t$ is induced by a weakly-continuous, one-parameter group $t \to \alpha_t$ of $*$ automorphisms of $\mathfrak{A}$. Suppose that the faithful representation ϕ is by operators on the d-dimensional Hilbert space $\mathscr{H}$. Let $\mathscr{H}'$ and $\mathscr{K}(= \mathscr{H} \otimes \mathscr{H}')$ be as in Lemma (4.12); and let $\psi(A) = \phi(A) \otimes I$, for each A in $\mathfrak{A}$. Now, the mapping $B \to B \otimes I$ is an algebraic $*$ isomorphism of $\mathscr{B}(\mathscr{H})$ into $\mathscr{B}(\mathscr{K})$ which is weakly and strongly-continuous on bounded subsets of $\mathscr{B}(\mathscr{H})$. Since the normal states of a von Neumann algebra are those which are strongly (or weakly) continuous on its unit ball [3; Théorème 1, p. 54], the normal states of $\phi(\mathfrak{A})^-$ and $\psi(\mathfrak{A})^-$ coincide under the mapping $B \to B \otimes I$. Thus ψ is a complete faithful (separable) representation of the dynamical system $(\mathfrak{A}, S_0, t \to v_t)$ with the same normal states as ϕ. Moreover, the automorphism $\psi\alpha_t\psi^{-1}$ of $\psi(\mathfrak{A})$ is extendable to a $*$ automorphism of $\psi(\mathfrak{A})^-$, since $\psi\alpha_t\psi^{-1}$ transforms the set of vector states of $\psi(\mathfrak{A})^-$ onto itself [9; Remark 2.2.3], by virtue of the fact that ψ is a representation of the dynamical system $(\mathfrak{A}, S_0, t \to v_t)$. From Lemma (4.12), this extension of $\psi\alpha_t\psi^{-1}$ is implemented by a unitary operator U_t on $\mathscr{K}$; and the theorem follows.

The proof of Theorem (3.8) will be reduced to a question about a representation of a group by (inner) $*$ automorphisms of a von Neumann algebra $\mathscr{R}$ and, then, concluded in Theorem (4.13). The representation, in question, is continuous relative to the bounded-weak (operator) topology (which we abbreviate as *bounded-weak* topology) on mappings of $\mathscr{R}$ into itself. A typical subbasic open set for this topology consists of all those mappings which take a given bounded set in $\mathscr{R}$ into a given weak (operator) open set in $\mathscr{R}$ (a typical bounded-weak open neighborhood of α' is $\{\alpha : |([\alpha(A) - \alpha'(A)]\mathbf{x}, \mathbf{x})| < 1$, for all A in the unit ball of $\mathscr{R}\}$).

Proof of Theorem (3.8). Since ϕ is faithful and the direct sum of factor representations, these factor representations form a separating family for $(\mathfrak{A}, S_0)$. Theorem (3.4) now guarantees that there is a one-parameter group $t \to \alpha_t$ of $*$ automorphisms of $\mathfrak{A}$ such that $[v_t(\rho)](A) = \rho(\alpha_t(A))$, for all A in $\mathfrak{A}$ and ρ in S_0. If E_α is a projection in $\phi(\mathfrak{A})'$ such that $\phi(\mathfrak{A})^- E_\alpha$ (acting on E_α) is a factor, then the central carrier of E_α is a minimal projection in the center of $\phi(\mathfrak{A})^-$; for each subprojection of this carrier in the center of $\phi(\mathfrak{A})^-$ lies in the center of the factor $\phi(\mathfrak{A})^- E_\alpha$. Thus, the center of $\phi(\mathfrak{A})^-$ is totally atomic (i.e. generated by its minimal projections). From Theorem (3.6), there is a complete, faithful, separable representation ψ of $(\mathfrak{A}, S_0, t \to v_t)$ with the same normal states as ϕ, such that the $*$ automorphism $\psi\alpha_t\psi^{-1}$ has an extension β_t to $\psi(\mathfrak{A})^-$ which is implemented by a unitary operator on the representation space $\mathscr{K}$ of ψ. This unitary operator can be chosen in $\psi(\mathfrak{A})^-$, from the hypothesis that $t \to v_t$ is inner relative to ϕ. Since $\psi(A) = \phi(A) \otimes I$, $\phi(\mathfrak{A})^-$ and $\psi(\mathfrak{A})^-$ are $*$ isomorphic. Thus the center of $\psi(\mathfrak{A})^-$ is totally atomic.

By assumption, $t \to v_t(\omega_{\mathbf{x}}\phi)$ is continuous relative to the norm topology on S_0, for each unit vector $\mathbf{x}$ in the representation space of ϕ. The same is true for each finite convex combination of vector states, i.e. $t \to v_t(\omega)$ is norm continuous, for each ω in S_ϕ. The weakly-continuous states of $\phi(\mathfrak{A})^-$ are norm dense in the set of normal states [3, Theorem 1, p. 54]; so that S_ϕ is norm dense in the normal states of ϕ. Since ϕ is complete its normal states lie in S_0. Moreover, each v_t is an isometry on S_0, since it is implemented by α_t, an isometric linear isomorphism of $\mathfrak{A}$ onto itself [6]. Thus $t \to v_t(\omega)$ is norm continuous for each normal state ω of ϕ (by a "three ε argument"). Since ϕ and ψ have the same normal states, it follows, in particular, that $t \to v_t(\omega_{\mathbf{y}}\psi)$ is norm continuous, for each unit vector $\mathbf{y}$ in $\mathscr{K}$. Thus $t \to \omega_{\mathbf{y}}\psi\alpha_t = \omega_{\mathbf{y}}\psi\alpha_t\psi^{-1}\psi = \omega_{\mathbf{y}}\beta_t\psi$ is norm continuous. Again, ψ being a $*$ isomorphism, the mapping $\rho \to \rho\psi$ of the continuous dual of $\psi(\mathfrak{A})$ onto that of $\mathfrak{A}$ is an isometry; so that $t \to \omega_{\mathbf{y}}\beta_t$ is norm continuous. It follows that $t \to \beta_t$ is a bounded-weak continuous, one-parameter group of inner $*$ automorphisms of the von Neumann algebra $\psi(\mathfrak{A})^-$ with totally-atomic center acting on a separable Hilbert space. We complete the proof with the aid of the theorem which follows. It establishes that such groups (and more general groups) of $*$ automorphisms of such von Neumann algebras are induced by strongly-continuous, unitary groups.

THEOREM (4.13). *If the topological group G is a simply-connected, compact or semi-simple Lie group or the additive group of real numbers and ϕ is a bounded-weak continuous representation of G by inner $*$ automorphisms of a von Neumann algebra $\mathscr{R}$ with totally-atomic center $\mathscr{C}$ acting on a separable Hilbert space $\mathscr{H}$, then there is a strongly-continuous, unitary representation $g \to U_g$ of G by operators in $\mathscr{R}$ such that $\phi(g)(A) = U_g A U_g^*$, for each A in $\mathscr{R}$ and each g in G.*

Proof. Let $\{P_j\}$ be the family of minimal projections in $\mathscr{C}$. By hypothesis $\Sigma_j P_j = I$; so that $\mathscr{H} = \Sigma \oplus \mathscr{H}_j$, where $\mathscr{H}_j = P_j(\mathscr{H})$. Since the center of $\mathscr{R}_j$ $(= \mathscr{R}P_j)$ is $\mathscr{C}P_j = \{bP_j : b$ a scalar$\}$; $\mathscr{R}_j$ (acting on $\mathscr{H}_j$) is a factor. Again, by hypothesis, there is a unitary operator V_g in $\mathscr{R}$ such that $\phi(g)(A) = V_g A V_g^*$. Since $P_j V_g = V_g P_j$, $V_g P_j$ is a unitary operator (in

$\mathscr{R}_j$) and $\phi(g)(\mathscr{R}_j) = \mathscr{R}_j$. Moreover, this mapping of G into inner $*$ automorphisms of $\mathscr{R}_j$ is a bounded-weak continuous representation of G. If we establish our theorem in the case where $\mathscr{R}$ is a factor, we will have a strongly-continuous, unitary representation $g \to U_g^{(j)}$ of G by operators in $\mathscr{R}_j$ such that $\phi(g)(A) = U_g^{(j)} A U_g^{(j)*}$, for each A in $\mathscr{R}_j$ and g in G. The direct sum $g \to U_g$ of the representations $g \to U_g^{(j)}$ has the desired properties.

We assume that $\mathscr{R}$ is a factor acting on the separable Hilbert space $\mathscr{H}$. The weak and strong topologies on the family of bounded operators agree on $\mathscr{U}$, the unitary group in this family (cf. [27; p. 3], for example) and provide it with the structure of a topological group. According to [4; Lemme 4], this topology on $\mathscr{U}$ is given by a metric in which it is a complete and separable (countable dense subset) space. Since $\mathscr{R}_u$, the group of unitary operators in $\mathscr{R}$, is the intersection of a weakly-closed set, $\mathscr{R}$, with $\mathscr{U}$, the same is true for $\mathscr{R}_u$. Now, [4; Lemme 3] establishes that if $\mathscr{U}_0$ is a closed subgroup of $\mathscr{R}_u$ there is a Borel subset $\mathscr{B}$ of $\mathscr{R}_u$ which meets each left coset of $\mathscr{U}_0$ in one and only one point ($\mathscr{B}$ is a Borel cross section for the canonical mapping η of $\mathscr{R}_u$ onto $\mathscr{R}_u/\mathscr{U}_0$). By changing a single point, if necessary, we may assume I is in $\mathscr{B}$.

In particular, the foregoing applies to the case where $\mathscr{U}_0$ is taken to be the center $\mathscr{C}_0$ $(= \{aI : |a| = 1\})$ of $\mathscr{R}_u$. Since $\mathscr{R}_u/\mathscr{C}_0$ is non-denumerable, $\mathscr{B}$ is non-denumerable and, so, Borel isomorphic with the unit interval [15; Théorème 2, p. 358]. Now, $\mathscr{R}_u$ is a complete separable metric topological group relative to the appropriate metric ($s(U, V) = \Sigma_m \|(U - V)\mathbf{x}_m\|/2^m\|\mathbf{x}_m\|$, where $\{\mathbf{x}_m\}$ is a countable dense subset of $\mathscr{H}$) with associated topology the strong (and weak) operator topology on $\mathscr{R}_u$. Hence $\mathscr{R}_u/\mathscr{C}_0$ is a separable topological group admitting the compatible metric, $s_0(U\mathscr{C}_0, V\mathscr{C}_0) = \inf\{s(U, aV) : |a| = 1\}$, relative to which it is complete. The canonical homomorphism η of $\mathscr{R}_u$ onto $\mathscr{R}_u/\mathscr{C}_0$ is continuous and bijective between $\mathscr{B}$ and $\mathscr{R}_u/\mathscr{C}_0$. From [14; Théorème 1, p. 253], η_0 $(= \eta|\mathscr{B})$ is a Borel isomorphism of $\mathscr{B}$ onto $\mathscr{R}_u/\mathscr{C}_0$.

The mapping τ' of $\mathscr{R}_u$ onto the group $\iota(\mathscr{R})$ of inner $*$ automorphisms of $\mathscr{R}$ defined by, $\tau'(U)(A) = UAU^*$, is a homomorphism with kernel $\mathscr{C}_0$. In fact, if $\tau'(U)(A) = U^*AU = A$, for each A in $\mathscr{R}$, then $U \in \mathscr{R}_u \cap \mathscr{R}_{u'} = \mathscr{C}_0$. Thus τ' induces an isomorphism τ of $\mathscr{R}_u/\mathscr{C}_0$ onto $\iota(\mathscr{R})$. We note that τ' is a continuous mapping when $\mathscr{R}$ is considered in its weak operator topology and $\iota(\mathscr{R})$ in the associated bounded-weak topology, so that τ is continuous (η being open). With $\mathbf{x}$ a unit vector in $\mathscr{H}$ and $\varepsilon > 0$ given, $|((U^*AU - V^*AV)\mathbf{x}, \mathbf{x})| \le |(U^*A(U - V)\mathbf{x}, \mathbf{x})| + |((U^* - V^*)AV\mathbf{x}, \mathbf{x})| \le 2\|(U - V)\mathbf{x}\| < \varepsilon$, for each A in the unit ball of $\mathscr{R}$, from the Cauchy–Schwarz Inequality, provided $\|(U - V)\mathbf{x}\| < \varepsilon/2$. We show that the bounded-weak topology on $\iota(\mathscr{R})$ is induced by the metric $d(\alpha, \alpha') = \sup\{\Sigma_m|((\alpha - \alpha')(A)\mathbf{x}_m, \mathbf{x}_m)|/2^m\|\mathbf{x}_m\|^2 : A$ in the unit ball of $\mathscr{R}\}$; so that $\iota(\mathscr{R})$ is a separable metrizable space relative to this topology. Note that $d(\alpha, \alpha') \le 2$, since $*$ automorphisms are norm preserving. If $\mathbf{x}$, a unit vector in $\mathscr{H}$, and $\varepsilon > 0$ are given, choose $\mathbf{x}_m$ so that $\|\mathbf{x} - \mathbf{x}_m\| < \varepsilon/8$. If $d(\alpha, \alpha') < \varepsilon/2^{m+2}\|\mathbf{x}_m\|^2$, then $|((\alpha - \alpha')(A)\mathbf{x}_m, \mathbf{x}_m)| < \varepsilon/4$, for each A in the unit ball of $\mathscr{R}$; so that $|((\alpha - \alpha')(A)\mathbf{x}, \mathbf{x})| = |\omega_{\mathbf{x}}((\alpha - \alpha')(A))| \le |(\omega_{\mathbf{x}} - \omega_{\mathbf{x}_m})[(\alpha - \alpha')(A)]| + |\omega_{\mathbf{x}_m}[(\alpha - \alpha')(A)]| < 2(\|\mathbf{x}\| + \|\mathbf{x}_m\|)\|\mathbf{x} - \mathbf{x}_m\| + \varepsilon/4 < \varepsilon$, for each such A. On the other hand, if $\varepsilon > 0$ is given, and α, α' in $\iota(\mathscr{R})$ are such that $|((\alpha - \alpha')(A)\mathbf{x}_j, \mathbf{x}_j)| < 2^j\|\mathbf{x}_j\|^2\varepsilon/2(m+1)$, for $j = 1, \ldots, m+1$, and all A in the unit ball of $\mathscr{R}$; where $2^{-m} < \varepsilon/2$; then $d(\alpha, \alpha') < \varepsilon$. Since τ is continuous and bijective, its image, $\iota(\mathscr{R})$, is a Borel set in its completion (relative to d) [14; p. 253 (1)] and τ is a Borel isomorphism. We conclude that $\eta_0^{-1}\tau^{-1}\phi$ $(= \theta)$ is a Borel mapping of G into $\mathscr{B}$.

Since $\theta(g)A\theta(g)^* = \phi(g)(A)$ for each g in G and A in $\mathscr{R}$, and since ϕ is a homomorphism of G into $\iota(\mathscr{R})$; $\theta(g_1 g_2)^{-1}\theta(g_1)\theta(g_2)$ $(= \gamma(g_1, g_2)I)$ is in $\mathscr{C}_0$. A straightforward computation, using the fact that $\theta(g_2)^{-1}\theta(g_1)^{-1}\theta(g_1 g_2)$ is in the center of $\mathscr{R}_u$, shows that $\gamma(g_2, g_3)\gamma(g_1 g_2, g_3)^{-1}\gamma(g_1, g_2 g_3)\gamma(g_1, g_2)^{-1} = 1$; so that γ is a 2-cocycle as a 2-cochain on G with coefficients in the circle group T_1 (trivial action on T_1) in the usual cohomology theory of groups [5].

Since θ is a Borel mapping, γ is a Borel mapping of $G \times G$ into T_1. In these circumstances, [17; Théorème 2] shows that the standard abstract group extension E (constructed by providing the Cartesian product $T_1 \times G$ with the multiplication $(a_1, g_1)(a_2, g_2) = (a_1 a_2 \gamma(g_1, g_2), g_1 g_2))$ of T_1 by G associated with γ has a unique locally compact separable topology relative to which there is a continuous (open) homomorphism of E onto G with (closed) kernel (topologically group isomorphic to) T_1. Now, [24; Theorem 1], [7; Theorem 4.4 or 1; Corollaire 2, p. 348] and [31; Lemma 3.4] tell us that when G is a simply-connected, compact or semi-simple Lie group or the additive group of real numbers the extension E of T_1 by G splits (i.e. E is the topological group direct product of T_1 and G relative to the given mapping of E onto G). Thus there is a continuous homomorphism $g \to (\xi(g)^{-1}, g)$ of G into E (the particular form for this homomorphism follows from the additional consequence of the splitting that composing the homomorphism with $(a, g) \to g$ yields the identity transform on G). The multiplication described on E together with the information that $g \to (\xi(g)^{-1}, g)$ is a homomorphism establishes $\gamma(g_1, g_2) = \xi(g_2)\xi(g_1 g_2)^{-1}\xi(g_1)$, for all g_1 and g_2 in G (i.e. γ is the coboundary of ξ). Now, [17; Théorème 2] tells us that the mapping of $T_1 \times G$ onto E (with its locally compact topology) is a Borel mapping; and, since the homomorphism of G into E is continuous, ξ is a Borel mapping of G into T_1.

It follows that $g \to U_g = \xi(g)^{-1}\theta(g)$ is a Borel mapping of G into $\mathscr{R}_u$. Moreover, $U_{g_1 g_2} = \xi(g_1 g_2)^{-1}\theta(g_1 g_2) = \gamma(g_1, g_2)\xi(g_2)^{-1}\xi(g_1)^{-1}\theta(g_1)\theta(g_2)\gamma(g_1, g_2)^{-1} = U_{g_1}U_{g_2}$, so that $g \to U_g$ is a Borel group representation of G. From [30, p. 67], $g \to U_g$ is a strongly-continuous, unitary representation of G, and by construction, it is by means of operators in $\mathscr{R}_u$ and gives rise to the representation ϕ of G in $\iota(\mathscr{R})$.

Remark (4.14). Since each $*$ automorphism of a type I von Neumann algebra which acts as the identity on the center is inner [13], the hypothesis that the representation ϕ of Theorem (4.13) is by inner automorphisms is automatically fulfilled in case they act as the identity on the center and $\mathscr{R}$ is of type I. Hence, if ϕ in Theorem (3.8) is a type I representation and the dynamical group acts as the identity on its center, it is generated by a Hamiltonian.

We conclude with a remark on representations of groups by $*$ automorphisms of a factor of type II_1, with coupling 1. The topological hypothesis may be added and the strong-continuity conclusion drawn very much as in Theorem (4.13). There is a corresponding corollary relating to dynamical systems.

Remark (4.15). If ϕ is a representation of a group G by $*$ automorphisms of a factor $\mathscr{R}$ of type II_1 with coupling 1 acting on a (separable) Hilbert space $\mathscr{H}$, then there is a unitary representation $g \to U_g$ of G on $\mathscr{H}$ such that $\phi(g)(A) = U_g A U_g^*$, for each A in $\mathscr{R}$. In fact, in this case, there is some unitary operator V_g on $\mathscr{H}$, such that $\phi(g)(A) = V_g A V_g^*$, for all A in $\mathscr{R}$. Let $\mathbf{x}_0$ be a trace vector for $\mathscr{R}$ and $\mathscr{R}'$. Then $V_g\mathbf{x}_0$ is a trace vector for $\mathscr{R}$ and $\mathscr{R}'$; so that the mapping $AV_g\mathbf{x}_0 \to A\mathbf{x}_0$ extends to a unitary operator V_g' in $\mathscr{R}'$. Thus $U_g\mathbf{x}_0 = \mathbf{x}_0$, where $U_g = V_g'V_g$ and $\phi(g)(A) = U_g A U_g^*$, for all A in $\mathscr{R}$. Now $U_g U_{g'} U_{gg'}^*$ lies in $\mathscr{R}'$ since it induces the identity automorphism on $\mathscr{R}$ (recall that ϕ is a representation of G). Since $U_g U_{g'} U_{gg'}^* \mathbf{x}_0 = \mathbf{x}_0$ and $\mathbf{x}_0$ is separating for $\mathscr{R}'$, $U_g U_{g'} U_{gg'}^* = I$; and $g \to U_g$ is a unitary representation of G.

REFERENCES

1. L. Calabi: Sur les extensions des groupes topologiques, *Ann. Mat. Pura Appl.* **32** (1951), 295–370.
2. J. Dieudonné: La dualité dans les espaces vectoriels topologiques, *Ann. Sci. École Norm. Sup.* **59** (1942), 107–139.
3. J. Dixmier: *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.
4. J. Dixmier: Dual et quasi-dual d'une algebre de Banach involutive, *Trans. Amer. Math. Soc.* **104** (1962), 278–283.
5. S. Eilenberg and S. MacLane: Cohomology theory in abstract groups I, *Ann. Math., Princeton* **48** (1947), 51–78.
6. I. Gelfand and M. Neumark: On the imbedding of normed rings into the ring of operators in Hilbert space, *Rec. Math. (mat. Sbornik) N.S.* **12** (1943), 197–213.
7. G. Hochschild: Group extensions of Lie groups, *Ann. Math., Princeton* **54** (1951), 96–109.
8. R. Kadison: Isometries of operator algebras, *Ann. Math., Princeton* **54** (1951), 325–338.
9. R. Kadison: Unitary invariants for representations of operator algebras, *Ann. Math., Princeton* **66** (1957), 304–379.
10. R. Kadison: Irreducible operator algebras, *Proc. Nat. Acad. Sci., Wash.* **43** (1957), 273–276.
11. R. Kadison: A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. of Math.* **56** (1952), 494–503.
12. I. Kaplansky: A theorem on rings of operators, *Pacif. J. Math.* **1** (1951), 227–232.
13. I. Kaplansky: Algebras of type I, *Ann. Math., Princeton* **56** (1952), 460–472.
14. C. Kuratowski: *Topologie I*, Warsaw, 1933.
15. C. Kuratowski: *Topologie I* (2nd edition), Warsaw, 1948.
16. G. Mackey: Lecture notes on the mathematical foundations of quantum mechanics, Harvard notes (mimeographed), 1960.
17. G. Mackey: Les ensembles boréliens et les extensions des groupes, *J. Math. Pures Appl.* **36** (1957), 171–178.
18. F. Murray and J. von Neumann: Rings of operators, *Ann. Math., Princeton* **37** (1936), 116–229.
19. R. Phillips: Integration in a convex linear topological space, *Trans. Amer. Math. Soc.* **47** (1940), 114–145.
20. I. Segal: Two-sided ideals in operator algebras, *Ann. Math., Princeton* **50** (1949), 856–865.
21. I. Segal: Irreducible representations of operator algebras, *Bull. Amer. Math. Soc.* **53** (1947), 73–88.
22. I. Segal: Postulates for general quantum mechanics, *Ann. Math., Princeton* **48** (1947), 930–948.
23. I. Segal: A class of operator algebras which are determined by groups, *Duke Math. J.* **18** (1951), 221–265.
24. A. Shapiro: Group extensions of compact Lie groups, *Ann. Math., Princeton* **50** (1949), 581–586.
25. S. Sherman: The second adjoint of a C^* algebra, *Proc. Int. Cong. Math.* **1** (1950), 470.
26. M. Stone: On one parameter groups in Hilbert space, *Ann. Math., Princeton* **33** (1932), 643–648.
27. M. Stone: On the foundations of harmonic analysis, *K. fysiogr. Sällsk. Lund Förh.* **21** No. 17 (1952), 155–192.
28. Z. Takeda: Conjugate spaces of operator algebras, *Proc. Jap. Acad.* **30** (1954), 90–95.
29. M. Takesaki: On the conjugate space of operator algebra, *Tôhoku Math. J.* **10** (1958), 194–203.
30. A. Weil: *L'intégration dans les groupes topologiques et ses applications*, Hermann et C[ie], Paris, 1940.
31. K. Iwasawa: On some types of topological groups, *Ann. Math., Princeton* **50** (1949), 507–558.

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