

NORMALCY IN OPERATOR ALGEBRAS

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1. Introduction. In [9, Theorem 5; 393], von Neumann proves that each weakly (equivalently, strongly) closed self-adjoint algebra of operators on a Hilbert space (von Neumann algebra) which contains the identity operator I enjoys the double commutant property. If we denote the algebra of all those bounded operators commuting with a given family \mathfrak{F} of operators by \mathfrak{F}' (called the commutant of \mathfrak{F}), this result may be phrased as: $\mathfrak{A} = \mathfrak{A}''$, for von Neumann algebras \mathfrak{A} containing I . It is one of the key results of the theory.

This double commutant result expresses an algebraic property (called normalcy) of the algebra of all bounded operators. In the process of determining which properties of the algebra of bounded operators (factor of type I) are shared by the other types of factors (von Neumann algebras whose center consists of scalar multiples of I), Murray and von Neumann raise the question of which factors are normal [4; 183]. They construct factors of type II which are not normal [4, Lemma 3.4.2; 209] and conjecture that no factor of type II is normal [4; 185]. In [8, Lemma 4.4.2 (iii)], non-normal factors of type III are exhibited. It is shown that all factors of type II are non-normal in [2]. The question of normalcy for a factor can be phrased in terms of the fixed algebra under groups of unitarily induced (inner) automorphisms of the factor. We speculated on the possibility that the normalcy assertion phrased so as to allow *-automorphisms might be valid for all factors of type II_1 . Singer disproved this in [10; 126]. A Galois theory result of this nature is established in [6], [7] for finite groups of outer automorphisms of finite factors.

A known and easy extension of von Neumann's result states that each von Neumann algebra of type I is normal—where one now tests only those von Neumann subalgebras satisfying the obvious necessary condition that they contain the center (see [1; 307, Exercise 13b], for example). We say that a von Neumann subalgebra \mathfrak{A}_0 of a von Neumann algebra \mathfrak{A} is normal in \mathfrak{A} when it has the double commutant property relative to \mathfrak{A} , i.e. $(\mathfrak{A}_0' \cap \mathfrak{A})' \cap \mathfrak{A} = \mathfrak{A}_0$. We shall prove that if \mathfrak{A}_0 is of type I, it is normal in \mathfrak{A} if and only if its center is normal in \mathfrak{A} (cf. Theorem 1). Dealing, then, with Abelian von Neumann subalgebras of \mathfrak{A} , we give some (easy) equivalent conditions to normalcy in \mathfrak{A} (cf. Theorem 2). We show that those Abelian von Neumann subalgebras which are totally-atomic over the center of \mathfrak{A} are normal in \mathfrak{A} (cf. Definition 3 and Theorem 4). In the last section, we give an example of an Abelian von Neumann subalgebra of a factor of type II_1 which is not normal in it.

Received September 3, 1961; presented to the American Mathematical Society January 22, 1962. This research was done while the author was an Alfred P. Sloan Research Fellow.

2. The normalcy results. Our main theorem follows.

THEOREM 1. *If \mathfrak{R}_0 is a von Neumann subalgebra of type I of the von Neumann algebra \mathfrak{R} , then \mathfrak{R}_0 is normal in \mathfrak{R} if and only if its center \mathfrak{C}_0 is normal in \mathfrak{R} .*

Proof. We suppose, first, that \mathfrak{C}_0 is normal in \mathfrak{R} . Let B lie in $(\mathfrak{R}'_0 \cap \mathfrak{R})' \cap \mathfrak{R}$. Let $\{E_\alpha\}$ be an orthogonal family of Abelian projections [3; 241] in \mathfrak{R}_0 each with central carrier P and with $\sum E_\alpha = P$. Choose one of the E_α , say E_0 ; and let V_0 be E_0 and V_α be a partial isometry in \mathfrak{R}_0 with E_0 as initial space and E_α as final space. (Recall that Abelian projections with the same central carrier are equivalent.)

We assert that $\sum_\alpha V_\alpha B V_\alpha^* (= B_c)$ lies in $(\mathfrak{C}'_0 \cap \mathfrak{R})' \cap \mathfrak{R}$, and hence, by hypothesis, in \mathfrak{C}_0 . Suppose, for the moment, that this has been established. We may conclude, then, that $E_0 B_c E_0 = E_0 B E_0$ is in \mathfrak{R}_0 , for each Abelian projection E_0 in \mathfrak{R}_0 which lies in a family such as $\{E_\alpha\}$. Since \mathfrak{R}_0 is of type I, each Abelian projection in \mathfrak{R}_0 is the sum of an orthogonal family of projections such as E_0 (intersect the given Abelian projection with each of the central portions of \mathfrak{R}_0 of type I_n). Thus $E B E$ is in \mathfrak{R}_0 , for each Abelian projection E in \mathfrak{R}_0 , since $B \in \mathfrak{C}'_0$.

If E and F are Abelian projections in \mathfrak{R}_0 with central carriers P and Q , respectively, then $EBF = PQEBPQF$. Now PQE and PQF are Abelian projections in \mathfrak{R}_0 with the same central carrier. Thus, to show that EBF lies in \mathfrak{R}_0 , with E and F Abelian projections in \mathfrak{R}_0 , it will suffice to show this when E and F have the same central carrier. Under this assumption, there is a partial isometry V in \mathfrak{R}_0 with initial space E and final space F . We have, $EBF = EB V V^* = EB V E V^*$. Since V lies in \mathfrak{R}_0 , it lies in $(\mathfrak{R}'_0 \cap \mathfrak{R})' \cap \mathfrak{R}$ as does B ; so that BV lies in $(\mathfrak{R}'_0 \cap \mathfrak{R})' \cap \mathfrak{R}$. From the preceding paragraph, $EB V E$ lies in \mathfrak{R}_0 ; so that $EBF (= EB V E V^*)$ does. Since \mathfrak{R}_0 is of type I, it is generated by its Abelian projections [1, Theorem 1; 123]. Thus B lies in \mathfrak{R}_0 .

It remains to prove that B_c lies in \mathfrak{C}_0 . Suppose that S in \mathfrak{R} commutes with \mathfrak{C}_0 ; and $V_\alpha^* S V_\gamma$ commutes with $E_0 B E_0$, for each α and γ . Then,

$$\begin{aligned} B_c S &= B_c P S P = B_c \sum_{\alpha, \gamma} E_\alpha S E_\gamma = \sum_{\alpha, \gamma} V_\alpha B V_\alpha^* E_\alpha S E_\gamma \\ &= \sum_{\alpha, \gamma} V_\alpha E_0 B E_0 V_\alpha^* S V_\gamma V_\gamma^* = \sum_{\alpha, \gamma} V_\alpha V_\alpha^* S V_\gamma E_0 B E_0 V_\gamma^* \\ &= \sum_{\alpha, \gamma} E_\alpha S V_\gamma B V_\gamma^* = P S \sum_\gamma V_\gamma B V_\gamma^* = P S B_c = S B_c, \end{aligned}$$

where $\sum_{\alpha, \gamma}$ denotes $\sum_\alpha \sum_\gamma$ taken in the strong topology.

If S is in \mathfrak{R}_0 then $E_0 S E_0$ is in $E_0 \mathfrak{R}_0 E_0$, which is Abelian, since E_0 is an Abelian projection in \mathfrak{R}_0 . Now the center of $E_0 \mathfrak{R}_0 E_0$ is $E_0 \mathfrak{C}_0 E_0 = E_0 \mathfrak{C}_0$ [1, Corollary; 19]; which is, accordingly, $E_0 \mathfrak{R}_0 E_0$. Since B commutes with \mathfrak{C}_0 , $E_0 B E_0$ commutes with $E_0 \mathfrak{C}_0 E_0$, and, hence, with $E_0 S E_0$. In particular, $E_0 B E_0$ commutes with $V_\alpha^* S V_\gamma = E_0 V_\alpha^* S V_\gamma E_0$, since $V_\alpha^* S V_\gamma$ lies in \mathfrak{R}_0 . Thus B_c commutes with \mathfrak{R}_0 as a consequence of B commuting with \mathfrak{C}_0 .

Assume now that T lies in \mathfrak{R} and commutes with \mathfrak{C}_0 . From the foregoing, T_c lies in \mathfrak{R} and commutes with \mathfrak{R}_0 . By hypothesis on B , T_c and B commute. Now $(V_\alpha^*TV_\gamma)_c$ and B commute, since $V_\alpha^*TV_\gamma$ lies in \mathfrak{R} and commutes with \mathfrak{C}_0 . Thus

$$\begin{aligned} E_0BE_0V_\alpha^*TV_\gamma &= E_0BV_\alpha^*TV_\gamma E_0 = E_0B(V_\alpha^*TV_\gamma)_cE_0 \\ &= E_0(V_\alpha^*TV_\gamma)_cBE_0 = V_\alpha^*TV_\gamma E_0BE_0. \end{aligned}$$

Since E_0BE_0 commutes with $V_\alpha^*TV_\gamma$, for each α and γ , and T commutes with \mathfrak{C}_0 , T commutes with B_c . Thus B_c commutes with each operator in \mathfrak{R} which commutes with \mathfrak{C}_0 . By hypothesis, B_c lies in \mathfrak{C}_0 ; and the proof of the sufficiency is complete.

If \mathfrak{R}_0 is normal in \mathfrak{R} and B lies in $(\mathfrak{C}'_0 \cap \mathfrak{R})' \cap \mathfrak{R}$, then B commutes with $\mathfrak{R}'_0 \cap \mathfrak{R} (\subseteq \mathfrak{C}'_0 \cap \mathfrak{R})$; so that B lies in \mathfrak{R}_0 . But since B lies in \mathfrak{R}'_0 (as $\mathfrak{R}_0 \subseteq \mathfrak{C}'_0 \cap \mathfrak{R}$), B lies in \mathfrak{C}_0 ; and \mathfrak{C}_0 is normal in \mathfrak{R} .

In view of the foregoing theorem, the question of normalcy of type I von Neumann subalgebras is reduced to the question of normalcy of Abelian von Neumann subalgebras. We take up this question.

THEOREM 2. *If \mathfrak{R} is a von Neumann algebra and \mathfrak{A} is an Abelian von Neumann subalgebra of \mathfrak{R} , the following statements are equivalent:*

- (a) \mathfrak{A} is normal in \mathfrak{R} .
- (b) \mathfrak{A} is the intersection of maximal Abelian (self-adjoint) subalgebras of \mathfrak{R} .
- (c) \mathfrak{A} is the intersection of all maximal Abelian subalgebras of \mathfrak{R} containing \mathfrak{A} .
- (d) The von Neumann algebra \mathfrak{R}_1 generated by \mathfrak{R}' and \mathfrak{A} has \mathfrak{A} as center.

In case the space upon which \mathfrak{R} acts is separable, the following may be added:

- (e) The direct integral reduction of \mathfrak{R}' relative to \mathfrak{A} yields factor representations almost everywhere.

Proof. (a) \rightarrow (b): If \mathfrak{A} is normal in \mathfrak{R} and B in \mathfrak{R} is not in \mathfrak{A} , then there is some operator C in \mathfrak{R} which commutes with \mathfrak{A} but not with B . Since \mathfrak{A} is a self-adjoint algebra, both $C + C^*$ and $i(C - C^*)$ commute with \mathfrak{A} , and at least one does not commute with B . We may assume that C is self-adjoint.

With the aid of Zorn's lemma, we choose a maximal Abelian subalgebra \mathfrak{A}_0 of \mathfrak{R} containing \mathfrak{A} and C . Since B and C do not commute, B is not in \mathfrak{A}_0 . Thus \mathfrak{A} is the intersection of maximal Abelian subalgebras of \mathfrak{R} . The intersection of all the maximal Abelian subalgebras of \mathfrak{R} containing \mathfrak{A} will be contained in this last intersection and contain \mathfrak{A} ; whence (b) \rightarrow (c).

(c) \rightarrow (d): Clearly \mathfrak{A} is contained in the center of \mathfrak{R}_1 . Suppose $\mathfrak{A} = \bigcap_\alpha \mathfrak{A}_\alpha$, where $\{\mathfrak{A}_\alpha\}$ is the family of maximal Abelian subalgebras of \mathfrak{R} containing \mathfrak{A} . If T is in the center of \mathfrak{R}_1 , then T commutes with \mathfrak{R}' . Thus T lies in \mathfrak{R} . Now each \mathfrak{A}_α commutes with \mathfrak{A} and with \mathfrak{R}' ; and, therefore, is contained in \mathfrak{R}'_1 . Since T is in the center of \mathfrak{R}_1 , T commutes with each \mathfrak{A}_α . But T lies in \mathfrak{R} , and each \mathfrak{A}_α is a maximal Abelian subalgebra of \mathfrak{R} . Thus T lies in $\bigcap_\alpha \mathfrak{A}_\alpha = \mathfrak{A}$; and \mathfrak{A} is the center of \mathfrak{R}_1 .

(d) \rightarrow (a): If \mathfrak{A} is the center of \mathfrak{R}_1 and T in \mathfrak{R} commutes with all operators in \mathfrak{R} which commute with \mathfrak{A} , then T commutes with \mathfrak{A} (since \mathfrak{A} is Abelian) and with \mathfrak{R}' (since T lies in \mathfrak{R}). Thus T lies in \mathfrak{R}'_1 . Now each operator in \mathfrak{R}'_1 commutes with \mathfrak{R}' , so lies in \mathfrak{R} ; and commutes with \mathfrak{A} . By hypothesis, then, each operator in \mathfrak{R}'_1 commutes with T . Thus T lies in the center \mathfrak{A} of \mathfrak{R}_1 ; and \mathfrak{A} is normal in \mathfrak{R} .

Suppose now that the Hilbert space upon which \mathfrak{R} acts is separable. We show that (d) and (e) are equivalent. According to [1, Theorem 2; 181], \mathfrak{R}_1 is decomposable relative to \mathfrak{A} . From [1, Theorem 3; 182], \mathfrak{A} is the center of \mathfrak{R}_1 if and only if the decomposition of \mathfrak{R}_1 relative to \mathfrak{A} yields a factor almost everywhere.

Of course, the maximal Abelian subalgebras of \mathfrak{R} are normal in \mathfrak{R} . These are precisely the subalgebras of \mathfrak{R} decomposition relative to which yield irreducible (special factor) representations almost everywhere. The preceding theorem identifies the normal Abelian subalgebras of \mathfrak{R} as those which yield arbitrary factor representations almost everywhere.

For an Abelian subalgebra of \mathfrak{R} to be normal in \mathfrak{R} , it is certainly necessary that it contain the center of \mathfrak{R} . This is not sufficient as we shall see by example in the next section. If the Abelian algebra is totally-atomic over the center, however, it is automatically normal in \mathfrak{R} . To make this concept precise and prove the stated result, we shall need the following:

DEFINITION 3. *If \mathfrak{C} is an Abelian von Neumann algebra and \mathfrak{R} is a von Neumann subalgebra of \mathfrak{C}' , we say that a non-zero projection E in \mathfrak{R} is minimal in \mathfrak{R} relative to \mathfrak{C} if no projection in \mathfrak{R} with the same central carrier as E in \mathfrak{C}' is properly less than E .*

Remarks. (a) If \mathfrak{C} consists of the scalar multiples of I , a minimal projection in \mathfrak{R} relative to \mathfrak{C} is just a minimal projection in \mathfrak{R} in the usual sense. Note also that each projection in \mathfrak{C} is a minimal projection relative to \mathfrak{C} .

(b) If \mathfrak{R} contains \mathfrak{C} , E is minimal in \mathfrak{R} relative to \mathfrak{C} , and P is a projection in \mathfrak{C} , then PE is 0 or is minimal in \mathfrak{R} relative to \mathfrak{C} . In fact, if $PE \neq 0$, has central carrier Q in \mathfrak{C}' , and F is a projection in \mathfrak{R} with central carrier Q in \mathfrak{C}' such that $F < PE$; then $F + (I - P)E$ has the same central carrier as E in \mathfrak{C}' and $F + (I - P)E < E$. This contradicts the minimality of E in \mathfrak{R} relative to \mathfrak{C} ; so that $PE = 0$ or PE is minimal in \mathfrak{R} relative to \mathfrak{C} .

(c) If \mathfrak{R} contains \mathfrak{C} and E and F are commuting minimal projections in \mathfrak{R} relative to \mathfrak{C} , then $EF = PE = PF$, where P is some projection in \mathfrak{C} . In fact, take P to be the central carrier of EF in \mathfrak{C}' . Then P is contained in the central carriers of both E and F . Thus both PE and PF have P as central carrier. Now PE and PF are both minimal in \mathfrak{R} relative to \mathfrak{C} , from the preceding remark; and both contain EF . But EF has central carrier P ; so that $EF = PE = PF$, by relative minimality of PE and PF .

THEOREM 4. *If \mathfrak{R} is a von Neumann algebra with center \mathfrak{C} and \mathfrak{A} is an Abelian*

von Neumann subalgebra of \mathfrak{A} which contains \mathfrak{C} and is generated by its minimal projections relative to \mathfrak{C} , then \mathfrak{A} is normal in \mathfrak{A} .

Proof. Let $\{E_\alpha\}$ be the set of minimal projections in \mathfrak{A} relative to \mathfrak{C} . Note that $E_\alpha B E_\alpha$ commutes with \mathfrak{A} for each B in \mathfrak{A} and each E_α . In fact, $E_\gamma E_\alpha = P E_\alpha = P E_\gamma$, for some projection P in \mathfrak{C} , from Remark (c). Thus $E_\gamma E_\alpha B E_\alpha = P E_\alpha B E_\alpha = E_\alpha B E_\alpha P = E_\alpha B E_\alpha E_\gamma$; and $E_\alpha B E_\alpha$ commutes with generators for \mathfrak{A} . If T lies in $(\mathfrak{A}' \cap \mathfrak{R})' \cap \mathfrak{A}$ then T commutes with $E_\alpha \mathfrak{A} E_\alpha$, as does $E_\alpha T E_\alpha$. Thus $E_\alpha T E_\alpha (= T E_\alpha)$ lies in the center $\mathfrak{C} E_\alpha$ of $E_\alpha \mathfrak{A} E_\alpha$. In particular, $T E_\alpha$ lies in \mathfrak{A} , for each E_α . The union of $\{E_\alpha\}$ is I , since any projection orthogonal to all E_α annihilates \mathfrak{A} and, in particular, I . Thus finite unions from $\{E_\alpha\}$ form a net which tends strongly to I . Since the union of two commuting projections E and F is $E + F - EF$, an elementary induction shows that the product of T and a finite union of E_α 's lies in \mathfrak{A} . It follows that T is a strong limit of operators in \mathfrak{A} ; and T lies in \mathfrak{A} . Thus \mathfrak{A} is normal in \mathfrak{A} .

We note especially that when \mathfrak{A} is a factor the above hypothesis on \mathfrak{A} implies that it is generated by its minimal projections.

3. An example. We describe an example of a factor of type II_1 and an Abelian von Neumann subalgebra of it which is not normal in it.

Let G be the free group on two generators a and b ; let \mathfrak{H} be $l_2(G)$; and let \mathfrak{M} be the von Neumann algebra generated by the "left translation operators" U_g on $l_2(G)$ defined by, $(U_g f)(g') = f(g^{-1}g')$. From [5, Lemma 6.2.2], \mathfrak{M} is a factor of type II_1 and each operator in \mathfrak{M} has the form $\sum_{g \text{ in } G} \lambda_g U_g$, where $\sum |\lambda_g|^2 < \infty$ and the operator sum converges in the strong operator topology independently of order of summation. Moreover, the result of applying addition and multiplication to these infinite sums with the usual formalism yields an infinite sum which represents the sum and product operators, respectively. The representation of operators in \mathfrak{M} in terms of these infinite sums is unique except for order.

Our example is based on the following:

LEMMA 5. *If $A (= \sum \lambda_g U_g)$ in \mathfrak{M} commutes with U_{a^n} , then $\lambda_g = 0$ unless $g = a^n$, for some integer n . The (Abelian) von Neumann algebra \mathfrak{A}_2 generated by U_{a^n} is distinct from the (Abelian) von Neumann algebra \mathfrak{A}_1 generated by U_a ; and \mathfrak{A}_1 is maximal Abelian in \mathfrak{M} .*

Proof. If U_{a^n} commutes with A , then $A = U_{a^n} A U_{a^{-n}} = \sum \lambda_g U_{a^n \cdot g \cdot a^{-n}} = \sum \lambda_{a^{-n} \cdot g \cdot a^n} U_g$. From uniqueness of the representation for A , we have $\lambda_g = \lambda_{a^{-n} \cdot g \cdot a^n}$, for each g in G . Thus $\lambda_g = \lambda_{a^{-m} \cdot g \cdot a^m}$, for each integer m and each g in G . Since $\sum |\lambda_g|^2$ is finite, $\lambda_g = 0$, or there are at most a finite number of distinct elements in $\{a^{-2m} g a^{2m}\}_{m=1,2,\dots}$. If this last holds, $a^{-2m} g a^{2m} = g$, for some $m \neq 0$. With g in the free group on the two generators a and b , this occurs if and only if $g = a^n$, for some integer n .

The elements $\sum \lambda_g U_g$ in \mathfrak{M} , with $\lambda_g = 0$ except when $g = a^n$, constitute the

Abelian von Neumann algebra \mathfrak{A}_1 . Now, from Zorn's lemma, \mathfrak{A}_2 is contained in some maximal Abelian von Neumann subalgebra. But the elements of \mathfrak{M} which commute with \mathfrak{A}_2 are precisely the operators in the Abelian von Neumann subalgebra \mathfrak{A}_1 . Thus \mathfrak{A}_1 is maximal Abelian in \mathfrak{M} and is the only maximal Abelian subalgebra of \mathfrak{M} containing \mathfrak{A}_2 .

To show that \mathfrak{A}_1 and \mathfrak{A}_2 are distinct, we observe that U_a in \mathfrak{A}_1 is not in \mathfrak{A}_2 . In fact, \mathfrak{A}_2 is the strong closure of the algebra of finite sums $B = \lambda_1 U_{a^{n_1}} + \cdots + \lambda_k U_{a^{n_k}}$, n_1, \dots, n_k integers. If x denotes the function in $l_2(G)$ which is 1 at the group identity and 0 elsewhere, then $\| (U_a - B)x \|^2 = 1 + \sum_{i=1}^k |\lambda_i|^2 \geq 1$. Thus $U_a \notin \mathfrak{A}_2$.

From the foregoing lemma, $\mathfrak{A}'_2 \cap \mathfrak{M} = \mathfrak{A}_1$ and $\mathfrak{A}'_1 \cap \mathfrak{M} = \mathfrak{A}_1$. Thus \mathfrak{A}_2 is not normal in \mathfrak{M} . With regard to Theorem 2(b), \mathfrak{A}_2 is contained in a unique maximal Abelian subalgebra \mathfrak{A}_1 of \mathfrak{M} and not equal to it.

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