TRIANGULAR OPERATOR ALGEBRAS.*

Fundamentals and Hyperreducible Theory.

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Chapter I. Introduction and Preliminaries.

1.1. Introduction. For the past three decades, the theory of self-adjoint operators and self-adjoint operator algebras has undergone a vigorous and moderately successful development. A large share of the credit for this moderate success must be given to the reasonably detailed theory of factors

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created by Murray and von Neumann [6], and to its later elaboration, the general theory of von Neumann algebras, by several mathematicians [1]. The von Neumann algebras are special enough so that incisive structural results can be established yet broad enough so that they can be related to the general self-adjoint operator algebras. The limitations which exist at present in the self-adjoint theory seem basically to reside in the special problems about factors which remain unanswered.

It is our hope that the theory we initiate in this tract will be capable of filling an analogous rôle in the study of non-self-adjoint operators and operator algebras. In finite dimensions, the class of operator algebras we study are the triangular operator algebras (the algebras of those matrices relative to given bases in given orderings with 0 entries below the diagonal). In general, the class of algebras we study is characterized by the simple property: $\mathcal{J}^* \cap \mathcal{J}$ is maximal abelian. We call such algebras "triangular"—in finite dimensions, they are subalgebras of the triangular matrices; with a maximality assumption, they are full algebras of triangular matrices. In infinite dimensions one would not expect a "classical" basis to be related to a given maximal triangular algebra in general. The "continuous" as well as the "discrete" appears in infinite dimensions. Even when this is taken into account, however, a large section of the theory must concern itself with maximal triangular algebras to which no ordered basis (in the appropriately general sense) can be said to be associated. Of course the ordering of the basis has a much more critical position in the infinite-dimensional theory than in the finite-dimensional theory (based mainly on the fact that there is just one total-ordering type associated with a given finite set). The hyperreducible maximal triangular algebras (those satisfying certain reducibility conditions) seem to be the correct generalization of the concept of "ordered (orthonormal) basis" in the same sense that maximal abelian (self-adjoint) algebra generalizes the concept of "(orthonormal) basis."

The theory of triangular algebras seems to us to provide the general framework within which the reducibility properties of a bounded operator can be studied—in particular, the questions centered about invariant subspaces and bringing operators to "triangular form." A satisfactory theory of triangular algebras might provide an effective tool for the analysis of infinite-dimensional representations of solvable Lie groups.

In Chapter II, our basic definitions and results as well as examples establishing the existence of various special classes of maximal triangular algebras are presented. Chapter III contains the detailed development of the theory of hyperreducible triangular algebras. These have the position in triangular theory which the abelian algebras occupy in the study of self-adjoint algebras. It is natural to expect, therefore, that their theory would be most accessible (though the abelian-hyperreducible analogy has very limited applicability). At another time we shall describe the general constructions, with triangular algebras—restriction to a projection, triangular direct sums, and triangular tensor products. These operations are related to the hyper-reducible theory. With the aid of these, some ideal theory for triangular algebras, and the results of [5], an example of a triangular algebra which is not strongly closed can be constructed.

1. 2. Preliminaries. Our Hilbert spaces are complex. Our maximal abelian algebras are the self-adjoint ones. All operators are bounded unless otherwise specified. The "multiplication algebra" associated with a measure space is the (maximal abelian) algebra of operators corresponding to multi-plications by essentially bounded measurable functions on its L_2 space. "Totally-atomic" maximal abelian algebras are those generated by minimal projections—the "non-atomic" ones are those without minimal projections (atoms). Each maximal abelian algebra is the direct sum of a non-atomic and totally-atomic one. The non-atomic algebras on separable spaces are unitarily equivalent to the multiplication algebra of the unit interval under Lebesgue measure. A separating vector for a maximal abelian algebra is one which is annihilated by no operator in the algebra other than 0.

If $\mathcal{T}$ is a set of operators and S a set of vectors, we denote by $[\mathcal{T}S]$ the closed linear space spanned by vectors Tx with T in $\mathcal{T}$ and x in S, and by $\mathcal{T}'$ the set of operators commuting with each of those of $\mathcal{T}$. We often use the same symbol to denote both a projection and its range. Invariance of the range of a projection, E, under an operator, T, is equivalent to $TE = ETE$. Moreover, E is invariant under T and T^* if and only if E commutes with T.

Chapter II. General Theory.

2. 1. Basic definitions and notation. A feature of the algebra of $n \times n$ matrices with entries on or above the diagonal relative to a particular basis and a particular ordering of that basis which embodies its characteristic property of having entries on just one side of the diagonal is the fact that its intersection with its adjoint is the set of diagonal matrices, i. e. the maximal abelian algebra associated with the basis. The fact that the algebra contains all such matrices is reflected in its maximality with respects to this inter-section property. These considerations lead us to

Definition 2.1.1. *If $\mathcal{M}$ is a factor and $\mathcal{A}$ a maximal abelian (self-adjoint) subalgebra of $\mathcal{M}$; a subalgebra, $\mathcal{J}$, of $\mathcal{M}$ will be said to be "triangular in $\mathcal{M}$" (or simply "triangular," when $\mathcal{M}$ is all bounded operators on the underlying Hilbert space) with diagonal $\mathcal{A}$ when $\mathcal{J} \cap \mathcal{J}^* = \mathcal{A}$. If $\mathcal{J}$ is not a proper subalgebra of another algebra which is triangular in $\mathcal{M}$ we shall say that $\mathcal{J}$ is maximal triangular in $\mathcal{M}$. The projections in $\mathcal{M}$ which are invariant under $\mathcal{J}$ are called "the hulls of $\mathcal{J}$". The intersection of all hulls of $\mathcal{J}$ containing a given projection, E, is called "the hull of E (in $\mathcal{J}$)" and is denoted by "$h_{\mathcal{J}}(E)$." The von Neumann algebra generated by the hulls of $\mathcal{J}$ is called "the core of $\mathcal{J}$."*

Remark 2.1.2. If $\mathcal{J}$ is triangular in $\mathcal{M}$ with diagonal, $\mathcal{A}$, and $\{\mathcal{J}_\alpha\}$ is a family of algebras, containing $\mathcal{J}$, which are triangular in $\mathcal{M}$, and is totally ordered by inclusion; then $\mathcal{J}_0$, the union of the family, is triangular in $\mathcal{M}$ with diagonal $\mathcal{A}$ and contains $\mathcal{J}$. In fact, $\mathcal{J}_\alpha \cap \mathcal{J}_\alpha^*$ is maximal abelian in $\mathcal{M}$, by assumption ($\mathcal{J}_\alpha$ is triangular), and contains $\mathcal{J} \cap \mathcal{J}^* (= \mathcal{A})$, which has also been assumed to be maximal abelian in $\mathcal{M}$. Thus $\mathcal{J}_\alpha \cap \mathcal{J}_\alpha^* = \mathcal{A}$, and $\mathcal{A}$ is the diagonal of each $\mathcal{J}_\alpha$. An operator in $\mathcal{J}_0 \cap \mathcal{J}_0^*$ lies in some $\mathcal{J}_\alpha$ and some $\mathcal{J}_{\alpha'}^*$. Since the family, $\{\mathcal{J}_\alpha\}$, is totally ordered by inclusion, the operator lies in one of $\mathcal{J}_{\alpha'} \cap \mathcal{J}_{\alpha'}^*$, $\mathcal{J}_\alpha \cap \mathcal{J}_\alpha^*$, and thus, in $\mathcal{A}$. Of course, $\mathcal{A}$ is contained in $\mathcal{J}_0 \cap \mathcal{J}_0^*$, so that $\mathcal{A} = \mathcal{J}_0 \cap \mathcal{J}_0^*$; and $\mathcal{J}_0$ is triangular in $\mathcal{M}$ (with diagonal $\mathcal{A}$). Applying Zorn's Lemma, we conclude the existence of a maximal triangular algebra in $\mathcal{M}$ containing $\mathcal{J}$. We have observed, in addition, that when one triangular algebra is contained in another, they have the same diagonal.

Remark 2.1.3. To test that an algebra $\mathcal{J}$ is triangular with diagonal $\mathcal{A}$, it suffices to show that each self-adjoint operator in $\mathcal{J}$ lies in $\mathcal{A}$ and that $\mathcal{A}$ is contained in $\mathcal{J}$. In fact, $\mathcal{J} \cap \mathcal{J}^*$ is a self-adjoint algebra containing each self-adjoint operator in $\mathcal{J}$ and generated (linearly) by these operators.

Remark 2.1.4. If $\mathcal{J}$ is triangular in $\mathcal{M}$ with diagonal $\mathcal{A}$, then each hull of $\mathcal{J}$ is invariant under $\mathcal{J}$, hence under $\mathcal{A}$; hence commutes with and therefore lies in $\mathcal{A}$ (since it lies in $\mathcal{M}$, by assumption, and $\mathcal{A}$ is maximal abelian (self-adjoint) in $\mathcal{M}$). Thus, the hull, $h(E)$, of each projection, E, lies in $\mathcal{A}$; and the core of $\mathcal{J}$ is an (abelian) von Neumann algebra contained in $\mathcal{A}$. It is easy to see that $h(E) = [\mathcal{J}\mathcal{M}'E]$.

In the theory of triangular algebras the core plays the rôle that the center has relative to the theory of self-adjoint operator algebras.

Remark 2.1.5. If $\mathcal{J}$ is a maximal triangular algebra with diagonal $\mathfrak{A}$, the center of $\mathcal{J}$ consists of scalar multiples of I. Suppose A is in the center of $\mathcal{J}$, then, since $\mathfrak{A}$ is maximal abelian, A lies in $\mathfrak{A}$, and, in particular, A is normal. With B in $\mathcal{J}$, $BA = AB$, whence, from Fuglede's Theorem [2], $AB^* = B^*A$, and B commutes with each spectral projection E of A. Thus E and $I - E$ are hulls in $\mathcal{J}$, an impossibility, since the hulls are totally ordered (cf. Lemma 2.3.3), unless E is 0 or I. It follows that A is a scalar multiple of I.

2.2. Some examples. The most familiar instances of maximal triangular algebras which are not finite dimensional arise from particular total orderings of an orthonormal basis for separable Hilbert space. They consist of all operators leaving invariant each of the subspaces generated by the basis vectors preceding a given one. These subspaces and those spanned by a basis vector and all basis vectors preceding it are the hulls—the diagonal is totally atomic (cf. Theorem 3.2.1). The operators leaving invariant the multiplication operators corresponding to the characteristic functions of intervals with left endpoint 0 on $L_2(0,1)$ relative to Lebesgue measure provide an example of a maximal triangular algebra with non-atomic diagonal. This same example relative to a measure with some atoms gives rise to a maximal triangular algebra with mixed diagonal. In each of these examples, two properties of the maximal triangular algebras are prominent: the core is equal to the diagonal and the hulls form a totally-ordered family (under the usual projection ordering). Examination of the finite-dimensional situation would lead us to suspect that these properties are valid for all maximal triangular algebras. In point of fact, however, the first does not hold in general (though it does for algebras with totally-atomic diagonals—cf. Theorem 3.2.1) while the second does (cf. Lemma 2.3.3). The theorem which follows provides us with the basis for specific examples of maximal triangular algebras whose core consists of the scalars (i. e. whose hulls are 0 and I).

We say that a unitary operator, U, acts ergodically on a von Neumann algebra, $\mathfrak{A}$, when $U\mathfrak{A}U^* = \mathfrak{A}$ and there are no projections in $\mathfrak{A}$ invariant under U other than 0 and I. In particular, no projections of $\mathfrak{A}$ other than 0 and I commute with U; however, this is not equivalent to the ergodicity of U on $\mathfrak{A}$. In fact, with $\{y_n\}_{n=0,\pm1,\cdots}$ an orthonormal basis, $F_n = [y_m \colon m \leqq n]$, and $\mathfrak{A}$ all bounded diagonal matrices; if we define U by: $Uy_n = y_{n-1}$; then U leaves each F_n invariant but commutes with no projection in $\mathfrak{A}$, other than 0 and I.

THEOREM 2.2.1. *If the unitary operator, U, acts ergodically on the infinite-dimensional, maximal abelian algebra, $\mathcal{A}$, then the algebra, $\mathcal{S}$, generated by $\mathcal{A}$ and U is triangular, and the set of hulls in $\mathcal{S}$ is $\{0, I\}$; so that the same is true of each maximal triangular algebra containing $\mathcal{S}$.*

Proof. Since U acts ergodically on $\mathcal{A}$, we know that 0 and I are the only hulls, once we know that $\mathcal{S}$ is triangular. Note that each element, T, of $\mathcal{S}$ has the form $A_0 + A_1U + \cdots + A_nU^n$, with A_j in $\mathcal{A}$, since $U^nA = U^nAU^{-n}U^n = A'U^n$. Assume that T is self-adjoint, so that

$$A_0 + A_1U + \cdots + A_nU^n = A_0^* + U^{-1}A_1^* + \cdots + U^{-n}A_n^*.$$

Multiplying both sides by U^n, renormalizing, and transposing, we have $0 = A_0' + A_1'U + \cdots + A_nU^{2n}$, where A_n is as before. Assume that $A_0 + \cdots + A_nU^n = 0$ is an equation of minimal degree for U over $\mathcal{A}$. Let E be the range projection of A_n and F any projection in $\mathcal{A}$ contained in E. Then, if $G = U^nFU^{-n} \nleqq F$, we have $0 \neq G - GF (= M)$; whence $0 \neq U^{-n}MU^n \leqq F$. Writing N for $U^{-n}MU^n$, we have $NA_n \neq 0$, since N is a non-zero projection in the range of A_n; and $NA_0 + \cdots + NA_nU^n = 0$. Thus $NA_0 \neq 0$, for otherwise, U would satisfy an equation of degree lower than n over $\mathcal{A}$. But

$$\begin{aligned} 0 = NA_0 + NA_1UN + \cdots + NA_nU^nN &= NA_0 + \cdots + A_nNU^nNU^{-n}U^n \\ &= NA_0 + \cdots + A_nNMU^n = NA_0 + A_1'U + \cdots + A_{n-1}'U^{n-1} \end{aligned}$$

(recall that $FM = 0$ and $N \leqq F$). Since $NA_0 \neq 0$, this last equation contradicts the minimal property of n. Thus U^n leaves each projection in $\mathcal{A}$ contained in E invariant (i.e. in the present instance, $G \leqq F$).

The range projections of A_0 and A_n must be identical, for if they are not, since they commute, one contains a non-zero projection, G, in $\mathcal{A}$ orthogonal to the other, so that one of GA_0, GA_n, is 0 while the other is not. In either case, U would satisfy an equation of degree less than n. Thus, E is the range projection of both A_0 and A_n. From our equation for U, we obtain $A_0U^{-n} + A_1U^{-(n-1)} + \cdots + A_n = 0$; and this is an equation of lowest degree for U^{-1} over $\mathcal{A}$. In fact, if there is one of lower degree, by taking adjoints and renormalizing, we locate an equation of degree less than n satisfied by U. From the result of the preceding paragraph, therefore, U^{-n} $(= U^{n*})$ leaves each subprojection of E in $\mathcal{A}$ invariant. We conclude that U^n commutes with each such subprojection. Thus, with $F \leqq E$ and F in $\mathcal{A}$,

$$F + U^{-1}FU + \cdots + U^{-(n-1)}FU^{n-1}$$

commutes with U. By ergodicity of U on $\mathcal{A}$, $F + \cdots + U^{-(n-1)}FU^{n-1}$ is a scalar multiple, $k_F I$, of I. Now each summand of

$$F + U^{-1}FU + \cdots + U^{-(n-1)}FU^{n-1}$$

is a projection, and these projections commute. Elementary spectral theory tells us that k_F is a positive integer $(F \neq 0)$. If $E \neq F$, $k_F + k_{E-F} = k_E$, so that one of k_F, k_{E-F} does not exceed $k_E/2$. If F and $E - F$ are not minimal in $\mathcal{A}$ and F, let us say, is such that $k_F \leqq k_E/2$, then we can choose F_1 in $\mathcal{A}$, $F_1 < F$ such that $k_{F_1} \leqq k_E/4$. Thus, if E contains no minimal projections, we can locate a projection, F_m, in $\mathcal{A}$, $F_m \leqq E$ with $k_{F_m} < k_E/2^m < 1$: a contradiction. Thus E contains a minimal projection, F, of $\mathcal{A}$. It follows that U^jFU^{-j} are minimal in $\mathcal{A}$, and $\mathcal{A}$ being abelian, are (mutually) orthogonal or identical. Let m be the least integer such that $U^mFU^{-m} = F$, so that $m \leqq n$, F, $UFU^{-1} + \cdots + U^{m-1}FU^{-(m-1)}$ are orthogonal, and

$$F + UFU^{-1} + \cdots + U^{m-1}FU^{-(m-1)}$$

commutes with U. This sum is a scalar multiple of I, non-zero, and a projection. It is, therefore, I; so that $\mathcal{A}$ is m-dimensional, contrary to hypothesis. It follows that U satisfies no polynomial equation over $\mathcal{A}$ and $\mathcal{S}$ is triangular. The maximal triangular algebras containing $\mathcal{S}$ can't have a larger family of hulls than $\mathcal{S}$ has; whence the hulls of such maximal triangular algebras are 0 and I.

This result and proof hold also in a factor.

Example 2.2.2. Let C be the unit circle with Lebesgue measure, $\mathcal{H}$ be $L_2(C)$, $\mathcal{A}$ be the multiplication algebra of $L_2(C)$, and U the unitary transformation of $\mathcal{H}$ induced by an irrational rotation of C. It is well known that an irrational rotation of C is ergodic with respect to Lebesgue measure, from which we deduce that U acts ergodically on $\mathcal{A}$. Of course, $\mathcal{A}$ is infinite dimensional, whence from Theorem 2.2.1, $\mathcal{A}$ and U generate a triangular algebra, $\mathcal{S}$. Each maximal triangular algebra containing $\mathcal{S}$ has 0 and I as its only hulls.

DEFINITION 2.2.3. *A triangular algebra whose only hulls are* 0 *and* I *will be said to be "irreducible."*

Irreducibility is equivalent to the core's consisting of scalars. It is clear, moreover, that the irreducible triangular algebras are those which act irreducibily on the underlying Hilbert space.

2. 3. Some basic lemmas. In this section we develop a criterion which guarantees the membership of a given bounded operator in a particular maximal triangular algebra. With the aid of this result, we show that the hulls of maximal triangular algebras in factors are totally ordered (under the projection ordering).

Remark 2. 3. 1. If N is a projection invariant under the algebra $\mathcal{S}$ and M is a projection orthogonal to N, then $MSN = 0$ for each S in $\mathcal{S}$ In fact, $MSN = MNSN = 0SN = 0$.

LEMMA 2. 3. 2. *If $\mathcal{A}$ is a self-adjoint operator algebra, $\mathcal{S}$ an operator algebra maximal with the property of having $\mathcal{A}$ as its intersection with its adjoint, N and M orthogonal projections with N invariant under, and in $\mathcal{S}$, and B an operator such that $B = NBM$, then B lies in $\mathcal{S}$.*

Proof. Since $B^2 = 0$, each operator, T, in $\mathcal{S}_0$, the algebra generated by $\mathcal{S}$ and B has the form,

$$(*)\quad S + \sum BS \cdots S + \sum S \cdots SB + \sum BS \cdots SB + \sum SB \cdots S,$$

where the terms S which appear are not necessarily the same but all lie in $\mathcal{S}$. We shall show that $\mathcal{S}_0 \cap \mathcal{S}_0^* = \mathcal{A}$, whence $\mathcal{S}_0 = \mathcal{S}$, by maximality, and B lies in $\mathcal{S}$. To this end, it will suffice, of course, to show that each self-adjoint operator T in $\mathcal{S}_0$ (so, in $\mathcal{S}_0 \cap \mathcal{S}_0^*$) lies in $\mathcal{A}$, since $\mathcal{S}_0 \cap \mathcal{S}_0^*$ is a self-adjoint algebra and, therefore, generated by its self-adjoint operators.

We assume that T in $\mathcal{S}_0$ is self-adjoint and has the form described in (*). By hypothesis, the range of B is contained in N, so that B leaves N invariant, and, thus, $\mathcal{S}_0$ leaves N invariant. From Remark 2. 3. 1, we conclude that $(I - N)TN = 0$, whence from the self-adjointness of T, $NT(I - N) = 0$. It follows that $T = NTN + (I - N)T(I - N)$. Now $(I - N)T(I - N) = (I - N)S(I - N)$, since $(I - N)B = (I - N)SB = (I - N)SNB = 0$, by hypothesis on B and invariance of N under S. With $(I - N)S(I - N)$ self-adjoint, we conclude that $(I - N)T(I - N)$ lies in $\mathcal{A}$. It suffices, therefore, to show that NTN lies in $\mathcal{A}$. But $NTN = NSN + \sum NBS \cdots BSN + \sum NS \cdots SBN + \sum NBS \cdots SBN + \sum NS \cdots BSN = NSN$, since $BN = BSN = 0$; and NSN lies in $\mathcal{A}$, being a self-adjoint operator (NTN) in $\mathcal{S}$.

In the above lemma, we may assume that $\mathcal{S}$ has the maximal property with respect to some algebra, $\mathcal{M}$, containing it, provided that the B in question lies in $\mathcal{M}$. If $\mathcal{M}$ is a factor, then with the notation of Lemma 2. 3. 2, we may state:

LEMMA 2. 3. 3. *If E and F are projections in $\mathcal{S}$ invariant under $\mathcal{S}$,* *then one of E, F contains the other (i.e. the invariant projections in $\mathcal{S}$ are totally ordered).*

Proof. From invariance of E under $\mathcal{S}$, and, in particular, F, we have $EFE = FE$; whence, taking adjoints, $FE = EF$. Both $F - EF$ and $E - EF$ are non-zero unless one of E, F contains the other. If both are non-zero, there is a non-zero partial isometry, V, in $\mathcal{M}$ with initial space in $F - EF$ and final space in $E - EF$. Thus $V = EV(F - EF)$, with E invariant under $\mathcal{S}$ and orthogonal to $F - EF$. It follows, from the preceding lemma, that V lies in $\mathcal{S}$. However, V maps a part of F in $F - EF$ into $E - EF$, orthogonal to F, contradicting the invariance of F under $\mathcal{S}$. Thus, one of E, F contains the other.

Note that if $\mathcal{A}$ is generated by its invariant projections, the first statement of the foregoing proof shows that $\mathcal{A}$ is abelian.

LEMMA 2.3.4. *If $\mathcal{T}$ is a maximal triangular algebra in a factor $\mathcal{M}$ with diagonal $\mathcal{A}$ and core $\mathcal{B}$, then $h(G) - G$ is the hull immediately preceding $h(G)$ in $\mathcal{T}$ if G is a minimal projection in $\mathcal{B}$. If E is a hull in $\mathcal{T}$ which has a hull, F, immediately preceding it, then $E - F$ is a minimal projection in $\mathcal{B}$.*

Proof. If N is a hull in $\mathcal{T}$ not containing G, then $N \leqq h(G)$, from Lemma 2.3.3, and $GN = 0$, from minimality of G; so that $N \leqq h(G) - G$. The union, F, of all such hulls, N, is clearly a hull, $F \leqq h(G) - G$, and F is a hull immediately preceding $h(G)$. If we have proved the last statement of this lemma, then $h(G) - F$ is a minimal projection in $\mathcal{B}$ containing G, from which $F = h(G) - G$, and $h(G) - G$ is a hull. It remains to establish the last assertion of this lemma.

If M is a non-zero proper subprojection of $E - F$ in $\mathcal{B}$ and $N = E - F - M$, we can find a partial isometry, V, with initial space a non-zero subprojection of M and final space in N. If P is a hull of $\mathcal{T}$ containing E, then V and V^* leave P invariant. If P does not contain E, then $P \leqq F$; so that V and V^* annihilate P. Thus V commutes with each hull of $\mathcal{T}$ and hence with $\mathcal{B}$. However, V does not commute with M and M was chosen in $\mathcal{B}$. Thus $E - F$ is minimal in $\mathcal{B}$.

2.4. Other directions. The method by which we established the existence of maximal triangular algebras (a Zorn's Lemma construction) would yield, as well, the existence of an algebra, $\mathcal{T}$, maximal with respect to the property that $\mathcal{T}^* \cap \mathcal{T}$ is a given self-adjoint algebra, $\mathcal{A}$. More particularly, we may choose $\mathcal{A}$ to be abelian or to be a von Neumann algebra—the case where $\mathcal{A}$ is maximal abelian is that of the maximal triangular algebras. (All this may be done in a given set of operators, e. g. a factor.) Without specifying $\mathcal{A}$, we can construct an algebra, $\mathcal{J}$, maximal with respect to the property that $\mathcal{J}^* \cap \mathcal{J}$ is abelian—and containing a given such algebra, $\mathcal{J}_0$. In fact, take $\mathcal{J}$ as the union of a maximal family of such algebras containing $\mathcal{J}_0$ and totally ordered by inclusion. Clearly, $\mathcal{J}$ is an algebra and $\mathcal{J}^* \cap \mathcal{J}$ is generated by its self-adjoint elements. If A_1 and A_2 are self-adjoint operators in $\mathcal{J}$, then $A_1 \in \mathcal{J}_1$, $A_2 \in \mathcal{J}_2$, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are in the maximal family of which $\mathcal{J}$ is the union. Say, $\mathcal{J}_1 \subseteq \mathcal{J}_2$, so that A_1 and A_2 lie in the abelian algebra, $\mathcal{J}_2^* \cap \mathcal{J}_2$. Thus $\mathcal{J}^* \cap \mathcal{J}$ is abelian. Of course $\mathcal{J}_0^* \cap \mathcal{J}_0 \subseteq \mathcal{J}^* \cap \mathcal{J}$, so that the abelian intersection with which we start may expand as we pass to the maximal algebra, $\mathcal{J}$. Indeed, it would appear possible that all such maximal algebras are maximal triangular (i. e. $\mathcal{J}^* \cap \mathcal{J}$ is maximal abelian). We shall note that this need not be the case—even in finite dimensions. Before doing this, however, we wish to point out the importance of these considerations for certain critical questions in the theory of triangular operator algebras.

The question of whether or not a bounded operator on a separable Hilbert space has proper invariant subspaces may be strengthened and weakened in various ways. In a stronger form, one might ask not just for proper invariant subspaces, but for a "thick" family of such subspaces. A sense in which we can make this precise is to require that there be a resolution of the identity consisting of invariant subspaces, and more hopefully, a resolution with simple spectrum. Phrased in the language of our theory, we may ask:

Question 2.4.1. Is each bounded operator contained in some hyper-reducible maximal triangular algebra (core = diagonal—cf. Definition 3.0)?

This would provide a "triangular form" for bounded operators. We are inclined to feel that there is little hope that this question has an affirmative answer. It raises, in a natural way, the following question:

Question 2.4.2. Is each bounded operator contained in some maximal triangular algebra?

This is a much broader question, allowing, as it does, the possibility that the operator falls in an irreducible maximal triangular algebra. From this we would not conclude the existence of a single proper invariant subspace, although in the general sense of our theory, we would have the operator in "triangular form" and might gain knowledge about it from an analysis of the maximal triangular algebra in question. (The ostensibly weaker demand that the operator lie in some triangular algebra is not really weaker, since each such algebra is contained in one which is maximal.) Suppose that B is a bounded operator and $\mathcal{J}_0$ the (commutative) algebra generated by B and I (we may use any of the standard closures of $\mathcal{J}_0$). Of course, $\mathcal{J}_0^* \cap \mathcal{J}_0$ is abelian, so that $\mathcal{J}_0$ is contained in an algebra, $\mathcal{J}$, maximal with respect to the property that $\mathcal{J}^* \cap \mathcal{J}$ is abelian. The example which follows shows that $\mathcal{J}^* \cap \mathcal{J}$ need not be maximal abelian.

Example 2.4.3. Let $\mathcal{J}_1$ be the algebra of all 3×3 matrices, (a_{ij}), with $a_{31} = a_{32} = a_{13} = a_{23} = 0$, and let H be the positive square root of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

For $\mathcal{J}$ we choose $H\mathcal{J}_1H^{-1}$. A computation shows that with A in $\mathcal{J}_1$, HAH^{-1} is self-adjoint if and only if the matrix, (a_{ij}), for A, has $a_{ij} = 0$, $i \neq j$, and $a_{22} = a_{33}$. Thus the self-adjoint operators in $\mathcal{J}$ are the image under an automorphism on 3×3 matrices of a 2-dimensional abelian set; and $\mathcal{J}^* \cap \mathcal{J}$ is not maximal abelian though abelian. It remains to show that $\mathcal{J}$ is maximal with respect to the property of having an abelian intersection with its adjoint. We do this with a dimension argument. Suppose that $\mathcal{S}$ is a subalgebra of the 3×3 matrices which has dimension n. Let $\mathcal{S} \cup \mathcal{S}^*$ be the linear space spanned by $\mathcal{S}$ and $\mathcal{S}^*$. Then

$$2n = \dim(\mathcal{S}^* \cap \mathcal{S}) + \dim(\mathcal{S}^* \cup \mathcal{S}) \leqq \dim(\mathcal{S}^* \cap \mathcal{S}) + 9.$$

Since $\mathcal{S}^* \cap \mathcal{S}$ is self-adjoint, it has dimension 3 or less if it is abelian; so that $\mathcal{S}$ has dimension 6 or less. The algebra generated by $\mathcal{J}_1$ and an operator not in it is easily seen to be at least 7 dimensional, whence the same is true for the automorph, $\mathcal{J}$, of $\mathcal{J}_1$. Thus, no algebra containing $\mathcal{J}$ properly can have an abelian intersection with its adjoint.

Presumably, another computation would yield an example of an algebra with the maximal property having the scalars as intersection with its adjoint. With the aid of the example just constructed, we can produce an example in which the disparity betwen the dimension of the abelian intersection and that of the algebra is greater. In fact, let $\mathcal{J}_0$ be the algebra of $m \times m$ matrices (corresponding to bounded operators, when $m = \infty$) whose entries are 3×3 matrices having each 3×3 entry below the diagonal the zero matrix and each 3×3 entry on the diagonal some matrix in $\mathcal{J}$. Each self-adjoint operator in $\mathcal{J}_0$ has non-zero entries only on the diagonal and these some self-adjoint element of $\mathcal{J}$. Thus $\mathcal{J}_0^* \cap \mathcal{J}_0$ is abelian and $2m$-dimensional.

With m finite, maximal abelian algebras have dimension $3m$ and $\mathfrak{T}_0$ has dimension $5m+\frac{9}{2}m(m-1)$. That $\mathfrak{T}_0$ is maximal with respect to the property of having an abelian intersection with its adjoint is immediate from the corresponding fact for $\mathfrak{T}$ and the fact that arbitrary entries appear above the diagonal.

A possible way of relating a given bounded operator to a maximal triangular algebra would be to replace the "part of the operator below the diagonal" by zero. Caution must be exercised with this process in infinite dimensions. Even in the classical case of the maximal triangular algebra arising from an orthonormal basis, $\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots$, the "super-diagonal" part of the operator may not be bounded. We may view this situation in terms of the group, $\mathfrak{I}$, of integers and $L_2(\mathfrak{I})$, relative to the disecrete measure on $\mathfrak{I}$ (Haar-Lebesgue measure). The Fourier-Plancherel transform establishes a unitary equivalence of $L_2(\mathfrak{I})$ with $L_2(C)$, where C is the circle group, which carries the maximal abelian algebra consisting of bounded L_2 convolution operators on $L_2(\mathfrak{I})$ onto the multiplication algebra of $L_2(C)$. If x_n is the function which is 1 at n and zero elsewhere on $\mathfrak{I}$, then the m,n-th entry of the matrix corresponding to convolution by f, relative to the basis $\{x_n\}$ for $L_2(\mathfrak{I})$, is $f(m-n)$, for $(f*x_n, x_m)=(f*x_n)(m)=f(m-n)$. If f_0 is 0 at positive integers and equal to f elsewhere on $\mathfrak{I}$, then the matrix for convolution by f_0 has as m,n-th entry (relative to $\{x_n\}$) $f_0(m-n)$, i.e. its matrix is the "super-diagonal part" of the matrix for f (when $n \geqq m$ the entry is $f(m-n)$, otherwise it is 0). However, while convolution by f may be a bounded operator, i.e. have Fourier-Plancherel transform a bounded measurable function on C, the transform of f_0 may be unbounded.

In general, then, we must not expect an operator, A, to have a decomposition, A_1+A_2, with A_1 in $\mathfrak{T}$ and A_2 in $\mathfrak{T}^*$, where $\mathfrak{T}$ is a given maximal triangular algebra. If such a decomposition exists, however, it is clearly unique up to an additive factor from the diagonal of $\mathfrak{T}$.

In the chapter which follows, we shall give a reasonably detailed description of the most accessible class of maximal triangular algebras. This description will include an effective test of maximality. When we leave this class, no such test is known to us; and we may ask:

Question 2.4.4. Is there an easily applicable test for the maximality of a triangular algebra?

We have in mind some test such as that afforded by the Double Commutant Theorem in the theory of von Neumann algebras for the property of being strongly closed.

Theorem 2.2.1 and the example which follows it make use of Zorn's Lemma so that we do not have an explicit description of the operators in the irreducible maximal triangular algebra which results.

Question 2.4.5. Is there an explicit construction of an irreducible maximal triangular algebra?

Questions 2.4.4 and 2.4.5 are admittedly vague though, nonetheless, important for this theory. The following very definite question is perhaps the most provoking sample from a long list of questions one could ask about the irreducible triangular algebras.

Question 2.4.6. Are there two (or more) irreducible maximal triangular algebras on separable Hilbert space which are not algebraically isomorphic?

Chapter III. Hyperreducible Algebras.

DEFINITION 3.0. *A triangular algebra whose hulls generate the diagonal is said to be hyperreducible.*

Note that each hull of a triangular algebra is a reducing subspace—the hyperreducible case is the one with the greatest possible reduction. Hyperreducible algebras are those for which the core is equal to the diagonal. We have noted (cf. Remark 2.1.5) that the core plays the rôle of the center, so that the hyperreducible algebras in the theory of triangular operator algebras would correspond to the abelian algebras of the self-adjoint theory. Through this analogy, we would expect the hyperreducible algebras to be the most tractable of the triangular algebras, and this is the case—though their theory is not nearly as complete at this time as the abelian self-adjoint theory.

3.1. The general structure. The following result gives, in very broad terms, the general structure of the maximal hyperreducible algebras.

THEOREM 3.1.1. *If $\{E_\alpha\}$ is a totally-ordered family of projections which generates the maximal abelian algebra, $\mathfrak{A}$; then $\mathfrak{T}$, the set of all bounded operators which leave each E_α invariant is a maximal triangular algebra, with core and diagonal $\mathfrak{A}$. If $\{E_\alpha\}$ is closed under unions and intersections then it is the set of hulls of $\mathfrak{T}$. Each hyperreducible maximal triangular algebra arises in this way.*

Proof. That $\mathfrak{T}$ is an algebra is clear. If A is a self-adjoint operator in $\mathfrak{T}$, then, since $AE_\alpha = E_\alpha A E_\alpha = (E_\alpha A E_\alpha)^* = E_\alpha A$, we conclude that A

commutes with and hence lies in $\mathfrak{A}$, so that $\mathfrak{T}$ is triangular with diagonal (and core, by hypothesis) equal to $\mathfrak{A}$.

Choose a maximal triangular algebra, $\mathfrak{T}_0$, containing $\mathfrak{T}$, and let B be an operator in $\mathfrak{T}_0$. For each operator, T, $E_\alpha T(I-E_\alpha)$ lies in $\mathfrak{T}$. Indeed, with $E_\beta \leqq E_\alpha$,

$$0 = E_\alpha T(I-E_\alpha)E_\beta = E_\beta E_\alpha T(I-E_\alpha)E_\beta,$$

while, with $E_\beta \geqq E_\alpha$,

$$E_\alpha T(I-E_\alpha)E_\beta = E_\beta E_\alpha T(I-E_\alpha)E_\beta.$$

In particular, $E_\alpha B^*(I-E_\alpha)$ lies in $\mathfrak{T}$ (hence in $\mathfrak{T}_0$). It follows that the self-adjoint operator,

$$E_\alpha B^*(I-E_\alpha) + (I-E_\alpha)BE_\alpha,$$

lies in $\mathfrak{T}_0$ and, so, in $\mathfrak{A}$. Commutativity with E_α then gives

$$E_\alpha B^*(I-E_\alpha) = (I-E_\alpha)BE_\alpha,$$

from which, by multiplying both sides by $(I-E_\alpha)$, we conclude that $(I-E_\alpha)BE_\alpha = 0$. Thus B leaves each E_α invariant, B lies in $\mathfrak{T}$, $\mathfrak{T} = \mathfrak{T}_0$, and $\mathfrak{T}$ is a maximal triangular algebra.

On the other hand, if $\mathfrak{T}$ is maximal triangular and hyperreducible with hulls $\{E_\alpha\}$ and diagonal $\mathfrak{A}$, then $\{E_\alpha\}$ generates the maximal abelian algebra, $\mathfrak{A}$, and is totally ordered. Thus, $\mathfrak{T}_0$, the set of operators leaving each E_α invariant, is maximal triangular and contains $\mathfrak{T}$. By maximality of $\mathfrak{T}$, we have, $\mathfrak{T} = \mathfrak{T}_0$.

If $\{E_\alpha\}$ is closed under union and intersection of its members, and E is a hull for $\mathfrak{T}$, then E_0 the union of, and E_1 the intersection of all E_α contained in and containing E, respectively, lie in $\{E_\alpha\}$. If $E_0 < E < E_1$ then, as in Lemma 2.3.4, a partial isometry with initial space in $E - E_0$ and range in $E_1 - E$ commutes with each E_α, hence with $\mathfrak{A}$, but does not leave $E - E_0$ invariant. Thus E is one of E_0, E_1, and E lies in $\{E_\alpha\}$. The hulls of $\mathfrak{T}$ are precisely the E_α, in this case.

3.2. Triangular algebras with totally-atomic diagonals. Throughout this section, we shall be discussing the maximal triangular algebra, $\mathfrak{T}$, over a maximal abelian algebra $\mathfrak{A}$ which is generated by its minimal projections. In this case, the structure of $\mathfrak{T}$ can be completely described. The main result is contained in:

THEOREM 3.2.1. *If $\mathfrak{T}$ is a maximal triangular algebra with diagonal*

$\mathcal{A}$ which is generated by its minimal projections, then $\mathcal{T}$ is hyperreducible. The total ordering of the hulls induces a total ordering on the minimal projections, $\{E_\alpha\}$, of $\mathcal{A}$ by means of the mapping from projections to their hulls (which is one-one on the minimal projections); two such triangular algebras are unitarily equivalent if and only if their sets of minimal projections are order isomorphic. Corresponding to each total-ordering type there is a maximal triangular algebra with a totally-atomic diagonal whose set of minimal projections has this order type.

Proof. Since $h(E_\alpha) = [\mathcal{T}E_\alpha]$, $E_\beta h(E_\alpha) \neq 0$ if and only if $[E_\beta \mathcal{T} E_\alpha] \neq 0$, i. e. $E_\beta T E_\alpha \neq 0$, for some T in $\mathcal{T}$. If, in addition, $E_\alpha h(E_\beta) \neq 0$, then $E_\alpha T' E_\beta \neq 0$, for some T' in $\mathcal{T}$. Now $E_\alpha T^* E_\beta$ is a scalar multiple of $E_\alpha T' E_\beta$, since E_α and E_β are one dimensional, so that $E_\alpha T^* E_\beta + E_\beta T E_\alpha$ lies in $\mathcal{A}$, and commutes with E_α, E_β,—contradicting $E_\beta T E_\alpha \neq 0$. Thus $E_\alpha h(E_\beta) = 0$, and $h(E_\beta) \leqq h(E_\alpha) - E_\alpha$. Since $h(E_\alpha) - E_\alpha = \sum E_\beta$, $h(E_\alpha) - E_\alpha = \vee\, h(E_\beta)$. Hence $h(E_\alpha) - E_\alpha$ is a hull, so that $E_\alpha (= h(E_\alpha) - [h(E_\alpha) - E_\alpha])$ lies in the core of $\mathcal{T}$. Since $\mathcal{A}$ is generated by $\{E_\alpha\}$, $\mathcal{T}$ is hyperreducible. If $h(E_\alpha) = h(E_\beta)$, then $h(E_\alpha) - E_\alpha = h(E_\beta) - E_\beta$; and $E_\alpha = E_\beta$. In fact, from Lemma 2. 3. 4, $h(E_\alpha) - E_\alpha$ and $h(E_\beta) - E_\beta$ are the hulls immediately preceding $h(E_\alpha)$ and $h(E_\beta)$, respectively. Thus the total ordering of the hulls induces a total ordering, $\ll$, of $\{E_\alpha\}$ by means of the one-one mapping, $h(E_\alpha) \to E_\alpha$. Since each projection in $\mathcal{A}$ is the sum of the minimal projections it contains, and $E_\beta \ll E_\alpha$ (i. e. $h(E_\beta) \leqq h(E_\alpha)$) if and only if $E_\beta \leqq h(E_\alpha)$; we have, $h(E_\alpha) = \sum_{E_\beta \ll E_\alpha} E_\beta$.

If $\mathcal{T}_1$ ond $\mathcal{T}_2$ are maximal triangular algebras with diagonals $\mathcal{A}_1$ and $\mathcal{A}_2$ which are generated by their sets $\{E_\alpha\}$, $\{F_\alpha\}$ of minimal projections, acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and the mapping $E_\alpha \to F_\alpha$ is an order isomorphism of $\{E_\alpha\}$ onto $\{F_\alpha\}$ relative to the $\ll$ ordering on these sets; then the unitary transformation, U, defined by mapping a unit vector in the range of E_α onto a unit vector in the range of F_α is such that $U\mathcal{A}_1U^{-1} = \mathcal{A}_2$ and $UE_\alpha U^{-1} = F_\alpha$. We shall show that $U\mathcal{T}_1U^{-1} = \mathcal{T}_2$. Indeed, from the description of $\mathcal{T}_1$ and $\mathcal{T}_2$ as the algebras of all operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, which leave the hulls in $\mathcal{T}_1$ and $\mathcal{T}_2$ invariant, it suffices to show that UEU^{-1} is a hull in $\mathcal{T}_2$ if E is a hull in $\mathcal{T}_1$ (by symmetry, then, F is a hull in $\mathcal{T}_1$ if UFU^{-1} is a hull in $\mathcal{T}_2$). Now $Uh(E_\alpha)U^{-1} = h(F_\alpha)$, since $h(E_\alpha) = \sum_{E_\beta \ll E_\alpha} E_\beta$, $h(F_\alpha) = \sum_{F_\beta \ll F_\alpha} F_\beta$, and $E_\beta \ll E_\alpha$ if and only if $F_\beta = UE_\beta U^{-1} \ll F_\alpha = UE_\alpha U^{-1}$. If E is a hull in $\mathcal{T}_1$, then $E = \bigvee_{E_\alpha \leqq E} h(E_\alpha)$, whence $UEU^{-1} = \bigvee_{UE_\alpha U^{-1} \leqq UEU^{-1}} Uh(E_\alpha)U^{-1} = \bigvee_{F_\alpha \leqq UEU^{-1}} h(F_\alpha)$; and UEU^{-1} is a hull in $\mathcal{T}_2$.

Let $\{\alpha\}$ be a set having a given total-order type, $\mathcal{H}$ a Hilbert space having dimension the cardinality of $\{\alpha\}$, and let $\{E_\alpha\}$ be a maximal orthogonal family of one-dimensional projections on $\mathcal{H}$ indexed by $\{\alpha\}$ and totally ordered by the relation, $\ll$, induced by this indexing. If $\mathcal{A}$ is the maximal abelian algebra generated by $\{E_\alpha\}$ and E is a projection in $\mathcal{A}$, define $h(E)$ to be $\vee\{E_\beta\colon E_\beta \ll E_\alpha$ for some $E_\alpha \leqq E\}$. Note that the set of β's involved in the defining sum for $h(E)$ *is* an initial segment of $\{\alpha\}$. Since initial segments of a totally-ordered set form a totally-ordered set, the projections in $\{h(E)\colon$ E a projection in $\mathcal{A}\}$ form a totally-ordered set. The definition implies that $h(E_\alpha) = \sum_{\beta \ll \alpha} E_\beta$; whence $h(E_\alpha) - E_\alpha = h(h(E_\alpha) - E_\alpha)$. Thus, each E_α lies in the algebra generated by $\{h(E)\}$, and this algebra is, therefore, $\mathcal{A}$. From Theorem 3.1.1, the set of operators, $\mathcal{T}$, leaving each $h(E)$ invariant is maximal, hyperreducible. Moreover, the equality, $h(E_\alpha) = \sum_{\beta \ll \alpha} E_\beta$, establishes the fact that h is an order isomorphism of $\{E_\alpha\}$ with $\{h(E_\alpha)\}$; and the proof is complete.

COROLLARY 3.2.2. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are maximal triangular algebras with totally-atomic diagonals and φ is an order isomorphism between their sets of hulls, then φ can be implemented by a unitary transformation which carries $\mathcal{T}_1$ onto $\mathcal{T}_2$.*

Proof. By virtue of Theorem 3.2.1, it will suffice to show that φ induces an order isomorphism between the sets of minimal projections. We recall that the mapping from minimal projections to their hulls is an order isomorphism between the minimal projections and their hulls. It remains to note that φ carries the hull, $h(G)$, of a minimal projection, G, in $\mathcal{T}_1$ onto such a hull in $\mathcal{T}_2$. Now $\varphi(h(G) - G)$ is a hull in $\mathcal{T}_2$ immediately preceding $\varphi(h(G))$; whence $\varphi(h(G))$ is the hull of a minimal projection $\mathcal{T}_2$ (cf. Lemma 2.3.4).

THEOREM 3.2.3. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are maximal triangular algebras with totally-atomic diagonals, $\mathcal{A}_1$ and $\mathcal{A}_2$, acting on Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and φ is an isomorphism of $\mathcal{T}_1$ onto $\mathcal{T}_2$ carrying $\mathcal{A}_1$ onto $\mathcal{A}_2$, then φ is implemented by a bicontinuous linear isomorphism of $\mathcal{H}_1$ onto $\mathcal{H}_2$. In particular, if $\mathcal{T}_1 = \mathcal{T}_2$ and φ is the identity transform on $\mathcal{A}_1$, then the implementing transformation lies in $\mathcal{A}_1$.*

Proof. Since $\varphi(\mathcal{A}_1) = \mathcal{A}_2$, φ carries the set of minimal projections, $\{E_\alpha\}$, of $\mathcal{A}_1$ onto that of $\mathcal{A}_2$. Moreover, since a projection, E, is invariant under T if and only if $TE = ETE$, φ preserves hulls. Now $E_\alpha \ll E_{\alpha'}$ if and only if

$E_\alpha \leqq h(E_{\alpha'})$; whence $\varphi(E_\alpha) \leqq \varphi(h(E_{\alpha'})) = h(\varphi(E_{\alpha'}))$, and $\varphi(E_\alpha) \ll \varphi(E_{\alpha'})$. Applying these considerations to φ^{-1}, we conclude that φ induces an order isomorphism of the minimal projections in $\mathfrak{T}_1$ onto those of $\mathfrak{T}_2$; so that there is a unitary transformation of $\mathcal{H}_1$ onto $\mathcal{H}_2$ carrying $\mathfrak{T}_1$ onto $\mathfrak{T}_2$ and implementing φ on $\{E_\alpha\}$ (hence on $\mathcal{A}_1$). Composing the inverse of this unitarily induced mapping with φ, we see that it suffices to consider the case where φ is an automorphism of $\mathfrak{T}_1$ which is the identity transform on $\mathcal{A}_1$. Since $E_{\alpha'}\mathfrak{T}_1 E_\alpha$ is one-dimensional if $E_{\alpha'} \ll E_\alpha$,

$$\varphi(E_{\alpha'}TE_\alpha) = E_{\alpha'}\varphi(T)E_\alpha = a_{\alpha'\alpha}E_{\alpha'}TE_\alpha, \text{ where } a_{\alpha'\alpha}$$

is some non-zero scalar (independent of T). If $E_{\alpha''} \ll E_{\alpha'}$, then

$$\begin{aligned} a_{\alpha''\alpha}E_{\alpha''}T'E_{\alpha'}TE_\alpha &= \varphi(E_{\alpha''}T'E_{\alpha'}TE_\alpha) \\ &= \varphi(E_{\alpha''}T'E_{\alpha'})\varphi(E_{\alpha'}TE_\alpha) = a_{\alpha''\alpha'}a_{\alpha'\alpha}E_{\alpha''}T'E_{\alpha'}TE_\alpha; \end{aligned}$$

whence $a_{\alpha''\alpha'}a_{\alpha'\alpha} = a_{\alpha''\alpha}$. Clearly $a_{\alpha\alpha} = 1$, for all α. Fix α'', and define b_α to be $a_{\alpha''\alpha}^{-1}$ (or $a_{\alpha\alpha''}$ if $\alpha \ll \alpha''$). Let $B = \sum_\alpha b_\alpha E_\alpha$. We assert that B is an invertible operator in $\mathcal{A}_1$. To establish this, we need show only that $\{|a_{\alpha'\alpha}|\}$ is bounded above, for applying this result to φ^{-1}, we conclude that $\{|a_{\alpha'\alpha}|^{-1}\}$ is bounded above; whence $\{|b_\alpha|\}$ and $\{|b_\alpha^{-1}|\}$ are bounded above. If $\{|a_{\alpha'\alpha}|\}$ is not bounded above, then for each positive integer, n, there exist α_n and α_n', with $\alpha_n' \ll \alpha_n$ such that $|a_{\alpha_n'\alpha_n}| \geqq n^3$. Now $\sum_n (1/n^2)T_n$ converges uniformly to an operator, T, in $\mathfrak{T}_1$, where T_n is a partial isometry (in $\mathfrak{T}_1$) with initial space E_{α_n} and range $E_{\alpha_n'}$. But

$$\|\varphi(T)\| \geqq \|E_{\alpha_n'}\varphi(T)E_{\alpha_n}\| = \|(a_{\alpha_n'\alpha_n}/n^2)T_n\| \geqq n,$$

whence $\varphi(T)$ is not bounded, a contradiction. Thus $\{|a_{\alpha'\alpha}|\}$ is bounded. With T in $\mathfrak{T}_1$,

$$\begin{aligned} E_{\alpha'}(BTB^{-1})E_\alpha &= B(E_{\alpha'}TE_\alpha)B^{-1} = b_{\alpha'}b_\alpha^{-1}E_{\alpha'}TE_\alpha \\ &= a_{\alpha''\alpha'}^{-1}a_{\alpha''\alpha}E_{\alpha'}TE_\alpha = a_{\alpha'\alpha}E_{\alpha'}TE_\alpha = \varphi(E_{\alpha'}TE_\alpha) = E_{\alpha'}\varphi(T)E_\alpha. \end{aligned}$$

It follows that $BTB^{-1} = \varphi(T)$, and the proof is complete.

An orthonormal basis for a Hilbert space determines and is determined by the maximal abelian algebra of bounded operators with this basis as eigenvectors (the diagonal matrices relative to this basis). It is natural and customary therefore, to think of the arbitrary maximal abelian algebra as a generalized basis. In the same way, an ordered basis for the Hilbert space corresponds to the maximal triangular, hyperreducible algebra with diagonal the maximal abelian algebra of this basis and atoms ordered by the basis. For this reason, we may think of the general maximal triangular, hyperreducible algebra as a generalized ordered basis, and we shall often refer to such an algebra as an "ordered basis." When the diagonal is totally atomic, we shall speak of a "discrete, ordered basis"; and when the atoms are ordered as the integers (positive, negative, or positive and negative), we shall speak of an "integer-ordered basis." The atoms of an integer-ordered basis correspond to those infinite, totally-ordered sets between each pair of elements of which, there are a finite number of elements.

3.3. Non-atomic hyperreducible algebras. In this section, we consider hyperreducible, maximal triangular algebras (on separable spaces) whose diagonals are non-atomic. We shall see that, unlike the totally-atomic case, all such algebras are unitarily equivalent and *a fortiori* isomorphic (cf. Theorem 3.3.1), but that not each order isomorphism between the hulls can be implemented by a unitary transformation.

THEOREM 3.3.1. *If $\mathfrak{T}$ is a hyperreducible, maximal triangular algebra with non-atomic diagonal, $\mathfrak{A}$, acting on a separable Hilbert space, then $\mathfrak{T}$ is unitarily equivalent to $\mathfrak{T}_0$, the algebra of all bounded operators on $L_2(0,1)$ (Lebesgue measure) leaving each F_λ invariant, where F_λ is the projection due to multiplication by the characteristic function, χ_λ, of $[0,\lambda]$.*

Proof. Since $\mathfrak{A}$ is abelian on a separable space, there is a unit vector, x, which is separating for $\mathfrak{A}$. The hulls being totally ordered and x separating, ω_x takes distinct values on distinct hulls so that we can index each hull, E, with $\omega_x(E)$. Let $\{E_\lambda\}$ be these hulls so indexed. Note that $E_0 = 0$, $E_1 = I$, and for each μ in $[0,1]$, there is an E_μ with $\bigwedge_{\lambda>\mu} E_\lambda = E_\mu = \bigvee_{\nu<\mu} E_\nu$. In fact, if this were not so, $\bigvee_{\nu<\mu} E_\nu$ would be the hull immediately preceding $\bigwedge_{\lambda>\mu} E_\lambda$ which would, according to Lemma 2.3.4, be the hull of a minimal projection. Thus $\{E_\lambda\}$ is a resolution of the identity, say $A = \int \lambda\, dE_\lambda$. Let $\mathfrak{A}_0$ be the multiplication algebra of $L_2(0,1)$ (Lebesgue measure), and let $\varphi(f(A))$ be T_f, the operator due to multiplication by f, a bounded measurable function on $[0,1]$. Because of the properties of $\{E_\lambda\}$, φ is an isomorphism of $\mathfrak{A}$ with $\mathfrak{A}_0$. The mapping $f(A)x \to f$ is isometric; for

$$\begin{aligned}(f(A)x, g(A)x) &= \int \bar{g}(\lambda) f(\lambda)\, d(E_\lambda x, x) = \int \bar{g}(\lambda) f(\lambda)\, d\lambda \\ &= (f, g),\end{aligned}$$

and therefore has a unitary extension, U, which is easily seen to implement φ.

Now $X_\lambda(A) = E_\lambda$, whence $\varphi(E_\lambda) = T_{X_\lambda} = F_\lambda$. From Theorem 3.1.1, $\mathfrak{T}$ is describable as the algebra of all bounded operators which leave each E_λ invariant, so that U carries $\mathfrak{T}$ onto $\mathfrak{T}_0$.

If f is an order isomorphism of $[0,1]$ onto $[0,1]$ the mapping $E_\lambda \to E_{f(\lambda)}$ will be an order isomorphism of the hulls of $\mathfrak{T}$ onto itself (using the notation of the preceding theorem). We may ask ourselves if this mapping can be implemented by a unitary transformation of $\mathfrak{T}$ onto $\mathfrak{T}$. The corresponding question in the totally-atomic case had an affirmative answer. In the present case, the answer is in the negative. An indication of why this is so will serve as a good introduction to the methods of the following section. Such a unitary transformation must carry $\int \lambda\, dE_\lambda$ onto $\int \lambda\, dE_{f(\lambda)}$ $(= \int f^{-1}(\mu)\, dE_\mu)$. We note that, under the hypothesis, f will be a homeomorphism of $[0,1]$ onto itself (the image of an interval is an interval). Thus, the C^*-algebras generated by both $\int \lambda\, dE_\lambda$ and $\int f^{-1}(\lambda)\, dE_\lambda$ correspond to the algebra of multiplications by continuous functions on $[0,1]$, both have simple spectrum $[0,1]$, and the spectral null sets in both cases are given by the constant function, 1. In the first case, these are the null sets of the integration process, $g \to \int g(\lambda)\, d\lambda$, and in the second, those of $g \to \int g(f^{-1}(\lambda))\, d\lambda$. The first, then, are the Borel subsets of $[0,1]$ of Lebesgue measure 0, and the second are the images under f of these. Our task, then, is to construct an f which does not preserve the Borel sets of measure 0. Such functions are described in the literature (cf., [3, p. 83]). For example, let f be defined by $f(\lambda) = (\lambda + g(\lambda))/2$, where g is the Cantor function.

3.4. The general diagonal. We consider, now, the case where the diagonal is not assumed pure in the sense of total atomicity or non-atomicity. Experience with the self-adjoint theory conditions us to expect that the general case is a simple matter of separating the diagonal into its totally-atomic and non-atomic parts. This is not so in the present theory, as will be evident from the results of this section. The order, which is a primary constituent of this investigation, places the atoms throughout the continuous portion of the diagonal in a manner which does not permit a separation consistent with this theory.

DEFINITION 3.4.1. *The "hull class," $h(\mathfrak{T})$, of an ordered basis, $\mathfrak{T}$, on a separable space with hulls, $\{E_\alpha\}$, and diagonal, $\mathfrak{A}$, is the class $\{\{(E_\alpha x, x)\}$: x a separating vector for $\mathfrak{A}\}$ of subsets of* $[0,1]$.

Remark 3.4.2. The mapping η, taking E_α onto $(E_\alpha x, x)$ is an order isomorphism since $\{E_\alpha\}$ is totally ordered and x is a separating vector for $\mathfrak{A}$.

The greatest lower bound, E_{α_0}, of a subset of $\{E_\alpha\}$ is its intersection and a strong limit point of it; whence $\eta(E_{\alpha_0})$ is a lower bound and a limit point of its image under η. Thus $\eta(E_{\alpha_0})$ is the greatest lower bound of this image. It follows that the greatest lower bound (and likewise the least upper bound) of each subset of $\{(E_\alpha x, x)\}$ lies in it, whence $\{(E_\alpha x, x)\}$ is closed. Each member of the hull class is a closed subset of $[0,1]$ containing 0 and 1.

Remark 3.4.3. With X in $h(\mathcal{T})$ and X' the complement of X in $[0,1]$, X' is open and thus, the sum of a countable family, $I_1, I_2, \cdots$, of disjoint open intervals. The left and right hand endpoints, l_k and r_k, respectively, of I_k lie in X (we refer to the set of these endpoints as "the edge of X," to the l_k as "the left edge," and to the r_k as "the right edge"); and l_k corresponds to the hull immediately preceding the hull which corresponds to r_k. Thus r_k corresponds to the hull of a minimal projection in $\mathcal{A}$ (cf. Lemma 2.3.4). Conversely, if G is a minimal projection in $\mathcal{A}$, then $h(G) - G$ is the hull immediately preceding $h(G)$, so that the (non-empty) interval $(\eta(h(G) - G), \eta(h(G)))$ is some I_k. There is a one-one correspondence effected by η between the hulls of minimal projections in $\mathcal{A}$ and the right edge points of X.

THEOREM 3.4.4. *Two separable ordered bases, $\mathcal{T}_1$ and $\mathcal{T}_2$, are unitarily equivalent if $h(\mathcal{T}_1) \cap h(\mathcal{T}_2) \neq \phi$ and only if $h(\mathcal{T}_1) = h(\mathcal{T}_2)$.*

Proof. A unitary equivalence between $\mathcal{T}_1$ and $\mathcal{T}_2$ preserves diagonals, hulls, and the separating vectors for the diagonals; whence $h(\mathcal{T}_1) = h(\mathcal{T}_2)$.

Suppose, now, that x and y are separating vectors for the diagonals of $\mathcal{T}_1$ and $\mathcal{T}_2$ which give rise to the same set, X, in both $h(\mathcal{T}_1)$ and $h(\mathcal{T}_2)$. Let us index the hulls of $\mathcal{T}_1$ and $\mathcal{T}_2$ by points of X in such a way that $(E_\lambda x, x) = \lambda = (F_\lambda y, y)$, for each of the hulls E_λ and F_λ in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The sets $\mathcal{A}_1$ and $\mathcal{A}_2$ of linear combinations of $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively, are self-adjoint algebras which are weakly dense in the diagonals of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, so that $[\mathcal{A}_1 x] = \mathcal{H}_1$ and $[\mathcal{A}_2 y] = \mathcal{H}_2$ (with $\mathcal{H}_1$ and $\mathcal{H}_2$ the Hilbert spaces upon which $\mathcal{T}_1$ and $\mathcal{T}_2$ act). The mapping taking $(\sum_{i=1}^{n} \alpha_i E_{\lambda_i})x$ onto $(\sum_{i=1}^{n} \alpha_i E_{\lambda_i})y$ is isometric, for

$$\|(\sum_{i=1}^{n} \alpha_i E_{\lambda_i})x\|^2 = (\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j E_{\lambda_i} E_{\lambda_j} x, x)$$

$$= (\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j F_{\lambda_i} F_{\lambda_j} y, y) = \|\sum_{i=1}^{n} \alpha_i F_{\lambda_i} y\|^2.$$

Thus, this mapping has a unitary extension, U. Note that,

$$UE_\lambda U^{-1}((\sum \alpha_i F_{\lambda_i})y) = UE_\lambda(\sum \alpha_i E_{\lambda_i})x = U(\sum \alpha_i E_\lambda E_{\lambda_i})x$$
$$= (\sum \alpha_i F_\lambda F_{\lambda_i})y = F_\lambda((\sum \alpha_i F_{\lambda_i})y);$$

whence $UE_\lambda U^{-1} = F_\lambda$, and U effects a unitary equivalence between $\mathfrak{T}_1$ and $\mathfrak{T}_2$ (cf. Theorem 3.1.1).

The concept of an orientation-preserving homeomorphism which carries the null sets of Lebesgue measure onto these null sets of the image will play an important role in our work. We shall call such a mapping "a Lebesgue order isomorphism."

THEOREM 3.4.5. *If X is a closed subset of $[0,1]$ containing 0 and 1, X' its complement in $[0,1]$ with connected components $I_1, I_2, \cdots, \mu$ is the Borel measure on $[0,1]$ defined by $\mu(S) = m(S \cap X) + \sum_{r_k \in S} m(I_k)$, where r_k is the right endpoint of I_k and m is Lebesgue measure on $[0,1]$, $\mathfrak{A}$ is the multiplication algebra of $L_2([0,1],\mu)$, $\mathfrak{T}$ is the algebra of all bounded operators on $L_2([0,1],\mu)$ leaving each E_λ and $E_{\lambda\cdot}$ invariant, where E_λ and $E_{\lambda\cdot}$ are the projections due to multiplication by the characteristic functions of the half-open interval, $[0,\lambda)$, and the closed interval, $[0,\lambda]$, respectively, then $\mathfrak{T}$ is an ordered basis with diagonal $\mathfrak{A}$ and hulls $\{E_\lambda, E_{\lambda\cdot}\}$, and $X \in h(\mathfrak{T})$.*

Proof. Each operator due to multiplication by the characteristic function of a closed or open interval, and hence, a Borel set, lies in the von Neumann algebra generated by $\{E_\lambda, E_{\lambda\cdot}\}$; whence this algebra is $\mathfrak{A}$. Of course, $\{E_\lambda, E_{\lambda\cdot}\}$ is a totally-ordered family of projections, and from Theorem 3.1.1, $\mathfrak{T}$ is an ordered basis with diagonal $\mathfrak{A}$. Let $\mathfrak{F}$ be a subset of $\{E_\lambda, E_{\lambda\cdot}\}$, E its intersection, and γ the right endpoint of the interval which is the intersection of the intervals corresponding to the projections of $\mathfrak{F}$. Clearly $E_\gamma \leqq E$. Now E corresponds to multiplication by the characteristic function of some μ-measurable subset, S, of $[0,1]$. If $\lambda > \gamma$, there is a λ', with $E_{\lambda'}$ or $E_{\lambda'\cdot}$ in $\mathfrak{F}$, such that $\gamma \leqq \lambda' < \lambda$; whence $E \leqq E_{\lambda'\cdot} \leqq E_\lambda$. Thus, $\mu(\{\lambda' : \lambda' \in S, \lambda' > \lambda\}) = 0$, and since this holds for each $\lambda > \gamma$, $\mu(\{\lambda : \lambda \in S, \lambda > \gamma\}) = 0$. Hence $E \leqq E_{\gamma\cdot}$, and $E = E_\gamma$ or $E_{\gamma\cdot}$. Similarly, the union of the projections in $\mathfrak{F}$ lies in $\{E_\lambda, E_{\lambda\cdot}\}$, and from Theorem 3.1.1, $\{E_\lambda, E_{\lambda\cdot}\}$ is the set of hulls of $\mathfrak{T}$.

Observe that each r_k lies in X, so that if $S \cap X = \phi$ then $\mu(S) = 0$. Note also that $\mu([0,\lambda_0]) = \lambda_0$, when $\lambda_0 \in X$, for

$$\mu([0,\lambda_0]) = m([0,\lambda_0] \cap X) + \sum_{r_k \leqq \lambda_0} m(I_k)$$
$$= m([0,\lambda_0] \cap X) + m([0,\lambda_0] - [0,\lambda_0] \cap X) = \lambda_0.$$

Moreover, $\mu([0,\lambda_0)) = \lambda_0$ or l_k if $\lambda_0 \notin \{r_k\}$ or $\lambda_0 = r_k$, respectively. Now, if $\lambda \notin X$, $(E_\lambda x, x) = \mu([0,\lambda)) = \mu([0,\lambda]) = (E_{\lambda'} x, x)$, where x is the constant function 1 on $[0,1]$. If λ_0 is the least upper bound of $[0,\lambda] \cap X$,

$$\mu([0,\lambda)) = \mu([0,\lambda_0]) + \mu((\lambda_0,\lambda)) = \mu([0,\lambda_0]) = \lambda_0 \in X.$$

Thus X is the member of the hull class of $\mathfrak{I}$ corresponding to the separating vector x.

The ordered bases arising from the constructions of the foregoing theorem contain representatives from each unitary equivalence class of ordered bases, since each ordered basis is unitarily equivalent to the ones constructed on each of the sets in its hull class. We must still say, however, when two ordered bases arising from the construction of Theorem 3. 4. 5 are unitarily equivalent (and which closed sets appear in a given hull class). Theorem 3. 4. 8 answers these questions. Several remarks will be of help.

Remark 3. 4. 6. If X and Y are closed subsets of $[0,1]$ containing 0 and 1, and f is an order isomorphism of X onto Y then f has an order-isomorphic extension mapping $[0,1]$ onto itself. In particular, the extension is a homeomorphism, and f is continuous on X. In fact, the right and left edgepoints of X are characterized as those points of X having an immediate predecessor and successor in X, respectively. Thus f maps such edgepoints onto the corresponding edgepoints for Y and can be extended linearly over the intervals of X'. It is routine to check that f, so extended, is an order isomorphism of $[0,1]$ onto itself.

Remark 3. 4. 7. With X, Y, and f, as above, construct μ and ν, measures on X and Y, respectively, as in Theorem 3. 4. 5. (We shall refer to μ and the ordered basis of Theorem 3. 4. 5 as "the canonical measure and ordered basis for X.") Denoting by f, again, the order-isomorphic extension of f constructed above, f is a Lebesgue order isomorphism on $[0,1]$ if and only if it is on X, if and only if f carries the null sets of μ onto those of ν. In fact, since f is linear on each of the countable number of connected components of X', it is a Lebesgue order isomorphism on each of these; so that f is a Lebesgue order isomorphism on $[0,1]$, if and only if it is on X. Now, the μ and ν null sets in X can be described as those subsets containing no right edgepoints and whose Lebesgue measure is zero, so that f is a Lebesgue order isomorphism on X if and only if f carries μ-null sets onto ν-null sets.

THEOREM 3. 4. 8. *The canonical ordered bases $\mathfrak{I}_1$ and $\mathfrak{I}_2$ for the sets X and Y, respectively, are unitarily equivalent if and only if X and Y are*

Lebesgue order isomorphic. The hull class of an ordered basis consists of the images of any one of its members under Lebesgue order isomorphisms of $[0,1]$ *onto itself.*

Proof. The hulls of $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively, where E_λ and F_λ are multiplication by the characteristic functions of $[0,\lambda]\cap X$ and $[0,\lambda]\cap Y$ with λ in X and λ in Y, respectively. Let f be an order isomorphism of X and Y, and define φ by: $\varphi(E_\lambda)=F_{f(\lambda)}$. The diagonals $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, contain strongly dense C^*-subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, consisting of multiplications by continuous functions on X and Y, respectively. Let T_g be the operator in $\mathfrak{A}_1$ corresponding to multiplication by g, and define $\varphi(T_g)$ to be $T_{g\circ f^{-1}}$, the operator in $\mathfrak{A}_2$ corresponding to multiplication by $g\circ f^{-1}$. If φ on $\{E_\lambda\}$ can be implemented by a unitary transformation, the unitary equivalence induced on $\mathcal{A}_1$, when restricted to $\mathfrak{A}_1$, is the mapping, φ, just defined; for T_g is approximable to within ϵ in bound by some finite linear combinations of the E_λ's, and the unitary equivalence transforms this linear combination (as φ) into one in the F_λ's approximating $T_{g\circ f^{-1}}$ to within ϵ in bound. On the other hand, a unitary transformation which implements φ on $\mathfrak{A}_1$ carries E_λ onto $F_{f(\lambda)}$; for E_λ is the greatest lower bound in $\mathcal{A}_1$ of the multiplications corresponding to continuous functions which are 1 on $[0,\lambda]\cap X$ and lie between 0 and 1 on $X-[0,\lambda]$, and φ on $\mathfrak{A}_1$ transforms this set onto the corresponding set for $F_{f(\lambda)}$. Thus, φ on $\{E_\lambda\}$ is implemented by a unitary transformation if and only if φ on $\mathfrak{A}_1$ is; and this last obtains, if and only if the two representations, ψ_1 and ψ_2, of $C(X)$ defined by: $\psi_1(g)=T_g$, $\psi_2(g)=T_{g\circ f^{-1}}$, are unitarily equivalent. Since the weak closures of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are maximal abelian, φ can be unitarily implemented if and only if φ has an isomorphic extension to these weak closures, and this occurs if and only if ψ_1 and ψ_2 have the same null sets (cf. [4, Corollary 2.3.1], for example). The constant function, 1, is a separating vector for $\mathcal{A}_1$ and $\mathcal{A}_2$, whence the null sets are easily computed as the μ-null sets in X for ψ_1 and f^{-1} of the ν-null sets in Y for ψ_2. By symmetry, we conclude that φ on $\{E_\lambda\}$ is implemented by a unitary transformation if and only if f carries the μ-null sets onto the ν-null sets.

Suppose now that $\mathcal{T}$ is an ordered basis with hulls $\{E_\alpha\}$, diagonal $\mathcal{A}$, and x is a separating vector for $\mathcal{A}$. Let X be $\{(E_\alpha x,x)\}$, so that $X\in h(\mathcal{T})$; and let $\mathcal{T}_1$ be the canonical ordered basis for X. From Theorem 3.4.5, $X\in h(\mathcal{T}_1)$; whence $\mathcal{T}_1$ and $\mathcal{T}$ are unitarily equivalent, by Theorem 3.4.4. If Y is a closed subset of $[0,1]$ containing 0 and 1, and $\mathcal{T}_2$ is the canonical ordered basis for Y, then $\mathcal{T}_2$ is unitarily equivalent to $\mathcal{T}$ if and only if $\mathcal{T}_2$

is unitarily equivalent to $\mathcal{J}_1$—which occurs if and only if there is a Lebesgue order isomorphism of X onto Y. On the other hand, $\mathcal{J}_2$ is unitarily equivalent to $\mathcal{J}$ if and only if Y is in the hull class of $\mathcal{J}$. Thus, $h(\mathcal{J})$ consists of the images of X under Lebesgue order isomorphisms of $[0,1]$ onto itself.

3. 5. Special cases and examples. While the preceding section gives a fairly complete account from the general viewpoint, there is still much to be done regarding special cases. By virtue of the results of that section, the remaining problem can be recast as that of finding detailed information concerning special hull classes. We have not ruled out the possibility that "order isomorphism" will suffice in place of "Lebesgue order isomorphism" for the classification of hull classes. (We shall do so in this section.) The example of § 3. 3 shows us that an order isomorphism need not be Lebesgue, but it is conceivable that its existence guarantees the existence of one which is Lebesgue. In fact, Theorem 3. 2. 1 shows that this is so in the totally-atomic case (in this instance, the order isomorphism itself is Lebesgue). In addition, the hull class of a non-atomic ordered basis consists of $[0,1]$, whence an order isomorphism guarantees the existence of one which is Lebesgue (e. g., the identity mapping), if one of the ordered bases is assumed non-atomic. Even after we have noted that this phenomenon does not hold in general, we may still inquire into the classification of hull classes by means of "order isomorphism" and special properties of sets—e. g. an order isomorphism class of measure 0 sets may form a total hull class (cf. Remark 3. 5. 9).

We begin with the description of a construction closely akin to the construction of the canonical measure for X.

LEMMA 3. 5. 1. *If X is a closed subset of $[0,1]$ containing 0 and 1, $I_1, I_2, \cdots$, the distinct connected components of $[0,1]-X$, r_k and l_k are the right and left endpoints of I_k, respectively, then the mapping θ of X into $[0,1]$ defined by:*

$$\theta(a) = a - \sum_{r_k \leqq a} m(I_k),$$

is continuous, order preserving, and identifies a and b if and only if $m([a,b] \cap X) = 0$. The image, Z, of θ is a closed interval.

Proof. Clearly, $|\theta(a) - \theta(b)| \leqq |a - b|$, so that θ is continuous and Z is compact. If a and b are in X and $a < r_k \leqq b$, then $[a,b]$ contains I_k; so that

$$\theta(b) - \theta(a) = b - a - \sum_{a < r_k \leqq b} m(I_k) = m([a,b] \cap X).$$

Thus, $\theta(a) \leqq \theta(b)$, and $\theta(a) = \theta(b)$ if and only if $m([a,b] \cap X) = 0$.

Let a' be $\sup\{a : a \in X \text{ and } \theta(a) \leqq c\}$, and b' be $\inf\{b : b \in X \text{ and } c \leqq \theta(b)\}$. Since X is closed, a' and b' lie in X; and since θ is continuous, $\theta(a') \leqq c \leqq \theta(b')$. However, $[a', b'] \cap X$ contains just a' and b' and, so, has measure 0. Thus $\theta(a') = \theta(b') = c$, and $c \in Z$. It follows that Z is a closed interval.

The set, Z, consists of 0 alone if and only if $\theta(1) = \theta(0) = 0$—which occurs if and only if in $([0,1] \cap X) = m(X) = 0$, i.e., $\sum m(I_k) = 1$. If $\mathcal{J}$ is an ordered basis with diagonal $\mathcal{A}$, X in $h(\mathcal{J})$ corresponding to the separating vector, x, and E_k is the minimal projection in $\mathcal{A}$ with $(h(E_k)x, x) = r_k$; then $l_k = ([h(E_k) - E_k]x, x)$, so that $m(I_k) = (E_k x, x)$, and $((\sum E_k)x, x) = \sum m(I_k) = 1$. This is equivalent to $\mathcal{A}$ being totally atomic.

LEMMA 3.5.2. *If X and Y are closed, measure-zero subsets of $[0,1]$ with infima a and c and suprema b and d, respectively, and f is an order isomorphism of the right edgepoints of X in $[a,b]$ onto those of Y in $[c,d]$, there is a Lebesgue order-isomorphic extension of f mapping $[a,b]$ onto $[c,d]$ and carrying X onto Y.*

Proof. Clearly, it suffices to consider the case where $a = c = 0$ and $b = d = 1$. Assuming this, let μ and ν be the canonical measures and $\mathcal{J}$ and $\mathcal{S}$ the canonical ordered bases for X and Y, respectively. From the comment preceding this lemma, $\mathcal{J}$ and $\mathcal{S}$ have totally-atomic diagonals. From Remark 3.4.3, the hulls of the minimal projections in $\mathcal{J}$ and $\mathcal{S}$ are order isomorphic with the right edgepoints of X and Y, respectively, and, so, to each other. According to Theorem 3.2.1, this isomorphism can be implemented by a unitary transformation carrying $\mathcal{J}$ onto $\mathcal{S}$. This unitary transformation carries the hulls of $\mathcal{J}$ onto those of $\mathcal{S}$ order isomorphically. By means of the order isomorphisms of X with the hulls of $\mathcal{J}$ and Y with those of $\mathcal{S}$, we arrive at a Lebesgue order isomorphism (recall that $0 = m(X) = m(Y)$) of X onto Y extending f. Remarks 3.4.6, 3.4.7 imply the existence of the desired extension of f to $[0,1]$.

If $\mathcal{A}$ is not totally atomic, we may normalize θ by composing it with multiplication by $1/m(Z)\,(=1/m(X))$ to get a mapping of X onto $[0,1]$ with the properties of θ noted in Lemma 3.5.1. We denote this new mapping by θ_X (in the case where $\mathcal{A}$ is not totally atomic). Writing p_k for $\theta_X(r_k)$ and $o(p_k)$ for the order type of the set of right edgepoints of X in $\theta_X{}^{-1}(p_k)$ (so that $o(p_k)$ is some denumerable total-order type), we shall say that X has a uniform edge of type τ if each $o(p_k) = \tau$. The construct consisting of $[0,1]$, the points p_k, and their associated order types, will be called "a point ordered interval" and the set of points $\{p_k\}$ its "atoms."

LEMMA 3.5.3. *If X and Y are closed subsets of $[0,1]$ containing* 0 *and* 1, $m(X) \neq 0 \neq m(Y)$, *f is a Lebesgue order isomorphism of X onto Y, and g is defined by $g(p) = \theta_Y(f(\theta_X^{-1}(p)))$, for each p in $[0,1]$, then g is a Lebesgue order isomorphism of $[0,1]$ onto itself, $\{g(p_k)\}$ are the atoms of the point ordered interval for Y, where $\{p_k\}$ are those of the interval for X, and $o(p_k) = o(g(p_k))$. If a mapping such as g is given, then there is a mapping, f, such that $g(p) = \theta_Y(f(\theta_X^{-1}(p)))$, for each p in $[0,1]$, which is a Lebesgue order isomorphism of X onto Y.*

Proof. As defined, g is not obviously single-valued, since $\theta_X^{-1}(p)$ may contain more than one point. However, if a and b are in $\theta_X^{-1}(p)$, then $m([a,b] \cap X) = 0$, from Lemma 3.5.1; and

$$0 = m(f([a,b] \cap X)) = m([f(a), f(b)] \cap Y),$$

since f is a Lebesgue order isomorphism, whence $\theta_Y(f(a)) = \theta_Y(f(b))$, again from Lemma 3.5.1. Thus g is single-valued. Since θ_Y, θ_X, and f are order preserving, g is. With p_k equal to $\theta_X(r_k)$ and $f(r_k)$ a right edgepoint of Y (since f is an order isomorphism), we have that $g(p_k)$ is $\theta_Y(f(r_k))$, an atom of the point ordered interval for Y.

In the proof of Lemma 3.5.1, we noted that $\theta(b) - \theta(a) = m([a,b] \cap X)$, so that $\theta_X(b) - \theta_X(a) = m([a,b] \cap X)/m(X)$. Thus, $m(\theta_X^{-1}([p,q])) = (q-p)m(X)$, and $m(\theta_X^{-1}(S)) = m(S)m(X)$, for each open subset, S, of $[0,1]$. By regularity of m, if $m(S') = 0$, then $m(\theta_X^{-1}(S')) = 0$. On the other hand, from the fact that $|\theta_Y(b) - \theta_Y(a)| \leqq |b-a|/m(Y)$, we conclude that if $m(Y_0) = 0$, for a subset, Y_0, of Y, then $m(\theta_Y(Y_0)) = 0$. Thus, if $m(S) = 0$, for S a subset of $[0,1]$, $m(\theta_X^{-1}(S)) = 0$, $m(f(\theta_X^{-1}(S))) = 0$, and $0 = m(\theta_Y(f(\theta_X^{-1}(S)))) = m(g(S))$.

What we have proved for g holds as well for the mapping $\theta_X f^{-1} \theta_Y^{-1}$, which is clearly g^{-1}. It follows that g is a Lebesgue order isomorphism of $[0,1]$ onto itself mapping atoms (for X) onto atoms (for Y). When we note that $\theta_Y^{-1}(g(p_k)) = f(\theta_X^{-1}(p_k))$ and f maps the right edgepoints of X onto those of Y (order isomorphically), we see that the right edgepoints of $\theta_Y^{-1}(g(p_k))$ are order isomorphic to those of $\theta_X^{-1}(p_k)$, whence $o(p_k) = o(g(p_k))$.

Suppose, now, that g is given with the properties described above. We show that there is a Lebesgue order isomorphism, f, of X onto Y such that $g = \theta_Y f \theta_X^{-1}$. We note first that if $\theta_X^{-1}(p)$ contains more than one point then p is an atom for X (i.e. p is some p_k). In fact, $\theta_X^{-1}(p) = [a,b] \cap X$, where $a = \inf\{a' : a' \in \theta_X^{-1}(p)\}$, $b = \sup\{b' : b' \in \theta_X^{-1}(p)\}$, a and b are in $\theta_X^{-1}(p)$, and $m([a,b] \cap X) = 0$ (cf. Lemma 3.5.1). Thus,

$$\sum_{I_k \subseteq [a,b]} m(I_k) = b - a \neq 0;$$

and in particular, there is some r_k in $[a, b]$. Of course, $\theta_X(r_k) = p = p_k$. If p is not an atom for X, then by assumption, $g(p)$ is not an atom for Y, whence $\theta_X^{-1}(p)$ and $\theta_Y^{-1}(g(p))$ each consist of a single point. Define $f(\theta_X^{-1}(p))$ to be $\theta_Y^{-1}(g(p))$, for such points, p. By assumption on g, $o(p_k) = o(g(p_k))$, which guarantees an order isomorphism of the right edgepoints of X in $\theta_X^{-1}(p_k)$ onto those of Y in $\theta_Y^{-1}(g(p_k))$. If $\theta_X^{-1}(p_k) = [a, b] \cap X$ and $\theta_Y^{-1}(g(p_k)) = [c, d] \cap Y$, the right edgepoints of $\theta_X^{-1}(p_k)$ in $[a, b]$ and $\theta_Y^{-1}(g(p_k))$ in $[c, d]$ are those of X which lie in $[a, b]$ and those of Y which lie in $[c, d]$, respectively. This is clear with the possible exception of the points a and c. However, a cannot be a right edgepoint of X for then the corresponding left edgepoint would lie in $\theta_X^{-1}(p_k)$ but not in $[a, b]$. Similarly, c is not a right edgepoint of Y. It follows from Lemma 3.5.2 that there is a Lebesgue order isomorphism of $[a, b] \cap X$ onto $[c, d] \cap Y$. We define f on $[a, b] \cap X$ to be this isomorphism. Clearly, $g = \theta_Y f \theta_X^{-1}$; and f is an order isomorphism. If $m(X_0) = 0$ for a subset, X_0, of X, then $m(\theta_X(X_0)) = 0$ and $m(\theta_Y^{-1}(g(\theta_X(X_0)))) = 0$, from the first part of this proof, since $m(g(\theta_X(X_0))) = 0$. Now $f(X_0)$ is contained in $\theta_Y^{-1}(g(\theta_X(X_0)))$; whence $m(f(X_0)) = 0$, and f is a Lebesgue order isomorphism.

By virtue of the preceding lemma, the study of hull classes is equivalent to the study of denumerable subsets of $[0, 1]$ each point of which has a denumerable total-order type associated with it, under Lebesgue order isomorphisms of $[0, 1]$ onto itself. The next lemma shows that all possibilities occur as point ordered intervals.

LEMMA 3.5.4. *If $\{p_k\}$ is a denumerable subset of $[0, 1]$ and $\{\tau_k\}$ is a set of denumerable total-order types, then there is a closed subset of $[0, 1]$ containing 0 and 1 whose point ordered interval has $\{p_k\}$ as its atoms and τ_k as the order type associated with p_k.*

Proof. We begin by showing that each non-zero interval contains a closed subset, with the interval endpoints as members, having measure 0, and whose right edgepoints have some preassigned denumerable order type, τ. If τ is the order type of a non-null finite set, this result is clear. We assume that τ is the order type of some infinite (denumerable) set, $\{a_j\}_{j=1,2,\cdots}$. We may assume, in addition, that the interval in question is $[0, 1]$ (translating and multiplying by a suitable scalar). For each positive integer, k, let b_k be $\sum_n 2^{-k_n}$, where $\{a_{k_n}\}$ is the subset of $\{a_j\}$ consisting of points not exceeding a_k; and let I_k be the open interval of length 2^{-k} with b_k as right endpoint. If $a_j < a_k$ then $b_k - b_j \geqq 2^{-k}$, by construction; whence I_k and I_j are disjoint.

Let X be the complement in $[0,1]$ of the union of the intervals, I_k. Clearly, X is closed, contains 0 and 1, has $\{b_k\}$ as its set of right edgepoints, and has measure 0 (since $\sum_{k=1}^{\infty} m(I_k) = \sum_{k=1}^{\infty} 2^{-k} = 1$). The comment establishing the disjointness of I_j and I_k, when $j \neq k$, also establishes the fact that the correspondence $a_j \rightarrow b_j$ is an order isomorphism of $\{a_j\}$ with $\{b_j\}$. Thus, the right edgepoints of X have order type τ.

If $\{p_k\}$ is a finite subset of $[0,1]$, there is no difficulty in establishing the conclusion of this lemma (when the result just proved is employed). We assume that $\{p_k\}$ is an infinite set. Let c_k be $(\sum_n 2^{-(k_n+1)}) + p_k/2$, where $\{p_{k_n}\}$ is the subset of $\{p_j\}$ consisting of those numbers which do not exceed p_k; and let J_k be the closed interval of length $2^{-(k+1)}$ with c_k as right endpoint (and d_k as left endpoint). Let X_k be a closed, measure 0 subset of J_k containing c_k and d_k and having right edgepoints in J_k with τ_k as order type. The complement, X, of $\bigcup_k (J_k - X_k)$ in $[0,1]$ is a closed subset of $[0,1]$ containing 0 and 1 (since neither of these points lies in $J_k - X_k$, $k = 1, \cdots$). If $p_j < p_k$, then $c_k - c_j \geqq 2^{-(k+1)} + (p_k - p_j)/2$, so that J_k and J_j are disjoint. The components of the complement of X in $[0,1]$ are, therefore, the aggregate of the components of the complement of each X_k in J_k, so that the right edgepoints of X in $[0,1]$ are those of each X_k in J_k. (Note that d_k is not a right edgepoint of X, by disjointness of J_k and J_j, with $j \neq k$.) Now, $\theta(1) = 1 - \sum_{j=1}^{\infty} 2^{-(j+1)} = 1/2$, so that $\theta_X = 2\theta$; and $\theta_X(c_k) = 2(p_k/2) = p_k$. Thus, each p_k is an atom for X; and since $m(X_k) = 0$, $\theta_X(X_k) = p_k$, so that $\{p_k\}$ is the set of atoms for X. The right edgepoints of X which θ_X maps onto p_k are those of X_k in J_k, and therefore has τ_k as order type; for any other right edgepoint of X lies in some X_j, with $j \neq k$, and $\theta_X(X_j) = p_j \neq p_k$. Thus X has a point ordered interval with $\{p_k\}$ as its set of atoms, each p_k associated with the denumerable, total-order type, τ_k.

The lemma which follows provides the key to the description of a special family of hull classes.

LEMMA 3.5.5. *If R and S are dense denumerable subsets of $[0,1]$ both containing 0 and 1, and m, M are numbers such that $0 < m < 1 < M$, then there exists a homeomorphism, f, of $[0,1]$ onto itself such that:*

(i) $f(0) = 0$, $f(1) = 1$;

(ii) *f maps R onto S;*

(iii) $m(x-y) \leqq f(x) - f(y) \leqq M(x-y)$, *for each $x \geqq y$ in $[0,1]$.*

Proof. If we construct a mapping, g, of R onto S satisfying (i) and (iii) for points x, y in R, then g is uniformly continuous and so has a unique continuous extension, f, to $[0,1]$. By density of R and continuity of f, (iii) holds; whence f is a homeomorphism of $[0,1]$ onto itself satisfying (i), (ii), (iii).

To construct such a g, we begin by enumerating the sets R and S as $r_1 = (0)$, $r_2(=1), r_3, \cdots$ and $s_1(=0), s_2(=1), s_3, \cdots$, respectively. When we refer to "the points of $\{a_1, \cdots, a_n\}$ adjacent to a", we shall mean those points a_j, a_k such that $a_j < a < a_k$ and $a_j < a_h < a_k$, for $1 \leqq h \leqq n$, implies $a_h = a$. We write (r, s, r', s') in place of the inequalities, $m < \dfrac{s' - s}{r' - r} < M$.

Define "an s link of length h," for s in S, to be a set of ordered pairs $\{(r_1, s_{n_1}), (r_2, s_{n_2}), \cdots, (r_h, s_{n_h})\}$ such that $(r_j, s_{n_j}, r_k, s_{n_k})$, for $j \neq k$; j, $k = 1, \cdots, h$; and $s_{n_h} = s$.

If $L = \{(r_1, s_{n_1}), \cdots, (r_k, s_{n_k})\}$ is an s_{n_k} link, $h > k$, and s_{n_h} is such that $(r_{n_j}, s_{n_j}, r_h, s_{n_h})$, for $j = 1, \cdots, k$, then there is an s_{n_h} link of length h of which L is a subset. In fact, with $k \leqq a < b < h$, suppose that we have found s_{n_a} such that $(r_j, s_{n_j}, r_a, s_{n_a})$, for $j = h, 1, 2, \cdots, a-1$. Let $r_{j'}$ and $r_{j''}$ be the points of $\{r_h, r_1, r_2, \cdots, r_{b-1}\}$ adjacent to r_b. The set of points, x, for which $(r_b, x, r_{j'}, s_{n_{j'}})$ and $(r_b, x, r_{j''}, s_{n_{j''}})$, is a non-null open set, since $(r_{j'}, s_{n_{j'}}, r_{j''}, s_{n_{j''}})$; so that it contains elements of the dense set, S. Let s_{n_b} be that element of lowest index. Since $r_{j'}$, $r_{j''}$ are adjacent to r_b and $(r_{j'}, s_{n_{j'}}, r_b, s_{n_b})$, $(r_{j''}, s_{n_{j''}}, r_b, s_{n_b})$, we have $(r_j, s_{n_j}, r_b, s_{n_b})$, for each $j = h, 1, \cdots, b-1$. In this way, we construct an s_{n_h} link containing L.

Suppose, now, that we have defined $g(r_1), \cdots, g(r_{n-1})$ so that $\{(r_1, g(r_1)), \cdots, (r_{n-1}, g(r_{n-1}))\}$ $(= L)$ is a link, and $g(r_1) = 0$, $g(r_2) = 1$. Let s be the element of $S - \{g(r_1), \cdots, g(r_{n-1})\}$ with least index; and let S_n be the set of elements of S paired with r_n in those s links containing L which have minimal length. We shall define $g(r_n)$ to be the element of S_n with least index, so that $\{(r_1, g(r_1)), \cdots, (r_n, g(r_n))\}$ is a link; but first, we must show that S_n is not empty. For this, it will suffice to prove that some s link containing L exists. Let $g(r_{j'})$ and $g(r_{j''})$ be the elements of $\{g(r_1), \cdots, g(r_{n-1})\}$ adjacent to s. As in the preceding paragraph (by density of R), there is an element r_h, of R such that $(r_{j'}, g(r_{j'}), r_h, s)$ and $(r_{j''}, g(r_{j''}), r_h, s)$; whence $(r_j, g(r_j), r_h, s)$, for all $j = 1, \cdots, n-1$. From our preceding comments, we conclude that there is an s link containing L.

If g does not map R onto S, let s be the element of S not in the range of g with least index; and suppose that each element of S with index less than that of s is contained among $g(r_1), g(r_2), \cdots, g(r_k)$. From the above, there is some s link, of length h, let us say, containing

$$\{(r_1, g(r_1)), \cdots, (r_k, g(r_k))\}.$$

Let $\{(r_1, g(r_1)), \cdots, (r_j, g(r_j)), (r_{j+1}, s_{n_{j+1}}), \cdots, (r_{h'}, s)\}$ be an s link of length not exceeding h for which j is maximal (since s is not in the range of g, there is some first $s_{n_{j+1}}$ which is not $g(r_{j+1})$). Certainly then $j \geqq k$, so that s is the element of $S - \{g(r_1), \cdots, g(r_j)\}$ with least index. By definition, $g(r_{j+1})$ occurs with r_{j+1} in an s link of length not exceeding h' and containing $\{(r_1, g(r_1)), \cdots, (r_j, g(r_j))\}$. This contradicts the maximal property of j; whence g maps R onto S, and the proof is complete.

Remark 3.5.6. The result of the preceding lemma is valid in the case where R and S both contain or both do not contain 0 or 1 (as can be seen by adjoining 0 or 1 to the sets—whichever is appropriate).

If X is a closed subset of $[0,1]$ containing 0 and 1, we shall say that two points, a and b, of X are "equivalent in X" when $m([a,b] \cap X) = 0$ (equivalently, when $\theta(a) = \theta(b)$—cf. Lemma 3.5.1).

THEOREM 3.5.7. *If τ is a denumerable order type, the family, $\mathfrak{F}$, of closed subsets, X, of $[0,1]$ containing 0 and 1, which are nowhere dense, have uniform edge of type τ, have non-zero Lebesgue measure, and for which 0 and 1 are equivalent in X to right edgepoints of X is the hull class of some ordered basis. The family of subsets having the same properties and for which 0 or 1 are not equivalent to a right edgepoint is also a hull class.*

Proof. If $X \in \mathfrak{F}$ and f is a Lebesgue order isomorphism of $[0,1]$ onto itself, then clearly, $f(X) \in \mathfrak{F}$. Thus, $\mathfrak{F}$ contains the hull class determined by X.

With $0 \leqq p < q \leqq 1$, $\theta^{-1}([p,q]) = [a,b] \cap X$, since θ is order preserving. Now X is nowhere dense, so that $[a,b] - X$ is non-null. Hence, a right edgepoint of X lies in $[a,b]$ and some atom for X lies in $[p,q]$. The set of atoms for each set of $\mathfrak{F}$ is everywhere dense in $[0,1]$ and contains 0 and 1 (in the case of the first statement of this theorem). According to Lemma 3.5.5, there is a Lebesgue order isomorphism of $[0,1]$ onto itself carrying the atoms for X onto those for Y, where X and Y are sets in $\mathfrak{F}$, and necessarily preserving the order types associated with these atoms, since by hypothesis, τ is associated with all the atoms for X and Y. Lemma 3.5.3 now applies, and we conclude that there is a Lebesgue order isomorphism of X and Y. Thus $\mathfrak{F}$ constitutes a complete hull family.

Remark 3.5.8. We note in the foregoing theorem that X being nowhere dense implies that its atoms are everywhere dense. Suppose that we are given that the atoms for X are everywhere dense in $[0,1]$. By definition of θ_X, an open interval in X maps in a one-one manner onto an interval in $[0,1]$ which contains no atoms for X. Thus X contains no open intervals, and being closed, X is nowhere dense.

Remark 3.5.9. The density of the atoms for X in $[0,1]$ is equivalent to their closure having measure 1. The unitary equivalence class of ordered bases corresponding to the hull class of Theorem 3.5.7 can also be described as the set of ordered bases whose point ordered intervals have a set of atoms whose closure has measure 1, contains 0 and 1 (similarly for the other three cases), and each atom is associated with a fixed order type, τ. In this framework, we can state a similar result for the case where the closure of the set of atoms has measure 0. In fact, the family of all denumerable subsets of $[0,1]$ equivalent to a given one under order isomorphisms of $[0,1]$ onto itself, all of whose closures have measure 0, and associated with each point of which is a fixed denumerable total-order type, τ, constitutes the family of point ordered intervals corresponding to a unitary equivalence class of ordered bases. Clearly, Lebesgue order isomorphisms of $[0,1]$ onto itself leave this family invariant; while the order isomorphism between the closures of two sets of the family is Lebesgue in these sets (since these closures have measure 0) and can, therefore, be extended to a Lebesgue order isomorphism of $[0,1]$ onto itself (cf. Remark 3.4.7). The order isomorphism of the sets of atoms, themselves, would not be sufficient; for unlike the situation of Remark 3.4.6, an order isomorphism between subsets of $[0,1]$ which are not closed need not be extendable to an order isomorphism of $[0,1]$ onto itself (and so, from Remark 3.4.6, not extendable to their closures). Indeed, the sets, $A=\{1/2-1/n, 1/2+1/n\}_{n=2,3,\cdots}$ and $B=\{1/4-1/n, 3/4+1/n\}_{n=4,5,\cdots}$ are order isomorphic but their closures in $[0,1]$ are not (the closure of A has a point, $1/2$, without immediate predecessor or successor, while the closure of B has no such point).

The examples which follow indicate some of the limitations to the possibility of simple characterization of hull classes. Making use of a non-uniform edge, the next example gives us our first instance of order isomorphic sets which do not belong to the same hull class.

Example 3.5.10. Let g be a homeomorphism of $[0,1]$ onto itself which carries some set of measure 0 onto a set of measure different from 0 (such as f, described in § 3.3). Let $r_1, r_2, \cdots$ be an enumeration of the rationals in $[0,1]$, and let τ_k be the total-order type corresponding to the totally-ordered sets with k elements (a finite set). If X and Y are closed subsets of $[0,1]$ which have $\{r_k, \tau_k\}$ and $\{g(r_k), \tau_k\}$ as point ordered intervals (cf. Lemma 3.5.4) then there is an order isomorphism, f, of X onto Y such that $g = \theta_Y f \theta_X^{-1}$, from Lemma 3.5.3. (Note that the second part of the proof of Lemma 3.5.3 applies to order isomorphisms, g, which are not Lebesgue to give order isomorphisms, f, which are not Lebesgue.) If there is a Lebesgue order isomorphism, f_0, of X onto Y, then there is a Lebesgue order isomorphism, g_0, of $[0,1]$ onto itself carrying r_k onto $g(r_{j_k})$, $k = 1, 2, \cdots$, such that $\tau_k = \tau_{j_k}$ and $g_0 = \theta_Y f_0 \theta_X^{-1}$, from Lemma 3.5.3. Since τ_k and τ_{j_k} are the order types of finite sets with k and j_k elements, respectively, $k = j_k$; whence $g_0(r_k) = g(r_{j_k}) = g(r_k)$. Both g and g_0 are continuous and $\{r_k\}$ is dense in $[0,1]$, so that $g = g_0$. But g_0 is Lebesgue and g is not. Thus X and Y, though order isomorphic, do not lie in the same hull class.

In the foregoing example, special use was made of the fact that the edge of the closed sets involved was not uniform. The next example describes a case in which two order isomorphic sets with uniform edge (of any type we wish) do not belong to the same hull class and whose sets of atoms have closures with the same measure (not 0 or 1, of course, in view of Theorem 3.5.7, and Remark 3.5.9).

Example 3.5.11. Let C_1 and C_2 be dense denumerable subsets of Cantor sets of measures 0 and 1/4, respectively, in $[0,1/2]$, both C_1 and C_2 containing $1/2$; and let g' be a homeomorphism of $[0,1/2]$ onto itself carrying C_1 onto C_2 (multiply by 1/2 in the preceding example). Let D_1 be a dense denumerable subset of $[5/8,1]$ containing $5/8$; and let D_2 be its image under the linear order preserving homeomorphism, g'', of $[5/8,1]$ onto $[7/8,1]$. The mapping, g, defined as g' on $[0,1/2]$, g'' on $[5/8,1]$, and linear from $[1/2,5/8]$ to $[1/2,7/8]$, is an order isomorphism of $[0,1]$ onto itself carrying $S_1 (= C_1 \cup D_1)$ onto $S_2 (= C_2 \cup D_2)$. Note that the closures of S_1 and S_2 have measure 3/8. If h is an order isomorphism of S_1 onto S_2, then $h(1/2)$ is a point of S_2 with an immediate successor and, hence, a point of C_2. If $h(1/2) \neq 1/2$, then some point of D_1 other than 5/8 maps onto 7/8. However, no point of D_1 other than 5/8 has an immediate predecessor, while 7/8 does. Thus $h(1/2) = 1/2$, $h(5/8) = 7/8$, $h(C_1) = C_2$, and $h(D_1) = D_2$. Since the closure of C_1 has zero measure and that of C_2 does not, no Lebesgue order isomorphism of $[0,1]$ onto itself carries S_1 onto S_2. From Lemma 3.5.3, X and Y, closed subsets of $[0,1]$ whose point ordered intervals have S_1 and S_2 for their sets of atoms, respectively, are not in the same hull class. However, there is an order isomorphism, viz. g, of $[0,1]$ onto itself carrying S_1 onto S_2, and again, from Lemma 3.5.3, X and Y are order isomorphic.

Added February 3, 1960: Question 2.4.1 has a negative answer as can be seen with the aid of a result of W. F. Donaghue, "The lattice of invariant subspaces of a completely continuous quasinilpotent transformation," *Pacific Journal of Mathematics*, vol. 7 (1957), pp. 1031-1035.

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