UNITARY INVARIANTS FOR REPRESENTATIONS OF OPERATOR ALGEBRAS

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CHAPTER I. INTRODUCTION AND BASIC CONCEPTS

1.1. Introduction

This paper will be concerned with certain developments in the spectral multiplicity (unitary invariants) theory of self-adjoint families of operators. The subject has its roots in the classical multiplicity solution of Hahn [12] and Hellinger [14] of the problem of describing those self-adjoint operators which are unitarily equivalent to a given self-adjoint operator. In their basic form, the results we obtain will provide a solution to this problem when the operators in question are not necessarily normal. These results were outlined in [19] (though, as stated there, they are incorrectly applied in the case of non-separable Hilbert spaces— we have made the revisions necessary to include the general case in this paper).

Since the publication of [12, 14], the question has been re-examined and several variations and improvements made on the original solution. One may mention in this connection Wecken [49], Nakano [31, 32], Plessner-Rohlin [39] and Halmos [13]. These improvements have brought into focus the critical rôle played by measure-theoretic constructs in the unitary determination of self-adjoint operators. Moreover, the theory developed in [31, 13] applies, almost without change, to the unitary determination of commutative C^*-algebras. (The following section contains a precise description of the terminology and notation we use.)

Concurrently with these later improvements in the spectral multiplicity theory of a single self-adjoint operator, Murray and von Neumann undertook an investigation of rings of operators, more particularly, factors [28, 29, 30, 33, 34], while Nakano carried out a multiplicity decomposition of abelian rings of operators in terms of maximal abelian algebras [32]. For all but the type III case, the Murray and von Neumann results reduced the unitary equivalence problem for factors to an algebraic problem and the determination of a "coupling constant" [30]. In recent years, Dixmier [6] and Kaplansky [22] have developed techniques for dealing with general rings of operators in much the same way as Murray and von Neumann dealt with factors. Making use of these techniques, Dye [7, 8], Griffin [10] and Pallu de la Barrière [51] carried the unitary equivalence theory developed by Murray and von Neumann to general rings of operators having no part of type III. Slightly before this, Segal [40, 41] put the results of [32] in a

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more precise and cogent form extending them to rings of type I and carrying out the classification of maximal abelian algebras in terms of numerical invariants by means of the Maharam classification of measure algebras [26]. In this connection, the work of Kaplansky in [23] is quite relevant.

By completely elementary means, Griffin established the surprising fact that an algebraic isomorphism between factors of type III on a separable Hilbert space can be implemented by a unitary transformation. Amplifying this and incorporating it in his previous work, Griffin was able to determine when an isomorphism between two rings of operators, having no portion of type II_∞ with a II_1 commutant, is implemented by a unitary transformation [11]. The gap in the case of a II_∞ ring with a II_1 commutant is filled in [18]. Totally, then, we have a unitary invariants theory for the general ring of operators in terms of algebraic isomorphism.

The complete (and, at present, rather smoothly functioning) classification of spatial types of rings of operators in terms of algebraic type together with the neatly formulated spectral multiplicity theory for a single self-adjoint operator [13, 31] (or, what amounts to the same thing, for a commutative C^*-algebra) made it seem hopeful that an attempt to classify the general C^*-algebra spatially in terms of its algebraic type (then followed by [17]) might prove successful.

To understand the nature of the question with which we deal it is important to make completely clear the distinction between the spatial classification of rings of operators and of C^*-algebras. The classical question and historically the first to be examined and settled in this area was the spatial classification of a single self-adjoint operator acting on a separable Hilbert space. Contrary to appearance, however, the spatial classification of abelian rings of operators in terms of algebraic type does not contain the classification of the single operator situation. A little thought shows that there is no apparent reason why the unitary equivalence of two (abelian) rings of operators, each generated by a self-adjoint operator having some fixed set as spectrum, should admit an implementation by a unitary transformation which carries the first of these operators onto the second. Indeed, in §5.2, we shall present an example in which such implementation is not possible under the most restricted circumstances. The fact is that the spatial classification of a single self-adjoint operator requires a more detailed analysis than the spatial classification of abelian rings of operators in terms of algebraic types (although the latter classification can be made a part of the former). An abelian ring of operators can be decomposed in terms of multiple copies of maximal abelian algebras, and algebraically isomorphic maximal abelian algebras are unitarily equivalent. Thus the abelian ring of operators is spatially characterized by cardinal numbers and the algebraic types of the maximal abelian algebras occurring in its decomposition. If the final classification in terms of numerical quantities is desired, the maximal abelian algebras must be analyzed in terms of the numerical quantities associated with its measure algebra [26, 40, 41]. In terms of algebraic types, however, this classification does not require measure-theoretic constructs.

In contrast to the case of abelian rings of operators, the spatial classification of abelian C^*-algebras includes the classification of the single-operator situation, both the C^*-algebra and single operator classifications requiring the determination of the spectrum and a canonically associated family of ideals of Borel subsets thereon. These remarks make it clear that the spatial classification of abelian C^*-algebras (rather than abelian rings of operators) is the "many-operator, commutative" extension of the classical multiplicity theory of a single self-adjoint operator. In accord with this, one would expect the spatial classification of (not-necessarily-commutative) C^*-algebras (rather than general rings of operators) to represent the non-commutative extension of classical multiplicity theory. This is the case, and the invariants we obtain which determine a C^*-algebra spatially are much like those of the commutative case (and yield them easily upon specialization).

Of course, the unitary equivalence of two C^*-algebras entails their algebraic equivalence. By classifying the representations of a given abstract C^*-algebra, we include a somewhat more general situation than we would by classifying the unitary equivalence classes of concretely represented C^*-algebras and at the same time embody the assumption of algebraic equivalence without the need for carrying added algebraic isomorphisms throughout the computations. (We are indebted to George Mackey for pointing out the advisability of dealing with representations under other circumstances and for bringing to our attention the outline of the commutative separable case in [24].)

According to [15], each abstract C^*-algebra is isomorphic with some canonically constructed linear space of functions on a compact Hausdorff space (the "pure state space" of the algebra). Thus a representation of the C^*-algebra gives rise to a representation of the associated function system. Speaking for the moment of the separable or countably-decomposable case, we associate a descending chain of ideals of Borel subsets of the compact Hausdorff space with such a representation in a canonical manner (cf. Definition 4.4.1). Our main theorem, The Unitary Invariants Theorem (and The Second Unitary Invariants Theorem) of §4.4, states that this chain of ideals completely characterizes the unitary equivalence class of the representation. Since the canonically associated function system and pure state space may be difficult to compute in a given specific situation, while some other associated function system may be readily at hand, we have broadened our approach to cover more general function systems. (This broadening is described in more detail in §2.4.) Thus the theory developed is not tied to the function representation on the pure state space.

The basically new concept underlying the theory is the association of an ideal of Borel subsets (cf. the "permanent null sets" of Definition 2.2.1) with a (possibly non-commutative) representation of a function system. The Extension Theorem, Theorem 2.2.5, provides the relation between the concrete C^*-algebra and its weak closure which permits the application of available ring-of-operator techniques. It is the key result, and, once it has been established, the remainder is the technically arduous but conceptually simple matter of reworking some of the theory of rings of operators into a form suitable for combination with it

(Chapters III and IV) and application to the proof of The Unitary Invariants Theorem, 4.4.2. We conclude the paper with a chapter on special cases, examples, applications, and computation techniques.

CONTENTS

1.2. Notation and basic concepts

When we refer to a Hilbert space we shall mean a complex Hilbert space without a separability restriction unless specifically noted. We employ the usual parentheses notation $(,)$ to denote the inner product and $\| x \|$ to denote the norm (length) of a vector x. We use the notation $\| A \|$ to denote the norm (bound) of a continuous (bounded) linear operator A, and all our operators will be bounded unless otherwise noted. The uniform topology on the family of bounded operators on a Hilbert space is the metric topology induced by $\| \ \|$.

In addition to the uniform topology, we shall have occasion to use the weak operator topology defined as the weakest topology on the bounded operators in which the linear functionals $A \to (Ax, y)$ are continuous (the class of functionals being that obtained by letting x and y range through the Hilbert space), and the strong topology defined as the weakest topology in which the linear mappings $A \to Ax$ of bounded operators into the Hilbert space (taken in its norm topology) are continuous. We recall the result of Dixmier [5] (cf. [27], in this connection) that the strong and weak closures of any convex set of operators coincide.

By a C^*-algebra we shall mean an abstract Banach algebra with a unit ele-

ment, over the complex numbers, with a conjugate-linear, involutory, anti-automorphism $A \rightarrow A^*$ satisfying $\| AA^* \| = \| A \|^2$. A well-known theorem of Gelfand-Neumark [9] asserts that a C^*-algebra has an isomorphic, norm-pre-serving, representation as a uniformly-closed algebra of operators on a Hilbert space such that the *-operation in the C^*-algebra is carried into the adjoint operation in the operator algebra—whence, the operator algebra is self-adjoint. (We say that a family of operators is self-adjoint when it contains the adjoint of each operator in it.) We refer to the general C^*-algebra as an "abstract C^*-algebra" and to a C^*-algebra of operators as a "concrete C^*-algebra", when it is necessary to make these distinctions.

By a state of a C^*-algebra we shall mean a linear functional, positive on posi-tive elements and 1 on the unit element. In [15], fairly general partially-ordered vector spaces are studied (the "states" being called "positive, normalized, linear functionals") and it is shown that such spaces are linearly isomorphic in an order-preserving manner with a linear space of real, continuous functions on some compact Hausdorff space. The compact Hausdorff space was obtained by taking the closure in the w^*-topology on the dual space of the partially-ordered vector space of the set of extreme points of the convex set formed by the states. We call such an extreme point a "pure state" of the vector space. We call the closure in question "the pure state space". Now the set of self-adjoint elements in a C^*-algebra is a particular example of such a partially-ordered vector space. We shall refer to the system consisting of the linear space of functions (or its com-plexification, which is obviously linearly isomorphic with the full C^*-algebra) and the compact Hausdorff space as "the representing function system of the C^*-algebra". The isomorphism between the C^*-algebra and the linear space of functions will be called "the canonical isomorphism". With regard to the states and pure states themselves, we recall that a state of a subspace, containing the order unit, of some partially-ordered vector space has a state extension to the full space, and pure states have pure state extensions. Those states of a concrete C^*-algebra of the form $A \rightarrow (Ax, x)$, with x a unit vector in the Hilbert space, will be called "vector states" and denoted by ω_x, as is now conventional.

A concrete C^*-algebra which is closed in the weak operator topology will be called "a ring of operators" (cf. [28]). Note that, by [5, 27], the strong as well as the weak closure of a concrete C^*-algebra is a ring of operators. We denote the weak (= strong) closure of $\mathfrak{A}$ by $\mathfrak{A}^-$. If the center of a ring of operators consists solely of scalar multiples of the identity operator I, it is called "a factor" (cf. [28]). If $\mathfrak{R}$ is a ring of operators, the family of operators each of which com-mutes with every operator in $\mathfrak{R}$ will be called the commutant of $\mathfrak{R}$ and be de-noted by $\mathfrak{R}'$. The commutant $\mathfrak{R}'$ is again a ring of operators, and when $\mathfrak{R}$ con-tains the identity operator, $\mathfrak{R} = \mathfrak{R}''$; in fact, with $\mathfrak{R}$ any self-adjoint algebra of operators containing the identity operator, $\mathfrak{R}''$ is the weak (and strong) closure of $\mathfrak{R}$ (cf. [35]). We employ the notation $\mathcal{S}'$ for the commutant of any set $\mathcal{S}$ of operators.

If $\mathcal{S}$ is a set of operators on the Hilbert space $\mathcal{H}$ and x is a vector in $\mathcal{H}$, we

denote by [$\mathfrak{S}x$] the closed linear manifold (or orthogonal projection on this manifold) generated by vectors of the form Ax with A in $\mathfrak{S}$. If $\mathfrak{A}$ is an arbitrary algebra of operators with weak (hence, strong) closure $\mathfrak{A}^-$, it is immediate that $[\mathfrak{A}x] = [\mathfrak{A}^-x]$, and, with $\mathfrak{A}$ self-adjoint, $[\mathfrak{A}x]$ belongs to $\mathfrak{A}'$, since $[\mathfrak{A}x]$ is invariant under $\mathfrak{A}$. We say that x is a generating vector for $[\mathfrak{A}x]$ (under $\mathfrak{A}$) and refer to $[\mathfrak{A}x]$ as a cyclic projection in $\mathfrak{A}'$. A projection E in a ring of operators $\mathfrak{R}$ will be said to be countably-decomposable (relative to $\mathfrak{R}$) when each (pairwise) orthogonal family of projections in $\mathfrak{R}$ contained in E has at most a countable number of non-zero elements. We say that $\mathfrak{R}$ is countably-decomposable when I is countably-decomposable (relative to $\mathfrak{R}$).

As we remarked earlier, much of our task will be that of reworking the theory of rings of operators into a form suitable for our purposes. We outline here some of the basic theory and results which we assume. To begin with, we assume the entire "comparison theory" for projections in a ring of operators. We recall that two projections E and F in a ring of operators $\mathfrak{R}$ are said to be equivalent (written $E \sim F$) when $V^*V = E$, $VV^* = F$, with V in $\mathfrak{R}$, in which case, V is a partially-isometric operator with initial space E and final space F. (We use the same symbol to denote an orthogonal projection and its range when no confusion can arise.) Using equivalence with subprojections, in an obvious way, one arrives at a partial ordering among the projections of $\mathfrak{R}$ (the notation $E \precsim F$, $E \prec F$ is employed in the usual manner). The basic result (cf. [6, 22, 25, 28]), which we shall refer to as "The Comparison Lemma", asserts that, corresponding to each pair of projections E and F in a ring of operators $\mathfrak{R}$ there are three central projections P, Q and R in $\mathfrak{R}$, uniquely determined by the properties: $PE \sim PF$, and P is the maximal central projection with this property, $QE \prec QF$, $RF \prec RE$, and $P + Q + R = I$.

A projection in a ring of operators $\mathfrak{R}$ is said to be infinite if it is equivalent to a proper subprojection of itself, finite otherwise. An infinite projection is said to be purely infinite when its product with each central projection is either 0 or infinite. In general, we qualify a property with the term "purely" when it persists under restriction to central projections in the manner indicated for "pure-infiniteness". Each ring of operators $\mathfrak{R}$ contains a central projection Q maximal with respect to the property of being purely-infinite. The projection Q will be called the purely-infinite portion of $\mathfrak{R}$, and $I - Q$, which is necessarily finite, will be called the finite portion of $\mathfrak{R}$.

We shall have occasion to make incidental reference to the "type decomposition" of rings of operators. We give a brief discussion of this decomposition—a detailed treatment of this and the basic comparison theory for projections may be found in [6, 22, 28]. A projection E in a ring of operators $\mathfrak{R}$ is said to be an abelian projection (in $\mathfrak{R}$) when $E \mathfrak{R} E$ is an abelian ring. Corresponding to each positive integer and cardinal n, there is a central projection P_n in $\mathfrak{R}$ maximal with respect to the property of being the sum of n orthogonal equivalent abelian projections in $\mathfrak{R}$. The projection P_n is called the portion of $\mathfrak{R}$ of type I_n ($\mathfrak{R}$ is said to be of type I_n when P_n is the unit element of $\mathfrak{R}$). The projection P equal to

the sum of the P_κ is called the portion of $\Re$ of type I (or the discrete portion of $\Re$). There is a central projection Q in $\Re$ maximal with respect to the property that each non-zero subprojection is infinite. The projection Q is called the portion of $\Re$ of type III (and $\Re$ is said to be of type III when Q is the unit projection in $\Re$). The projection $I - P - Q$ is called the portion of $\Re$ of type II (or the continuous portion of $\Re$). The finite portion of $\Re(I - P - Q)$ is called the portion of type II_1 and the infinite portion, the portion of type II_∞.

The theory of rings of operators has a natural dividing line which separates the algebraic theory from the spatial theory. Those results involving the ring itself are algebraic and those involving the ring and its commutant are spatial. The theory we have had to rework is mostly the spatial theory (except for our dimension theory), and, for the most part, those results we use are proved. We shall, however, make use of two basic spatial results which we state here. The first is Lemma 9.3.3 of [28] and we note that (despite the preliminary comments made there) the proof and result apply to general rings of operators on non-separable Hilbert spaces. This result states that if $\Re$ and $\Re'$ are a ring of operators and its commutant, respectively, and if x and y are vectors in the Hilbert space upon which they act such that $[\Re'x] \underset{>}{\overset{<}{\sim}} [\Re'y]$ then $[Rx] \underset{>}{\overset{<}{\sim}} [Ry]$.

The second result states that a *-isomorphism between two rings of operators which have, together with their commutants, joint generating vectors, is implemented by a unitary transformation. We had originally intended to include a proof of this, but it is so elegantly and simply presented in [3] that we have only sketched the proof of an allied result in §2.4 and shall refer to that note instead. We shall call the result just stated "The Unitary Implementation Theorem".

A result of Kaplansky, [21], proves of great technical value to us. It states that the unit sphere in the strong (= weak) closure of an algebra of operators invariant under the adjoint operation is the strong closure of the unit sphere in this algebra. We note that Kaplansky's argument establishes that the self-adjoint operators in one unit sphere are strong limits of the self-adjoint operators in the other unit sphere and that the analogous result holds for the subfamily of all positive operators and its strong closure.

A normal state of a concrete C^*-algebra is a normal state of its weak closure in the sense of [4], i.e., a state which has as the limit of its values on each of the operators of a bounded increasing directed sequence in the weak closure, the value of the state on the least upper bound of the directed sequence. Equivalently (though not obviously so) a normal state of a ring of operators is a state which is completely additive on the lattice of orthogonal projections in the ring, and is a state which is strongly continuous on the unit sphere in the ring.

The various mappings we consider will be adjoint preserving (when applied to systems of functions, we mean by this that real functions are carried onto self-adjoint operators). By a representation of an abstract C^*-algebra (as a concrete C^*-algebra), we shall mean then an algebraic homomorphism which preserves the adjoint operation (and as an inessential but convenient normalization, we assume that the unit in one algebra is carried onto the unit in the other algebra). Our isomorphisms, homomorphisms, etc., between C^*-algebras, will

be adjoint preserving (even when this is not specifically indicated by referring to them as *-isomorphisms, etc.). Employing the identity of the various characterizations of a normal state, Kaplansky's Lemma, and some elementary convergence theory for operators, it is a relatively easy matter to show that a *-isomorphism between rings of operators is strongly (weakly) continuous when restricted to the unit spheres, each being taken in its strong (or weak) topology.

We include, in this section, a lemma concerning well-ordered families of projections, which will be of some use to us later.

LEMMA 1.2. *If* $\{P_\alpha\}$ *is a set of projections indexed by a well-ordered family* S *and* $P_\alpha \leqq P_\beta$ *when* $\alpha \leqq \beta$, *then the family* $\{Q_\alpha\}$, $Q_\alpha = P_\alpha - \bigcup_{\beta<\alpha} P_\beta$, *consists of mutually orthogonal projections with sum equal to the union* P *of* $\{P_\alpha\}$.

PROOF. If $\alpha < \beta$ then by definition Q_β is orthogonal to P_α, and Q_α is contained in P_α, so that Q_β and Q_α are orthogonal to each other. From the remark just made it is clear that $\bigcup_\alpha Q_\alpha \leqq P$. We assert that $P_\beta = \bigcup_{\alpha \leqq \beta} Q_\alpha$ for each β in S, whence $\bigcup_\alpha Q_\alpha$ contains P, and the proof is complete. We establish this fact by transfinite induction. If 1 denotes the first element in S then $Q_1 = P_1$, by definition, and the assertion holds for index 1. We assume that it has been established for all $\alpha < \beta$. But $Q_\beta = P_\beta - \bigcup_{\alpha<\beta} P_\alpha = P_\beta - \bigcup_{\alpha<\beta} Q_\alpha$, so that $P_\beta = \bigcup_{\alpha \leqq \beta} Q_\beta$, as we wished to show.

It should be noted that the analogous statement need not hold if S is simply-ordered but not well-ordered. The case in which S is the real line and $\{P_t\}$ any purely continuous spectral resolution (say, the resolution of the operator "multiplication by x" on $L_2(0, 1)$ under Lebesgue measure) illustrates this failure, since each Q_t is 0 in this case.

Two representations ϕ_1 and ϕ_2 of $\mathfrak{A}$ as algebras of operators $\mathfrak{A}_1$ and $\mathfrak{A}_2$ acting on the Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, are said to be unitarily equivalent when there exists a unitary transformation U of $\mathcal{K}_1$ onto $\mathcal{K}_2$ such that $U\phi_1(A)U^{-1} = \phi_2(A)$ for each A in $\mathfrak{A}$. If $\mathfrak{A} = \mathfrak{A}_1$ and ϕ_1 is the identity isomorphism, we say, in this case, that the isomorphism ϕ_2 of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ is unitarily implemented by U. If there exists a U carrying the family $\mathcal{S}_1$ onto $\mathcal{S}_2$, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are unitarily equivalent.

CHAPTER II. REPRESENTATIONS AND EXTENSION INVARIANTS

The constructs which provide the key to the multiplicity decomposition of representations of C^*-algebras are defined and their properties listed in this chapter. The first section deals with the various types of mappings we shall consider; the second section, with extension invariants in general and permanent null sets of representations in particular; the third section, with results related to and clarifying certain features of The Extension Theorem of the preceding section; and the fourth section, with techniques for carrying the general considerations of this chapter over to an extension of our later results.

2.1. Mappings of function systems

By a "function system", we shall mean the pair $(\mathfrak{L}, X)$ consisting of a complex (real) linear subspace, $\mathfrak{L}$, closed under complex conjugation of functions and

containing the constant functions, of the linear space, $C(X)$, of complex (real)-valued, continuous functions on the compact Hausdorff space X, and the compact space X. When we speak of "order" in referring to functions of $\mathfrak{L}$, we shall mean the usual pointwise partial-ordering of the real functions in $\mathfrak{L}$.

DEFINITION 2.1.1. *An "order-representation" of a function system* $(\mathfrak{L}, X)$ *is a linear, order, adjoint and identity preserving mapping of* $\mathfrak{L}$ *upon some linear space of operators acting on a Hilbert space. An "order-homomorphism" is an order-representation* ϕ *whose kernel is linearly generated by positive functions and such that* $\phi^{-1}(A)$ *contains a positive element for each positive operator* A *in the image of* ϕ.

Several elementary remarks will be of later use. In view of the order-preserving nature of an order-representation, such a mapping is continuous, since it is norm-decreasing on real functions (the supremum norm being used for functions). The kernel of an order-representation is generated by its positive elements if and only if each real element of the kernel is dominated by a positive element of the kernel. In fact, if such domination is possible then for each real f in the kernel $\mathcal{K}$ there is a positive g in $\mathcal{K}$ such that $g \geqq f$, so that $f = g - (g - f)$; and $\mathcal{K}$ is generated by its positive elements (certainly by its real elements, since an order-representation is adjoint preserving). On the other hand, if $\mathcal{K}$ is generated by its positive elements and f is a real function in $\mathcal{K}$, then $f = g_1 + \cdots + g_n - (h_1 + \cdots + h_n) = g - h$, with g and h positive and in $\mathcal{K}$ (we may assume coefficients are real by equating real parts), so that $g \geqq f$.

By an "operator system", we shall mean a (complex) C^*-algebra or the real Jordan algebra of self-adjoint elements in such an algebra (with $AB + BA$ as product), the terms "abstract" and "concrete" being appended when it is desired to indicate whether or not action on a specific Hilbert space is intended. The term "representation" as applied to such systems will mean a linear, adjoint (where applicable), product or Jordan-product, and identity preserving mapping onto another such system. It follows from the fact that $\phi(|A|) = |\phi(A)|$, that a representation ϕ of an operator system is an order-homomorphism.

As with C^*-algebras alone, the general operator system has a canonically associated function system (its representing function system on its pure state space [15]). When discussing a representation of an operator system, we shall, when occasion demands, pass to the mapping of its associated function system obtained by composing the representation with the canonical mapping of the function system upon the operator system, without further statement and using the same symbol for both mappings.

2.2. Permanent null sets and The Extension Theorem

We describe our key extension invariant in:

DEFINITION 2.2.1. *If* $(\mathfrak{L}, X)$ *is a function system and* ρ *is a state of* $\mathfrak{L}$, *we say that a Borel subset* S *of* X *is a "permanent null set of* ρ" *when each positive extension of* ρ *to* $C(X)$ *induces a regular Borel measure on* X *of which* S *is a null set.*

The collection $\mathfrak{N}_\rho$ of permanent null sets of ρ we call "the null ideal of ρ". If ϕ is an order-representation of $\mathfrak{L}$ as a linear space of operators $\mathfrak{A}$ acting on the Hilbert space $\mathfrak{K}$, we call the intersection $\mathfrak{N}_\phi$ of the null ideals $\mathfrak{N}_{\omega\phi}$ of the states $\omega\phi$ of $\mathfrak{L}$, as ω ranges through the normal states of $\mathfrak{A}$, "the null ideal of ϕ" and the sets in $\mathfrak{N}_\phi$, "the permanent null sets of ϕ". (We denote by $\mathfrak{N}_\phi'$ the intersection as ω ranges through the vector states of $\mathfrak{A}$ and refer to "the vector null ideal of ϕ" and "the permanent vector null sets of ϕ".)

When the order-representation in question is clearly indicated, we shall denote the null ideal of $\omega\phi$ by $\mathfrak{N}_\omega$ and of $\omega_x\phi$ by $\mathfrak{N}_x$. We note that the null ideal of a state of $\mathfrak{L}$ is its null ideal when it is considered to be an order-representation. Although we have agreed to pass freely between an operator system $\mathfrak{A}$ and its representing function system $(\mathfrak{L}, X)$, we must recognize that, when $\mathfrak{A}$ is concretely represented as acting on a Hilbert space, the canonical isomorphism ι between $\mathfrak{L}$ and $\mathfrak{A}$ possesses a null ideal $\mathfrak{N}_\iota$, in view of the above definition. We shall call this ideal "the canonical null ideal of $\mathfrak{A}$" and denote it by $\mathfrak{N}(\mathfrak{A})$.

The distinction we have made between the null ideal $\mathfrak{N}_\phi$ and the vector null ideal $\mathfrak{N}_\phi'$ is not a vacuous one. In Example 5.2.4, we give an instance of $\mathfrak{N}_\phi$ being properly smaller than $\mathfrak{N}_\phi'$—it being clear, of course, that $\mathfrak{N}_\phi$ is always contained in $\mathfrak{N}_\phi'$. In many cases, however, $\mathfrak{N}_\phi$ and $\mathfrak{N}_\phi'$ coincide—for example, when the weak closure of the image $\mathfrak{A}$ of ϕ is an operator system possessing a separating vector. (In this case, each normal state is a vector state, by [3]).

The following lemma will be important for the proof of The Extension Theorem to follow and for the construction of the example to which we have just alluded.

LEMMA 2.2.2. *If ϕ is an order-homomorphism of the function system $(\mathfrak{L}, X)$ onto the concrete operator system $(\mathfrak{A}, \mathfrak{K})$ then a Borel subset S of X does not lie in $\mathfrak{N}_\phi$ (in $\mathfrak{N}_\phi'$) if and only if there exists a normal (vector) state ω' of $\mathfrak{A}$ such that the integration process ω due to the regular Borel measure induced on X by some state extension of $\omega'\phi$ from $\mathfrak{L}$ to $C(X)$ has the property that $\omega(f) = \omega(f\chi_S)$, for each bounded Borel function f on X, where χ_S is the characteristic function of S. If S consists of a single point p in X, then $\{p\}$ does not lie in $\mathfrak{N}_\phi$ (in $\mathfrak{N}_\phi'$) if and only if the state $f \to f(p)$ of $\mathfrak{L}$ is induced by a normal (vector) state of $\mathfrak{A}^-$.*

PROOF. If a state such as ω' exists, then, taking f to be 1, $1 = \omega(1) = \omega(\chi_S)$, so that S does not lie in $\mathfrak{N}_\phi$ (in $\mathfrak{N}_\phi'$), by definition of $\mathfrak{N}_\phi$ (of $\mathfrak{N}_\phi'$).

If S does not lie in $\mathfrak{N}_\phi$ (in $\mathfrak{N}_\phi'$), there is a normal (vector) state ω_0' of $\mathfrak{A}$ such that the integration process ω_0 induced by some state extension of $\omega_0'\phi$ to $C(X)$ assigns to χ_S a number t greater than 0. Let ω_1 be defined by $\omega_1(f) = \omega_0(f\chi_S)/t$ for f in $C(X)$, so that ω_1 is a state of $C(X)$; and let ω be the integration process due to ω_1, so that $\omega(g) = \omega_0(g\chi_S)/t$ for g a bounded Borel function on X. Applying the Schwarz inequality to the integration process ω_0, we have

$$0 \leq |\omega_0(h\chi_S)| \leq \omega_0(h)^{\frac{1}{2}}\omega_0(h\chi_S)^{\frac{1}{2}} = \omega_0'(\phi(h))^{\frac{1}{2}}\omega_0(h\chi_S)^{\frac{1}{2}} = 0,$$

with h a positive function in the kernel $\mathfrak{K}$ of ϕ. Thus ω_1', the restriction of ω_1 to $\mathfrak{L}$, annihilates $\mathfrak{K}$, since $\mathfrak{K}$ is generated by its positive elements, so that $\omega_1' = \omega'\phi$, with ω' a linear functional on $\mathfrak{A}$. Since ϕ is an order-homomorphism, the

representative f of the positive operator $\phi(f)$ in $\mathfrak{A}$ may be chosen positive, whence $\omega'[\phi(f)] = \omega_1(f) \geqq 0$, and ω' is a state of $\mathfrak{A}$. Moreover,

$$\omega'[\phi(f)] = \omega_1(f) = \frac{1}{t}\,\omega_0(f\chi_S) \leqq \frac{1}{t}\,\omega_0(f) = \frac{1}{t}\,\omega_0'[\phi(f)],$$

that is, $\omega' \leqq \omega_0'/t$. Now ω', being dominated by a multiple of a normal (vector) state ω_0' of $\mathfrak{A}^-$, is weakly continuous on the unit sphere of $\mathfrak{A}$, whence, as a special case of the remark following this lemma (by [7; Lemma 2.2]), ω' has a normal (vector) state extension to $\mathfrak{A}^-$. The integration process ω stems from ω' and

$$\omega(f) = \frac{1}{t}\,\omega_0(f\chi_S) = \omega(f\chi_S),$$

which establishes the first assertion of this lemma.

With $S = \{p\}$ and ω as above,

$$\omega(f) = \omega(f\chi_p) = f(p)\omega(\chi_p) = f(p),$$

so that the state $f \to f(p)$ of $\mathfrak{L}$ is induced by a normal (vector) state of $\mathfrak{A}^-$ if and only if $\{p\}$ does not lie in $\mathfrak{N}_\phi$ (in $\mathfrak{N}_\phi'$)—which completes the proof.

REMARK 2.2.3. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are operator systems acting on the Hilbert spaces $\mathfrak{IC}_1$ and $\mathfrak{IC}_2$, respectively, and ϕ is a positive linear mapping of $\mathfrak{A}_1$ into $\mathfrak{A}_2$, then ϕ has an extension mapping $\mathfrak{A}_1^-$ into $\mathfrak{A}_2^-$, which is weakly continuous on the unit sphere in $\mathfrak{A}_1^-$ if and only if the states $A \to (\phi(A)x, x)$, x a unit vector in $\mathfrak{IC}_2$, of $\mathfrak{A}_1$ are each weakly continuous on the unit sphere $\mathcal{S}_1$ in $\mathfrak{A}_1$ at 0. Indeed, the necessity of this condition is obvious from the definition of "weak operator topology". On the other hand, if the condition obtains then ϕ is weakly continuous on $\mathcal{S}_1$ at 0 since the functionals $B \to (Bx, x)$, x in $\mathfrak{IC}_2$, define a subbase for the weak operator topology in $\mathfrak{A}_2$. Now the weak operator topology determines a topological-linear and uniform structure, and ϕ, being linear and continuous at 0 on any sphere about 0, is continuous relative to the uniform structure on $\mathcal{S}_1$. However, the unit sphere in $\mathfrak{A}_1^-$ is compact in the weak operator topology (as a consequence of The Tychonoff Theorem on the compactness of the product of compact sets) and is equal to $\mathcal{S}_1^-$, by [21, 5, 27]. Thus ϕ has a (unique) weakly continuous extension ϕ_0 to $\mathcal{S}_1^-$ (see e.g., [2; Theorem 1, p. 101]). The extension ϕ_0 of ϕ from $\mathfrak{A}_1$ to $\mathfrak{A}_1^-$ is now apparent. We note that ϕ_0 is order-preserving on $\mathfrak{A}_1^-$, since, with A positive in $\mathfrak{A}_1^-$, we have $(\phi_0(A)x, x)$ is a limit point of non-negative real numbers $(\phi(B)x, x)$, B positive in $\mathfrak{A}_1$. Moreover, for ϕ to have the extension in question, it will suffice to show that the states $A \to (\phi(A)x, x)$ above are weakly continuous at 0 on the set $\mathcal{S}_1^+$ of positive operators in the unit sphere of $\mathfrak{A}_1$. In fact, if this is the case, it follows at once that the states $A \to (\phi(A)x, x)$ are strongly continuous at 0 on $\mathcal{S}_1^+$; whence, the composite mapping $A \to (A^+, -A^-) \to (\phi(A^+)x, x) - (\phi(-A^-)x, x) = (\phi(A)x, x)$, where $A^+ = (|A| + A)/2$ and $A^- = (A - |A|)/2$, is strongly continuous at 0 on the self-adjoint part $\mathcal{S}_{1*}$ of $\mathcal{S}_1$, hence at 0 on $k\mathcal{S}_{1*}$ for each $k > 0$, and hence, by linearity, on all of $\mathcal{S}_{1*}$. From [5, 27] and linearity of $A \to (\phi(A)x, x)$, we conclude its weak continuity on $\mathcal{S}_{1*}$ and hence on $\mathcal{S}_1$.

The following definition introduces the refinement in the concept of "null ideal of a representation" necessary to handle the extension problem when the images of the representations under consideration have higher than countable decomposability character. This refinement is analogous to (and, in fact, gives a slight improvement of) the (best) refinement employed in the description of the multiplicity invariants of a self-adjoint operator (or commutative C^*-algebra) on a non-separable Hilbert space.

DEFINITION 2.2.4. "*A (vector) null ideal band*" *of a representation ϕ of a function system as a concrete operator system* $(\mathfrak{A}, \mathfrak{K})$ *is a collection* $\{\mathfrak{N}_{\phi|E_\alpha}\}(\{\mathfrak{N}'_{\phi|E_\alpha}\})$ *of null ideals* $\mathfrak{N}_{\phi|E_\alpha}(\mathfrak{N}'_{\phi|E_\alpha})$ *of restrictions* $\phi \mid E_\alpha$ *of ϕ to countably-decomposable projections E_α in* $\mathfrak{A}^-$ *(i.e., $\phi \mid E_\alpha(f) = E_\alpha\phi(f)E_\alpha$), where $\{E_\alpha\}$ is an orthogonal family with sum I. We say that an ideal band b_1 of a representation of a function system is contained in an ideal band b_2 of another representation of the same function system when there is a one to one mapping of b_1 onto b_2 such that the image of each null ideal in b_1 contains that null ideal.*

THEOREM 2.2.5. (The Extension Theorem). *An order-homomorphism ϕ_1 and an order-representation ϕ_2 of a function system $(\mathfrak{L}, X)$ onto operator systems $(\mathfrak{A}_1, \mathfrak{K}_1)$ and $(\mathfrak{A}_2, \mathfrak{K}_2)$, respectively, induce an order-representation ϕ of $\mathfrak{A}_1^-$ into $\mathfrak{A}_2^-$ such that $\phi\phi_1 = \phi_2$ and which is weakly continuous on the unit sphere of $\mathfrak{A}_1^-$ if some null ideal band $b_1 = \{\mathfrak{N}_{\phi_1|E_\alpha}\}$ of ϕ_1 is contained in a vector null ideal band $b'_2 = \{\mathfrak{N}'_{\phi_2|F_\alpha}\}$ of ϕ_2. A mapping ϕ as described carrying E_α onto F_α, for each α, exists only if b_1 is contained in $b_2 = \{\mathfrak{N}_{\phi_2|F_\alpha}\}$. With ϕ_1 and ϕ_2 order-homomorphisms, ϕ is an order-isomorphism—hence a C^*-isomorphism, and, thus, the direct sum of a $*$-isomorphism and a $*$-anti-isomorphism of $\mathfrak{A}_1^-$ onto $\mathfrak{A}_2^-$ if either of the conditions $b_1 = b_2$ or $b'_1 = b'_2$ are satisfied and only if $b_1 = b_2$. In particular, if $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$ are countably-decomposable, $\mathfrak{N}_{\phi_1}$, $\mathfrak{N}_{\phi_2}$ and $\mathfrak{N}'_{\phi_1}$, $\mathfrak{N}'_{\phi_2}$ are null ideal bands and vector null ideal bands, respectively, and may be used in place of their respective null ideal bands in the preceding statements.*

PROOF. We assume first that ϕ_1 is an order-homomorphism, ϕ_2 an order-representation, and that the null ideal band $b_1 = \{\mathfrak{N}_{\phi_1|E_\alpha}\}$ of ϕ_1 is contained in the vector null ideal band $b'_2 = \{\mathfrak{N}'_{\phi_2|F_\alpha}\}$ of ϕ_2 (i.e., $\mathfrak{N}_{\phi_1|E_\alpha} \subset \mathfrak{N}'_{\phi_2|F_\alpha}$, for each α). We prove that the kernel $\mathfrak{K}_1$ of ϕ_1 is contained in the kernel $\mathfrak{K}_2$ of ϕ_2. Since $\mathfrak{K}_1$ is generated by its positive elements, it suffices to show that each positive function f in $\mathfrak{K}_1$ lies in $\mathfrak{K}_2$. The set S of points p in X for which $f(p) > 0$ is an open Borel set, and, for each normal state ω of $E_\alpha\mathfrak{A}_1^-E_\alpha$, $\omega[\phi_1 \mid E_\alpha(f)] = 0$, so that S is a null set of the regular Borel measure induced on X by each positive extension of $\omega\phi_1 \mid E_\alpha$ from $\mathfrak{L}$ to $C(X)$, i.e., $S \in \mathfrak{N}_\omega$. This being true for each such ω, $S \in \mathfrak{N}_{\phi_1|E_\alpha}$; and thus $S \in \mathfrak{N}'_{\phi_2|F_\alpha}$, so that $(\phi_2(f)x, x) = 0$ for each x in the range of F_α. Since $\phi_2(f)$ is positive, it follows that $\phi_2(f)x = 0$ for each such x, and therefore $\phi_2(f)F_\alpha = 0$. Hence $\phi_2(f) = \phi_2(f)\sum_\alpha F_\alpha = \sum_\alpha \phi_2(f)F_\alpha = 0$, and f is in $\mathfrak{K}_2$.

Since $\mathfrak{K}_1 \subset \mathfrak{K}_2$, we conclude that ϕ defined on $\mathfrak{A}_1$ by: $\phi[\phi_1(g)] = \phi_2$ (g) is well-defined. Moreover, if $A \geq 0$ and A is in $\mathfrak{A}_1$, we can choose a positive f in $\mathfrak{L}$ such that ϕ_1 (f) $= A$, since ϕ_1 is an order-homomorphism, and, thus, $\phi(A) = \phi_2$ (f) is positive, i.e., ϕ is an order-representation.

In view of the remark preceding this theorem, it suffices to establish the weak continuity on S_1^+, the positive operators in the unit sphere of $\mathfrak{A}_1$, at 0 of each state $A \to (\phi(A)z, z)$ of $\mathfrak{A}_1$, z a unit vector in $\mathfrak{IC}_2$, to show that ϕ has an order-preserving extension mapping $\mathfrak{A}_1^-$ into $\mathfrak{A}_2^-$ which is weakly continuous on the unit sphere S_1^- of $\mathfrak{A}_1^-$. Suppose we have established that the linear functional $A \to (\phi(A)z_\alpha, z_\alpha)$ on $\mathfrak{A}_1$ is weakly continuous on S_1^+ at 0, for each α, with $z_\alpha = F_\alpha z$; and let $\varepsilon > 0$ be given. Since $1 = \| z \|^2 = \sum_\alpha \| z_\alpha \|^2$, z_α is 0 for all but a countable number of α, say $\alpha = 1, 2, \cdots$. Then $1 = \| z \|^2 = \sum_k \| z_k \|^2$, whence, it is possible to choose a positive integer N such that $\sum_{k=N+1}^{\infty} \| z_k \|^2 < \varepsilon^2/36$. Let $x_1^{(j)}, \cdots, x_{k_j}^{(j)}$ be a finite set of vectors in $\mathfrak{IC}_1$ such that if $| (Ax_i^{(j)}, x_i^{(j)}) | < 1$ for $i = 1, \cdots, k_j$ and A in S_1^+, then $| (\phi(A)z_j, z_j) | < \varepsilon/2N^2$ (such vectors exist by weak continuity of the functional $A \to (\phi(A)z_j, z_j)$ at 0 on S_1^+). Suppose now that B is in S_1^+ and that $| (Bx_i^{(j)}, x_i^{(j)}) | < 1$ for $i = 1, \cdots, k_j$ and $j = 1, \cdots, N$. Let $z' = z_1 + \cdots + z_N$ and $z'' = z - z'$. Then

$$| (\phi(B)z, z) | \leq | (\phi(B)z', z') | + 2 | (\phi(B)z', z'') | + | (\phi(B)z'', z'') |$$

$$\leq \sum_{j,k=1}^{N} | (\phi(B)z_j, z_k) | + 2 \| z'' \| + \| z'' \|$$

$$\leq \sum_{j,k=1}^{N} (\phi(B)z_j, z_j)^{\frac{1}{2}} (\phi(B)z_k, z_k)^{\frac{1}{2}} + \varepsilon/2$$

$$\leq \sum_{j,k=1}^{N} \varepsilon/2N^2 + \varepsilon/2 = \varepsilon,$$

whence $A \to (\phi(A)z, z)$ is weakly continuous at 0 on S_1^+ for each unit vector z in $\mathfrak{IC}_2$, and ϕ has the desired extension. (Note that with B positive $[v, w] = (\phi(B)v, w)$ is a positive semi-definite inner product on $\mathfrak{IC}_2$, and that we have applied the Schwarz Inequality to this inner product to obtain the third inequality above.)

It remains, therefore, to prove that the state ω' of $\mathfrak{A}_1$ defined by $\omega'(A) = (\phi(A)z, z)$, with z a unit vector in the range of F_0 and F_0 in $\{F_\alpha\}$, is weakly continuous at 0 on S_1^+. We shall do this by extending ω' from $\mathfrak{A}_1$ to a certain subclass of $\mathfrak{A}_1^-$ and establishing the countable additivity of this extension. The subclass in question is obtained from $\mathfrak{A}_1$ by a process like that employed in defining the Baire function class with the difference that we shall consider only monotone (directed) sequences. (Sequences are all we shall use in this connection, but we extend the discussion to include directed sequences because of its independent interest.)

We begin by observing that it is possible to set up a correspondence between ordinals (attention may be restricted to ordinals whose cardinals do not exceed 2^{cd}, with d the dimension of $\mathfrak{IC}_1$) and a class of subsets of $\mathfrak{A}_1^-$, each containing $\mathfrak{A}_{1*}$, the self-adjoint operators in $\mathfrak{A}_1$, so that if $\beta > 0$ corresponds to $\mathfrak{A}_1^\beta$ then $\mathfrak{A}_1^\beta$ consists of all operators A_β which are the strong limit of monotone directed sequences $\{A_\gamma\}$ in $\bigcup_{\delta<\beta} \mathfrak{A}_1^\delta$; and so that $\mathfrak{A}_1^0 = \mathfrak{A}_{1*}$. Moreover, the correspondence with these properties is unique. (We note, and shall make use of the fact, that in the foregoing statement and in all that follows concerning the operator classes $\mathfrak{A}_1^\beta$, the results remain valid if we replace "directed sequences" by "sequences". In this case we denote the classes by $\mathfrak{A}_{1s}^\beta$).

Suppose that we have mapped each ordinal $\delta < \beta$ into a subset $\mathfrak{A}_1^\delta$ of $\mathfrak{A}_1^-$ so that the above properties hold for the ordinals less than β. Defining $\mathfrak{A}_1^\beta$ to be the set of operators in $\mathfrak{A}_1^-$ which are strong limits of monotone directed sequences of operators in $\bigcup_{\delta<\beta} \mathfrak{A}_1^\delta$ and applying transfinite induction we have our assertion. The uniqueness of the correspondence is clear from its properties (by a transfinite induction). We shall call $\mathfrak{A}_1^\beta$ "the βth monotone extension class of $\mathfrak{A}_1$", and $\bigcup_\beta \mathfrak{A}_1^\beta$ will be called "the monotone closure of $\mathfrak{A}_1$" and denoted by $\mathfrak{A}_1^m$. (We use $\mathfrak{A}_{1s}^m$ for the monotone sequential closure of $\mathfrak{A}_1$.) If A is an operator in $\mathfrak{A}_1^m$, then by "the class of A" we shall mean the least ordinal β such that A is in $\mathfrak{A}_1^\beta$.

It is apparent from the defining properties that if $\mathfrak{A}_1^\beta = \mathfrak{A}_1^{\beta+1}$ then $\mathfrak{A}_1^\gamma = \mathfrak{A}_1^\beta$ for all $\gamma > \beta$, and that $\mathfrak{A}_1^\beta$ must repeat before the cardinality of β exceeds that of the subsets of all bounded operators on $\mathfrak{K}_1$. Moreover, $\mathfrak{A}_1^m$ is closed with respect to the process of taking strong monotone directed sequential limits, for if β' is the least ordinal such that $\mathfrak{A}_1^{\beta'} = \mathfrak{A}_1^{\beta'+1}$ and A is the strong limit of the monotone directed sequence $\{A_\gamma\}$ of operators in $\mathfrak{A}_1^m$ $(= \mathfrak{A}_1^{\beta'})$ then A is in $\mathfrak{A}_1^{\beta'+1}$ $(= \mathfrak{A}_1^m)$, by definition.

An elementary transfinite induction shows us that each class $\mathfrak{A}_1^\beta$ is closed with respect to multiplication by real scalars and addition of I or $-I$. We prove that $\mathfrak{A}_1^m$ is closed with respect to addition so that $\mathfrak{A}_1^m$ is a linear space. In fact, $\mathfrak{A}_1^0$ is closed with respect to addition. Suppose it is known that the sum of two operators of class less than β is in $\mathfrak{A}_1^m$, and let A and B be the strong limits of the monotone directed sequences of operators $\{A_\delta\}$, $\{B_\gamma\}$, respectively, each A_δ and B_γ of class less than β. Now $A_\delta + B$ lies in $\mathfrak{A}_1^m$, for each δ, since it is the strong limit of the directed sequence (in γ) $\{A_\delta + B_\gamma\}$ of operators $A_\delta + B_\gamma$ which lie in $\mathfrak{A}_1^m$, by inductive assumption. Thus $A + B$, the strong limit of $\{A_\delta + B\}$, a monotone directed sequence of operators in $\mathfrak{A}_1^m$, lies in $\mathfrak{A}_1^m$; and $\mathfrak{A}_1^m$ is closed under addition, by transfinite induction.

We observe next that $\mathfrak{A}_1^m$ is closed in the uniform (norm) operator topology. In fact suppose $\{A_n\}$ tends uniformly to A with each A_n in $\mathfrak{A}_1^m$. Replacing (A_n) by a suitable subsequence, which we again call (A_n), we can arrange to have $\| B_n \| = \| A_n - A_{n-1} \|$, $n = 1, 2, \cdots$ (with $A_0 = 0$) less than $1/2^n$, so that $B_n < I/2^n$ and

$$I + \sum_{n=1}^k \left(B_n - \frac{I}{2^n} \right) = \sum_{n=1}^k B_n + \sum_{n=k+1}^\infty \frac{I}{2^n} = A_k + \frac{I}{2^k}$$

is a monotone decreasing sequence tending uniformly to A. Thus A lies in $\mathfrak{A}_1^m$, and $\mathfrak{A}_1^m$ is closed in the uniform operator topology.

We prove that $\mathfrak{A}_1^m$ is closed under the Jordan product $(A, B) \to AB + BA$, to which end it will suffice to show that the square of a positive operator in $\mathfrak{A}_1^m$ lies in $\mathfrak{A}_1^m$. Indeed, if this is done, choosing k larger than $\| B \|$, with B some operator in $\mathfrak{A}_1^m$, we have $B^2 = (kI - B)^2 - k^2I + 2kB$ lies in $\mathfrak{A}_1^m$. Thus, with B and C in $\mathfrak{A}_1^m$, we have $BC + CB = (B + C)^2 - B^2 - C^2$ in $\mathfrak{A}_1^m$. (We recall to the reader that $0 \leqq A \leqq B$ does not imply $A^2 \leqq B^2$—even for 2×2 matrices—so that, what otherwise might be the obvious approach to proving that $\mathfrak{A}_1^m$ is closed

with respect to squaring, fails.) To begin, we show, by transfinite induction, that the inverse of a positive invertible operator in $\mathfrak{A}_1^m$ lies in $\mathfrak{A}_1^m$. This is true for operators of class 0. Suppose that we have established it for operators of class less than β, and let A be a positive invertible operator in $\mathfrak{A}_1^m$ which is the strong limit of the monotone directed sequence (A_γ), each A_γ of class less than β. Multiplying by some positive scalar, we may assume that A and each A_γ has norm less than 1, so that $(I - A_\gamma)$ is a monotone directed sequence of positive invertible operators, $I - A_\gamma$ of the same class as A_γ, by the foregoing results. Thus $([I - A_\gamma]^{-1})$ is a monotone directed sequence of operators in $\mathfrak{A}_1^m$, by inductive assumption (monotone—from the fact that $B^{-1} \leq C^{-1}$ if B and C are positive invertible operators with $C \leq B$). Now $([I - A_\gamma]^{-1})$ has $(I - A)^{-1}$ as strong limit (least upper or greatest lower bound), so that $(I - A)^{-1}$ lies in $\mathfrak{A}_1^m$. By the same token, $(I - tA)^{-1}$ lies in $\mathfrak{A}_1^m$, for all positive t not exceeding 1. If we have established that all powers of A not exceeding $n \geq 1$ lie in $\mathfrak{A}_1^m$ then

$$\frac{1}{t^{n+1}} [(I - tA)^{-1} - (I + tA + t^2 A^2 + \cdots + t^n A^n)] = A^{n+1}(I - tA)^{-1}$$

lies in $\mathfrak{A}_1^m$, and $A^{n+1}(I - tA)^{-1}$ tends monotonely to A^{n+1} as t tends to 0. Thus A^{n+1} lies in $\mathfrak{A}_1^m$, so that, all positive integral powers of A lie in $\mathfrak{A}_1^m$, by induction. This, together with The Spectral Theorem, The Weierstrass Approximation Theorem, and the fact that $\mathfrak{A}_1^m$ is closed in the uniform operator topology implies that A^{-1} lies in $\mathfrak{A}_1^m$. Our assertion that the inverse of a positive invertible operator in $\mathfrak{A}_1^m$ lies in $\mathfrak{A}_1^m$ follows now, by transfinite induction. If A is a positive operator of norm less than 1 in $\mathfrak{A}_1^m$, then $(I - tA)^{-1}$ lies in $\mathfrak{A}_1^m$, when $0 < t \leq 1$; and the argument given above, with n equal to 1, shows that A^2 lies in $\mathfrak{A}_1^m$. Thus $\mathfrak{A}_1^m$ is closed under the Jordan product as asserted.

We imitate the monotone extension class construction in the function space on X. More precisely, let $\mathfrak{B}$ be the partially-ordered vector space of bounded Borel functions on X taken in their usual pointwise ordering; and let $\tilde{B}$ be the quotient space of $\mathfrak{B}$ modulo the order-ideal $\mathfrak{N}$ of functions in $\mathfrak{B}$ which vanish on the complement of a set in $\mathfrak{N}_{\phi_1}$. We prove, by transfinite induction, that it is possible to assign to each ordinal β a set of elements $\tilde{L}^\beta$ in $\tilde{B}$ and a one to one mapping ϕ_1^β of $\tilde{L}^\beta$ onto $\mathfrak{A}_{1s}^\beta$ in such a manner that:

(a) $\tilde{L}^0 = \tilde{L}$, the image of $\mathfrak{L}$ under the quotient mapping $g \to \tilde{g}$ of $\mathfrak{B}$ onto $\tilde{B}$, and $\phi_1^0 (\tilde{f}) = \phi_1(f)$, f in $\mathfrak{L}$.

(b) an element $\tilde{f}$ in $\tilde{B}$ lies in $\tilde{L}^\beta$, $\beta > 0$, if and only if there exists a monotone sequence $(\tilde{f}_n)$, $\tilde{f}_n$ in $\tilde{L}^{\gamma_n}$ with $\gamma_n < \beta$, tending to $\tilde{f}$, i.e., each sequence of representatives (f_n) tends to f pointwise on the complement of a set in $\mathfrak{N}_{\phi_1}$. Moreover $\phi_1^\beta (\tilde{f})$ is the strong limit of $(\phi_1^{\gamma_n} [\tilde{f}_n])$.

(c) if ρ' is a normal state of $\mathfrak{A}_1^-$ and ρ is the integration process due to the regular Borel measure induced on X by any positive extension of $\rho'\phi_1$ from $\mathfrak{L}$ to $C(X)$, then $\rho(f) = \rho'[\phi_1^\beta(\tilde{f})]$, for each $\tilde{f}$ in $\tilde{L}^\beta$.

(d) with $\tilde{f}$ and $\tilde{g}$ in $\tilde{L}^\beta$, $\phi_1^\beta (\tilde{f}) \geq \phi_1^\beta (\tilde{g})$ if and only if $\tilde{f} \geq \tilde{g}$. The correspondence with these properties is unique.

Condition (a) dictates our assignment to 0; viz., $\tilde{L}^0 = \tilde{L}$ and $\phi_1^0(\tilde{f}) = \phi_1(f)$, for $\tilde{f}$ in $\tilde{L}$. Regarding this definition, we note that $\mathfrak{N} \cap \mathfrak{L} = \mathfrak{K}_1$, so that ϕ_1^0 is well-defined. In fact, with f in $\mathfrak{L}$ and $\mathfrak{N}$, $(\phi_1(f)x, x) = 0$ for each vector x in $\mathfrak{K}_1$, and thus $\phi_1(f) = 0$, or f lies in $\mathfrak{K}_1$. With f positive and in $\mathfrak{K}_1$, $\tau[\phi_1(f)] = 0$ for each normal state τ of $\mathfrak{A}_1$, so that the set of points where f differs from 0 (and so, exceeds 0) is a null set of ϕ_1, and f lies in $\mathfrak{N}$. It follows that $\tilde{L} \cong (\mathfrak{L} + \mathfrak{N})/\mathfrak{N} \cong \mathfrak{L}/\mathfrak{K}_1 \cong \mathfrak{A}_1$, the isomorphisms involved being linear and, under the present assumptions, order-isomorphisms—the resultant mapping of $\tilde{L}$ upon $\mathfrak{A}_1$ is ϕ_1^0. Indeed, the isomorphism from $\mathfrak{L}/\mathfrak{K}_1$ to $\mathfrak{A}_1$ is an order-isomorphism by hypothesis (ϕ_1 is an order-homomorphism). If A is a positive operator in $\mathfrak{A}_1$, there is a positive g in $\mathfrak{L}$ such that $\phi_1(g) = A$, so that $\tilde{g} \geqq 0$ and $\phi_1^0(\tilde{g}) = A$. If $\tilde{f}$ is a positive element in $\tilde{L}$, there is a function n in $\mathfrak{N}$ such that $f + n \geqq 0$, so that $(\phi_1(f)x, x) \geqq 0$ for each x in $\mathfrak{K}_1$, and $\phi_1^0(\tilde{f}) = \phi_1(f) \geqq 0$. This establishes (d) for ϕ_1^0 (conditions (a), (b) and (c) are clear, in this case).

Suppose now that we have constructed $\tilde{L}^\alpha$, ϕ_1^α satisfying conditions (a) through (d), for each $\alpha < \beta$, and have established their uniqueness. According to (b), $\tilde{L}^{\alpha'}$ is contained in $\tilde{L}^\alpha$ if $\alpha' \leqq \alpha$, and, by uniqueness, ϕ_1^α is an extension of $\phi_1^{\alpha'}$ if $\alpha' \leqq \alpha$. Let $\tilde{L}^\beta$ be the set of all $\tilde{f}$ in $\tilde{B}$ which are the limits of monotone sequences of elements $(\tilde{f}_n)$ in $\tilde{L}^{\alpha_n}$, $\alpha_n < \beta$. Since the ϕ_1^α are extensions of one another and $(\tilde{f}_n)$ is monotone, bounded by $\tilde{f}$ in $\tilde{B}$ and hence, by some constant, it follows from inductive hypothesis (d), that $(\phi_1^{\alpha_n}(\tilde{f}_n))$ is bounded and monotone and, so, has a strong limit A which we define to be $\phi_1^\beta(\tilde{f})$. Let $\tilde{g}$ be the limit of the monotone sequence $(\tilde{g}_n)$, $\tilde{g}_n$ in $\tilde{L}^{\beta_n}$, $\beta_n < \beta$, with $(\phi_1^{\beta_n}(\tilde{g}_n))$ tending strongly to B. Suppose $A \nleqq B$, and let ρ be the integration process due to the regular Borel measure induced by X by some positive extension of $\rho'\phi_1$, where ρ' is a normal state of $\mathfrak{A}_1$ such that $\rho'(A) > \rho'(B)$. By inductive assumption (c), $\rho(\tilde{f}_n) = \rho'[\phi_1^{\alpha_n}(\tilde{f}_n)]$ and $\rho(\tilde{g}_n) = \rho'[\phi_1^{\beta_n}(\tilde{g}_n)]$. The Monotone Convergence Theorem and normalcy of ρ' imply that $\rho(\tilde{f}_n)$ tends to $\rho(\tilde{f})$ and to $\rho'(A)$, while $\rho(\tilde{g}_n)$ tends to $\rho(\tilde{g})$ and to $\rho'(B)$. (Recall that, by definition of $\mathfrak{N}_{\phi_1}$, f_n and g_n tend monotonely and almost everywhere to f and g, respectively, relative to ρ-measure.) Thus $\rho(\tilde{f}) = \rho'[\phi_1^\beta(\tilde{f})] = \rho'(A)$, and $\rho(\tilde{g}) = \rho'[\phi_1^\beta(\tilde{g})] = \rho'(B)$, so that $\rho(\tilde{f}) > \rho(\tilde{g})$, and $\tilde{g} \ngeqq \tilde{f}$. Thus if $\tilde{g}$ and $\tilde{f}$ lie in $\tilde{L}^\beta$ and $\tilde{g} \geqq \tilde{f}$ then $\phi_1^\beta(\tilde{g}) \geqq \phi_1^\beta(\tilde{f})$. Taking $\tilde{g} = \tilde{f}$, we see that ϕ_1^β is well-defined (i.e., the definition of $\phi_1^\beta(\tilde{f})$ is independent of the sequence $(\tilde{f}_n)$ chosen). The foregoing argument also establishes (c) for ϕ_1^β (conditions (a) and (b) being apparent).

To establish the remaining half of (d) for $\tilde{L}^\beta$ and ϕ_1^β, suppose that $\phi_1^\beta(\tilde{f}) \geqq \phi_1^\beta(\tilde{g})$, with $\tilde{f}$ and $\tilde{g}$ in $\tilde{L}^\beta$. If $\tilde{f} \ngeqq \tilde{g}$ then the set S of points p in X such that $g(p) > f(p)$ does not lie in $\mathfrak{N}_{\phi_1}$ and is a Borel subset of X, whence, according to Lemma 2.2.2, there exists a normal state ρ' of $\mathfrak{A}_1$ such that integration ρ due to the regular Borel measure induced on X by some positive extension of $\rho'\phi_1$ from $\mathfrak{L}$ to $C(X)$ has the property that $\rho(f) = \rho(f\chi_S)$, for each f in $\mathfrak{B}$, where χ_S is the characteristic function of S. Thus $\rho(\tilde{g}) > \rho(\tilde{f})$. However, by (c),

$$\rho(\tilde{g}) = \rho'(\phi_1^\beta[\tilde{g}]) \leqq \rho'(\phi_1^\beta[\tilde{f}]) = \rho(\tilde{f}),$$

a contradiction. Thus $\tilde{f} \geqq \tilde{g}$, and (d) is established for $\tilde{L}^\beta$ and ϕ_1^β. It follows that ϕ_1^β is a one to one mapping.

If A is an element of class β in $\mathfrak{A}_{1s}^m$ and (A_n) is a monotone sequence of elements in $\bigcup_{\alpha<\beta} \mathfrak{A}_{1s}^m$ tending strongly to A, we can choose $\tilde{f}_n$ (uniquely) in $\bigcup_{\alpha<\beta} \tilde{L}^\alpha$, by inductive hypothesis, such that $\phi_1^\beta (\tilde{f}_n) = A_n$. From (d) we have that $(\tilde{f}_n)$ is monotone, so that (f_n) converges on the complement $X - S$ of a set S in $\mathfrak{N}_{\phi_1}$ to a function f in $\mathfrak{G}$ (define f to be 0 on S). The element $\tilde{f}$ of $\tilde{B}$ is in $\tilde{L}^\beta$ and $\phi_1^\beta (f) = A$, by definition of $\tilde{L}^\beta$ and ϕ_1^β. Thus ϕ_1^β maps $\tilde{L}^\beta$ onto $\mathfrak{A}_1^m$, and the transfinite induction is complete, so that the assignment of ordinals β to classes $\tilde{L}^\beta$ and mappings ϕ_1^β with the stated properties exists.

Let $\tilde{L}^m$ be the union of all the $\tilde{L}^\beta$, and observe that, since the ϕ_1^β's are extensions of one another, they define a mapping $\tilde{\phi}_1$ of $\tilde{L}^m$ onto $\mathfrak{A}_{1s}^m$. Aside from the properties of $\tilde{L}^m$ and $\tilde{\phi}_1$ which are an immediate consequence of the properties of $\tilde{L}^\beta$ and ϕ_1^β, it is also the case that $\tilde{L}^m$ is a linear space and $\tilde{\phi}_1$ is a linear isomorphism of $\tilde{L}^m$ onto $\mathfrak{A}_{1s}^m$—as follows easily from the same transfinite induction argument employed to establish the linear character of $\mathfrak{A}_1^m$.

Returning now to the state ω' whose weak continuity at 0 on S_1^+ we wish to prove, we note that $\omega'\phi_1 = \omega_z\phi\phi_1 = \omega_z\phi_2$, so that $\omega'\phi_1$ has each set of $\mathfrak{N}_{\phi_1}$ as permanent null set (since $\mathfrak{N}_{\phi_1} \subset \mathfrak{N}_{\phi_1|E_0} \subset \mathfrak{N}_{\phi_2|F_0}' \subset \mathfrak{N}_z$, by hypothesis). Let ω'' be the integration process due to the regular Borel measure induced on X by some positive extension of $\omega'\phi_1$ from $\mathfrak{L}$ to $C(X)$. From the preceding remark, ω'' annihilates $\mathfrak{N}$ and therefore induces a state $\tilde{\omega}''$ of $\tilde{B}$. Define ω on $\mathfrak{A}_{1s}^m$ by: $\omega(\tilde{\phi}_1(\tilde{f}) = \tilde{\omega}''(\tilde{f})$. Since $\tilde{\phi}_1$ is a linear, order-isomorphism of $\tilde{L}^m$ onto $\mathfrak{A}_{1s}^m$, ω is a state of $\mathfrak{A}_{1s}^m$; and with $\tilde{f}$ in $\tilde{L}^0$,

$$\omega[\tilde{\phi}_1(\tilde{f})] = \tilde{\omega}''(\tilde{f}) = \omega'[\phi_1(f)] = \omega'[\tilde{\phi}_1(\tilde{f})],$$

so that ω is an extension of ω'. If (A_n) is a monotone sequence in $\mathfrak{A}_{1s}^m$ with A as strong limit and $(\tilde{f}_n)$ is the monotone sequence in $\tilde{L}^m$ tending to $\tilde{f}$ such that $\tilde{\phi}_1(\tilde{f}_n) = A_n$ and $\tilde{\phi}_1(\tilde{f}) = A$, The Monotone Convergence Theorem implies that $\tilde{\omega}''(\tilde{f}_n) = \omega(A_n)$ tends to $\tilde{\omega}''(\tilde{f}) = \omega(A)$. In particular, we conclude that $\omega(\sum_n E_n) = \sum_n \omega(E_n)$ for each countable orthogonal family of projections $\{E_n\}$ in $\mathfrak{A}_{1s}^m$. (The apparent generality in the choice of a positive extension of $\omega'\phi_1$ is illusory, for it is easy to show that the integration process ω'' restricted to $\tilde{L}^m$ is the same for each such extension.)

It is possible, at this point, to complete the proof under the assumption that $\mathfrak{A}_1^-$ is countably-decomposable (or just that $\mathfrak{A}_{1s}^m$ is countably-decomposable) with the hypothesis $\mathfrak{N}_{\phi_1} \subset \mathfrak{N}_{\phi_2}'$. In fact, with G a non-zero projection in $\mathfrak{A}_{1s}^m$ and x a unit vector in the range of G (so that $\omega(G) \leqq (Gx, x) = 1$), if G does not have the property that $\omega(F) \leqq (Fx, x)$ for each projection F in $\mathfrak{A}_{1s}^m$ contained in G then, by Zorn's Lemma, there exists a non-null, maximal, orthogonal family $\{G_n'\}$ of projections in $\mathfrak{A}_{1s}^m$ less than G such that $\omega(G_n') > (G_n'x, x)$. Let $G_0 = G - \sum_n G_n'$, so that G_0 lies in $\mathfrak{A}_{1s}^m$. Since

$$(Gx, x) \geqq \omega(G) \geqq \sum_n \omega(G_n') > \sum_n (G_n'x, x) = \left(\sum_n G_n' x, x,\right)$$

we have that $G_0 \neq 0$. From the maximal property of $\{G_n'\}$, we have that $\omega(F) \leqq$ $(Fx,\ x)$ for each projection F in $\mathfrak{A}_{1s}^m$ contained in G_0. We conclude that each non-zero projection G in $\mathfrak{A}_{1s}^m$ contains a non-zero projection G_0 in $\mathfrak{A}_{1s}^m$ such that, for some unit vector x in the range of G, $\omega(F) \leqq (Fx,\ x)$ for each projection F in $\mathfrak{A}_{1s}^m$ contained in G_0.

Let $\{G_n\}$ be a maximal orthogonal family of non-zero projections in $\mathfrak{A}_{1s}^m$ with associated unit vectors x_n such that $\omega(F) \leqq (Fx_n,\ x_n)$ for each projection F in $\mathfrak{A}_{1s}^m$ contained in G_n. From maximality, $\sum_n G_n = I$. It follows at once from the fact that each positive operator can be approximated as closely as desired, in the uniform operator topology, by positive finite linear combinations of its spectral projections (The Spectral Theorem), and the fact that ω, being a state of $\mathfrak{A}_{1s}^m$, is continuous on $\mathfrak{A}_{1s}^m$ in the uniform operator topology, that $\omega(A) \leqq$ $(Ax_n,\ x_n)$ if the range of the positive operator A in $\mathfrak{A}_{1s}^m$ is contained in G_n.

Let $\varepsilon > 0$ be assigned, and let $H_k = \sum_{j=k+1}^\infty G_j$. Note that $\omega(H_k) = \sum_{j=k+1}^\infty \omega(G_j)$, so that $\omega(H_k)$ tends monotonely to 0. Choose k so that $\omega(H_k) < \varepsilon^2/4$, and let A be an operator in s_1^+ such that $(Ay_j,\ y_j) < \varepsilon^2/4k^2, j = 1,\ \cdots,\ k$, where $y_j = G_j x_j$. Observe that

$$\omega[(H_k + \lambda A)^2] = \omega(H_k) + \lambda\omega(H_k A + A H_k) + \lambda^2\omega(A^2) \geqq 0$$

for each real λ, so that

$$[\tfrac{1}{2} \mid \omega(H_k A + A H_k) \mid]^2 \leqq \omega(H_k)\omega(A^2) < \varepsilon^2/4.$$

In addition,

$$0 \leqq (G_j A^{\frac{1}{2}} + \lambda A^{\frac{1}{2}})(A^{\frac{1}{2}}G_j + \lambda A^{\frac{1}{2}}) = G_j A G_j + \lambda(G_j A + A G_j) + \lambda^2 A,$$

for all real λ, whence, as before,

$$\mid \tfrac{1}{2}\omega(G_j A + A G_j) \mid^2 \leqq \omega(G_j A G_j)\omega(A) \leqq \omega(G_j A G_j) \leqq (Ay_j,\ y_j) < \varepsilon^2/4k^2,$$

for $j = 1,\ \cdots,\ k$. Thus

$$2\omega(A) = \sum_{j=1}^k \omega(G_j A + A G_j) + \omega(H_k A + A H_k) < \sum_{j=1}^k \frac{\varepsilon}{k} + \varepsilon = 2\varepsilon,$$

and $\omega(A) < \varepsilon$. It follows that ω is weakly continuous at 0 on s_1^+, and the proof, in the countably-decomposable case, is complete.

Unfortunately the representations $\{\phi_1 \mid E_\alpha\}$ are not such that (not necessarily order-homomorphisms) the above result is applicable to them "piece by piece" so as to give the general result. Further, the fact that E_0 is not necessarily in $\mathfrak{A}_{1s}^m$, so that $E_0\mathfrak{A}_1 E_0$ and $E_0\mathfrak{A}_{1s}^m E_0$ need not be algebras, lends to the technical complications. We shall require a good deal more to settle the general case, but we wish to note that when we obtain a completely-additive extension of ω to $\mathfrak{A}_1^-$, as we shall, the argument of the preceding two paragraphs may be applied almost unchanged to complete the proof. The families of projections involved will then be uncountable, but, with $\{G_\beta\}$ replacing $\{G_n\}$, $1 = \omega(I) = \sum_\beta \omega(G_\beta)$ so

that at most a countable number G_1, G_2, $\cdots$ of $\{G_\beta\}$ have $\omega(G_\beta) \neq 0$. Defining H_k to be $\sum_{\beta\neq1,\ldots,k} G_\beta$, the proof then proceeds as above.

In order to deal with the general case, it will be necessary to obtain more information about $\mathfrak{A}_{1s}^m$. Specifically, we shall prove now that $E\,\mathfrak{A}_{1s}^m\,E = E\,\mathfrak{A}_{1s}^{-}E$, if E is a countably-decomposable projection in $\mathfrak{A}_1^{-}$. To this end, it will suffice, of course, to show that $EA'E$ lies in $E\,\mathfrak{A}_{1s}^m\,E$, for each operator A' in $\mathfrak{A}_1^{-}$ which is positive and has norm 1. Let A be the positive square root of A', and let $\{P_n\}$ be an orthogonal family of projections cyclic under $\mathfrak{A}_1'$ with unit generating vectors $\{z_n\}$, respectively, and sum E. According to [21], it is possible to choose A_n in $\mathbb{S}_1^{+}$ so that $\max \{\; \|\,(A_n - A)A^k z_j\,\| :k = 0, 1, 2; j = 1, \cdots, n\} < \frac{1}{2}^{n+1}$. Since $\mathbb{S}_1^{-}$ is compact in the weak-operator topology, (A_n) has as weak limit point some positive operator B in $\mathbb{S}_1^{-}$. In view of the choice of A_n, we have

$$\max \{\; \|\,(A_n - A_{n+1})A^k z_j\,\| :k = 0, 1, 2; j = 1, \cdots, n\} < \frac{1}{2}^n.$$

Thus, with $B_n = (A_{n+1} - A_n)^{+}$, $n = 1, 2, \cdots$, and $B_0 = A_1$, $\sum_{n=1}^{\infty} B_n A^k z_j$ converges absolutely, for $k = 0, 1, 2$. In addition, $A_n A^k z_j$ tends to $A^{k+1} z_j$, for $k = 0, 1, 2$. (Cf. [20] with the argument below.)

Let $T_k = (I + \sum_{n=0}^{k} B_n)^{-1}$ and note that, since $0 \leq T_k^{-1} \leq T_{k+1}^{-1}$, $0 \leq T_{k+1} \leq T_k$, so that the strong limit T of (T_k) lies in $\mathfrak{A}_{1s}^m$ and is positive. For each k, $T(\sum_{n=0}^{k} B_n)T \leq I$. In fact for fixed v in $\mathfrak{K}_2$, $(T(\sum_{n=0}^{k}B_n)Tv, v)$ is approximated as closely as desired by $([\sum_{n=0}^{k} B_n](I + \sum_{n=0}^{r} B_n)^{-1}v, (I + \sum_{n=0}^{r} B_n)^{-1}v)$, for sufficiently large r. This last term does not exceed $([\sum_{n=0}^{r} B_n](I + \sum_{n=0}^{r} B_n)^{-1}v,$ $(I + \sum_{n=0}^{r} B_n)^{-1}v)$ for $r \geq k$, and $(\sum_{n=0}^{r} B_n)(I + \sum_{n=0}^{r} B_n)^{-2} \leq I$, by Spectral Theory—whence, our assertion. With t a positive integer, $T(\sum_{n=t}^{k} B_n)T \leq T(\sum_{n=0}^{k} B_n)T \leq I$. Moreover, $T(\sum_{n=t}^{k} B_n)T$ is monotone increasing with k, so that its strong limit, C_t, is less than or equal to I and a positive operator in $\mathfrak{A}_{1s}^m$. Of course, $C_t \geq C_{t+1} \geq 0$, so that C_t has a positive strong limit C in $\mathfrak{A}_{1s}^m$. In addition, $T(\sum_{n=0}^{k} B_n)T + C_{k+1} = C_0$, so that

$$C_0 + T(\sum_{n=1}^{k} (A_{n+1} - A_n)^{-})T = T[A_1 + \sum_{n=1}^{k} ((A_{n+1} - A_n)^{+}$$
$$+ (A_{n+1} - A_n)^{-})]T + C_{k+1} = T[A_1 + \sum_{n=1}^{k} (A_{n+1} - A_n)]T$$
$$+ C_{k+1} = TA_{k+1}T + C_{k+1},$$

which is monotone decreasing and positive, and so has a positive, strong limit in $\mathfrak{A}_{1s}^m$. This limit must be $TBT + C$ since, TA_kT tends weakly to TBT and C_k tends strongly to C. Thus TBT lies in $\mathfrak{A}_{1s}^m$.

We observe next that, for each w in $\mathfrak{K}_1$, by choosing r large enough, we can approximate $(T(A^k z_j + \sum_{n=0}^{\infty} B_n A^k z_j), w)$, $k = 0, 1, 2$, as closely as desired by $(A^k z_j + \sum_{n=0}^{r} B_n A^k z_j, (I + \sum_{n=0}^{r} B_n)^{-1}w) = (A^k z_j, w)$; so that

$$T(A^k z_j + \sum_{n=0}^{\infty} B_n A^k z_j) = A^k z_j, \; k = 0, 1, 2; j = 1, 2, \cdots,$$

and the range projection F of T is such that $FA^k z_j = A^k z_j$.

The real-valued function, g_n, defined as $1/t$ for $t \geq 1/n$ and 0 for $t < 1/n$ is the pointwise limit of a monotone decreasing sequence of continuous functions,

so that $g_n(T)$ lies in $\mathfrak{A}_{1s}^n$ (by Spectral Theory). Thus, defining G_n to be $g_n(T)T$, for $n > 1$ and G_1 to be 0, G_n is a spectral projection for T lying in $\mathfrak{A}_{1s}^m$, and $g_n(T)TBTg_m(T) + g_m(T)TBTg_n(T) = G_nBG_m + G_mBG_n$ lies in $\mathfrak{A}_{1s}^m$, since $\mathfrak{A}_{1s}^m$ is closed under Jordan multiplication. Moreover, (G_m) is monotone increasing to F, which lies in $\mathfrak{A}_{1s}^m$.

Let $H_n = G_{n+1} - G_n$, $n = 1, 2, \cdots$, so that $\sum_n H_n = F$. We have that

$$(H_mBH_n + H_nBH_m)^2 = H_mBH_nBH_m + H_nBH_mBH_n,$$

for $m \neq n$, as well as $H_mBH_mBH_m$, lie in $\mathfrak{A}_{1s}^m$ and are positive. Thus

$$\sum_{n=1}^{\infty}(H_mBH_nBH_m + H_nBH_mBH_n) = H_mBFBH_m + \sum_{n=1}^{\infty}H_nBH_mBH_n$$

converges and lies in $\mathfrak{A}_{1s}^m$; whence

$$H_m(H_mBFBH_m + \sum_{n=1}^{\infty}H_nBH_mBH_n)H_m = H_mBFBH_m + H_mBH_mBH_m,$$

and hence H_mBFBH_m lies in $\mathfrak{A}_{1s}^m$. Now

$$(H_mBH_n + H_n)(H_nBH_m + H_n)$$

$$= H_mBH_nBH_m + H_mBH_n + H_nBH_m + H_n \geqq 0,$$

so that $H_nBH_mBH_n + H_mBH_nBH_m + H_mBH_n + H_nBH_m + H_n$ is positive and lies $\mathfrak{A}_{1s}^m$. It follows that $\sum_{n=1}^{\infty}H_nBH_mBH_n + H_mBFBH_m + H_mBF + FBH_m + F$ lies in $\mathfrak{A}_{1s}^m$, so that, from the foregoing, $H_mBF + FBH_m$ lies in $\mathfrak{A}_{1s}^m$. Again

$$(H_mBF + FBH_m)^2 = H_mBFBH_m + H_mBH_mBF + FBH_mBH_m + FBH_mBF$$

and

$$H_mBH_m(H_mBF + FBH_m) + (H_mBF + FBH_m)H_mBH_m$$

$$= H_mBH_mBF + 2H_mBH_mBH_m + FBH_mBH_m$$

lie in $\mathfrak{A}_{1s}^m$; so that $H_mBH_mBF + FBH_mBH_m$, and hence FBH_mBF lie in $\mathfrak{A}_{1s}^m$ (recall that H_mBFBH_m is in $\mathfrak{A}_{1s}^m$). Now FBH_mBF is positive so that $\sum_{m=1}^{\infty}FBH_mBF = FBFBF = S$ lies in $\mathfrak{A}_{1s}^m$. Since $(A_nA^kz_j)$, $k = 0, 1, 2$ tends strongly to $A^{k+1}z_j$ and weakly to BA^kz_j, $BA^kz_j = A^{k+1}z_j$. We have $Sz_j = FBFBFz_j = A^2z_j = A'z_j$, so that $S\mathfrak{A}_1'z_j = A'\mathfrak{A}_1'z_j$, and $SP_j = A'P_j$. It follows that $SE = S\sum_j P_j = A'\sum_j P_j = A'E$, and that $EA'E = ESE$ lies in $E\mathfrak{A}_{1s}^m E$. Our assertion, $E\mathfrak{A}_1^- E = E\mathfrak{A}_{1s}^m E$, with E a countably-decomposable projection in $\mathfrak{A}_1^-$, is established.

Since E_0, the projection in $\{E_\alpha\}$ corresponding to F_0 in $\{F_\alpha\}$, is countably-decomposable, we have, from the preceding work, that $E_0\mathfrak{A}_{1s}^m E_0$ and $E_0\mathfrak{A}_1^- E_0$ coincide. We define ω_0 on $E_0\mathfrak{A}_{1s}^m E_0$ by taking $\omega_0(E_0AE_0)$, with A in $\mathfrak{A}_{1s}^m$, to be $\omega(A)$, and establish that ω_0 is a (well-defined) completely additive state of $E_0\mathfrak{A}_{1s}^m E_0$. In fact, let A be an operator in $\mathfrak{A}_{1s}^m$ such that $E_0AE_0 \geqq 0$, and let $\tilde{f}$ in $\bar{L}^m$ be such that $\bar{\phi}_1(\tilde{f}) = A$. Let f be a representative of $\tilde{f}$ in $\mathfrak{G}$ and S the set of points p in X for which $f(p) < 0$. We assert that S is in $\mathfrak{N}_{\phi_1|E_0}$; for if this is not the case, there is a normal state ρ' of $E_0\mathfrak{A}_1^- E_0$ such that the integration process

ρ due to the regular Borel measure on X induced by some positive extension of $\rho'\phi_{1|E_0}$ from $\mathfrak{L}$ to $C(X)$ does not have S as a null set. Now ρ'' defined on $\mathfrak{A}_1^-$ by $\rho''(B) = \rho'(E_0BE_0)$ is a normal state of $\mathfrak{A}_1^-$ and $\rho''\phi_1 = \rho'\phi_1 \mid E_0$. By Lemma 2.2.2, we have that there is a normal state τ' of $\mathfrak{A}_1^-$ such that integration τ due to the regular Borel measure induced on X by some positive extension of $\tau'\phi_1$ has the property; $\tau(g\chi_s) = \tau(g)$, for g in $\mathfrak{B}$ and χ_s the characteristic function of S; and τ' is dominated on $\mathfrak{A}_1$ by a positive multiple of ρ''. From [21], we can find a directed sequence $\{A_\mu\}$ of elements in $\mathfrak{S}_1^+$ tending strongly to $I - E_0$, so that $0 \leqq \tau'(A_\mu) \leqq k\rho''(A_\mu)$ and $(\rho''(A_\mu))$ tends to $\rho''(I - E_0) = 0$. Thus $\tau'(I - E_0) = 0$. Now $\tau(f) = \tau(f\chi_S) < 0$, while

$$\tau(f) = \tau'[\tilde{\phi}_1(\tilde{f})] = \tau'(A) = \tau'[E_0AE_0 + (I - E_0)AE_0$$
$$+ E_0A(I - E_0) + (I - E_0)A(I - E_0)] = \tau'[E_0AE_0] \geqq 0$$

(the first equality coming from property (c) of $\tilde{\phi}_1$, and the last equality coming from an application of The Schwarz Inequality to τ' and the fact that $\tau'(I - E_0) = 0$). Thus S is in $\mathfrak{N}_{\phi_1|E_0}$ as asserted, and therefore in $\mathfrak{N}_z = \mathfrak{N}_{\omega'}$. It follows that

$$\omega(A) = \omega[\tilde{\phi}_1(\tilde{f})] = \bar{\omega}''(\tilde{f}) = \omega''(f) \geqq 0.$$

We conclude from this that ω_0 is well-defined, for if $E_0AE_0 = 0$, then $\omega(A) \geqq 0$ and $\omega(-A) \geqq 0$, so that $\omega(A) = \omega_0(E_0AE_0) = 0$, and that ω_0 is a state of $E_0\mathfrak{A}_{1s}^mE_0$. For the complete additivity of ω_0, we observe that if $\sum_\mu E_0A_\mu E_0$ is a convergent sum of positive operators, $E_0A_\mu E_0$, A_μ in $\mathfrak{A}_1^-$, then at most a countable number of $E_0A_\mu E_0$ are different from 0. In fact, by countable-decomposability of E_0, there exists a countable, orthogonal family $\{P_n\}$ of cyclic projections in $\mathfrak{A}_1^-$ having sum E_0 with generating vectors $\{y_n\}$, respectively. For each n, $\sum_\mu(E_0A_\mu E_0y_n, y_n)$ converges and each $(E_0A_\mu E_0y_n, y_n)$ is real and non-negative. Thus, at most a countable number of terms are different from 0, whence $E_0A_\mu E_0y_n = E_0A_\mu y_n = 0$ with at most a countable number of exceptions. It follows that $E_0A_\mu\mathfrak{A}_1' y_n = 0$ and $E_0A_\mu P_n = 0$ for all n with at most a countable number of A_μ excepted. Now, if $E_0A_\mu P_n = 0$ for all n then $E_0A_\mu \sum P_n = E_0A_\mu E_0 = 0$. Let $A_1, A_2, \cdots$ be all A_μ such that $E_0A_\mu E_0 \neq 0$ so that $\omega_0(\sum_\mu E_0A_\mu E_0) = \omega_0(\sum_n E_0A_nE_0)$ and $\sum_\mu\omega_0(E_0A_\mu E_0) = \sum_n\omega_0(E_0A_nE_0)$. Suppose now that the A_μ lie in $\mathfrak{A}_{1s}^m$ and that $\sum_\mu E_0A_\mu E_0 = \sum_n E_0A_nE_0 = E_0AE_0$ with $E_0A_\mu E_0$ positive; and let f, f_n be functions in $\mathfrak{B}$ such that $\tilde{\phi}_1(\tilde{f}_n) = A_n$, $\tilde{\phi}_1(\tilde{f}) = A$. Since $E_0A_nE_0 \geqq 0$, it follows from the preceding remarks that the set S_n of points p in X such that $f_n(p) < 0$ is in $\mathfrak{N}_{\phi_1|E_0}$, so that f_n and f are greater than or equal to 0 almost everywhere relative to $\mathfrak{N}_{\phi_1|E_0}$ and to ω''—integration. Moreover, $E_0AE_0 \geqq \sum_{n=1}^m E_0A_nE_0$, so that $f \geqq \sum_{n=1}^m f_n$, $\mathfrak{N}_{\phi_1|E_0}$—almost everywhere, and $g = \sum_{n=1}^\infty f_n$ is an ω''—integrable function (by The Monotone Convergence Theorem) $\mathfrak{N}_{\phi_1|E_0}$—almost everywhere less than or equal to f, with $\omega''(g) = \sum_{n=1}^\infty \omega''(f_n) \leqq \omega''(f)$. If S_0, the set of points p in X for which $(f - g)(p) > 0$, does not lie in $\mathfrak{N}_{\phi_1|E_0}$, then, as before, there is a normal state τ' of $E_0 \mathfrak{A}_1^-E_0$ such that the

integration process τ has the property $\tau(h\chi_{S_0}) = \tau(h)$ for each h in $\mathfrak{G}$. In particular, $\tau(f - g) > 0$ so that $\tau(g) = \sum_{n=1}^{\infty} \tau(f_n) = \sum_{n=1}^{\infty} \tau'[E_0\tilde{\phi}_1(\tilde{f}_n)E_0] = \sum_{n=1}^{\infty} \tau'(E_0A_nE_0) = \tau'(\sum_{n=1}^{\infty} E_0A_nE_0) = \tau'(E_0AE_0) < \tau(f) = \tau'(E_0AE_0)$. Thus

$$\omega''(g) = \sum_{n=1}^{\infty}\omega''(f_n) = \sum_{n=1}^{\infty}\omega(A_n) = \sum_{n=1}^{\infty}\omega_0(E_0A_nE_0) = \sum_{\mu}\omega_0(E_0A_\mu E_0)$$

$$= \omega''(f) = \omega_0(E_0AE_0) = \omega_0\left(\sum_{\mu}E_0A_\mu E_0\right),$$

and ω_0 is a completely additive state of $E_0 \, \mathfrak{A}_{1s}^m \, E_0 \, (= E_0 \, \mathfrak{A}_1^- E_0)$.

Finally, we define a completely additive state extension of ω from $\mathfrak{A}_{1s}^m$ to $\mathfrak{A}_1^-$, which we again denote by ω, as the composition of the completely additive, positive, linear mapping $A \rightarrow E_0AE_0$ of $\mathfrak{A}_1^-$ onto $E_0\mathfrak{A}_{1s}^m \, E_0$ and the completely additive state ω_0 of $E_0 \, \mathfrak{A}_{1s}^m \, E_0$. (The definition of ω_0 makes it apparent that the state just defined is an extension of ω.) That ω' is weakly continuous at 0 on $\mathfrak{S}_1^+$ follows now, as indicated before, by the argument employed in establishing the same fact with $\mathfrak{A}_1^-$ countably-decomposable.

The remaining assertions of this theorem are easily proved. In fact, with both ϕ_1 and ϕ_2 order-homomorphisms and a null ideal band of ϕ_1 equal to a null ideal band of ϕ_2, the foregoing results imply that there exist mappings ϕ and ψ such that $\phi\phi_1 = \phi_2$ and $\psi\phi_2 = \phi_1$, with ϕ and ψ weakly continuous on $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively. Thus $\psi\phi$ is the identity mapping on $\mathfrak{A}_1$, and $\phi\psi$ is the identity mapping on $\mathfrak{A}_2$. Moreover ϕ and ψ have unique extensions $\tilde{\phi}$ and $\tilde{\psi}$ to $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$, respectively. Since $\tilde{\psi}\tilde{\phi}$ and $\tilde{\phi}\tilde{\psi}$ are weakly continuous on $\mathfrak{S}_1^-$ and $\mathfrak{S}_2^-$, respectively, and agree with the identity mappings on $\mathfrak{S}_1$ and $\mathfrak{S}_2$, we have that $\tilde{\phi}$ and $\tilde{\psi}$ are inverse to one another. Now $\tilde{\phi}$, being an order-isomorphism of $\mathfrak{A}_1^-$ onto $\mathfrak{A}_2^-$, is a C^*-isomorphism, as follows from [17] and hence the direct sum of a *-isomorphism and a *-anti-isomorphism, as follows from [16]. (The foregoing remains valid under the assumption that a vector null ideal band of ϕ_1 is equal to a vector null ideal band of ϕ_2).

With ϕ_1 an order-homomorphism, ϕ_2 an order-representation, an order-representation ϕ of $\mathfrak{A}_1^-$ onto $\mathfrak{A}_2^-$ which is weakly continuous on $\mathfrak{S}_1$, such that $\phi\phi_1 = \phi_2$, and such that $\phi(E_\alpha) = F_\alpha$ with $\{E_\alpha\}$, $\{F_\alpha\}$ orthogonal families of countably-decomposable projections in $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$, respectively, with sum I exists, only if the null ideal band $\{\mathfrak{N}_{\phi_1|E_\alpha}\}$ is contained in the null ideal band $\{\mathfrak{N}_{\phi_2|F_\alpha}\}$. In fact, if S lies in $\{\mathfrak{N}_{\phi_1|E_\alpha}\}$ and ω_α is a normal state of $F_\alpha\mathfrak{A}_2^-F_\alpha$ then $\omega_\alpha(F_\alpha AF_\alpha) = \omega_\alpha(A)$ and $\omega_\alpha\phi(B) = \omega_\alpha\phi(E_\alpha BE_\alpha)$, since $\omega_\alpha(I - F_\alpha) = 0$ and $\omega_\alpha\phi(I - E_\alpha) = \omega_\alpha(I - F_\alpha) = 0$. Now $\omega_\alpha\phi$ is normal on $E_\alpha\mathfrak{A}_1^-E_\alpha$ and

$$\omega_\alpha\phi_2 \mid F_\alpha = \omega_\alpha\phi_2 = \omega_\alpha\phi\phi_1 = \omega_\alpha\phi\phi_1 \mid E_\alpha,$$

so that S is a permanent null set of $\omega_\alpha\phi$. It follows that S lies in $\mathfrak{N}_{\phi_2|F_\alpha}$, and our assertion is established. With ϕ assumed to be an order-isomorphism as well, it is a C^*-isomorphism [17], and carries countably-decomposable projections onto countably-decomposable projections, so that, by the preceding result, each null ideal band of ϕ_1 is a null ideal band of ϕ_2. The proof is complete.

We point out that the "null ideal band" refinement of the "null ideal" con-

cept and the attendant technical complications of the above proof seem un-avoidable in the non-countably-decomposable case as we shall illustrate in Example 5.2.3.

2.3. General remarks on the extension problem

We make some comments concerning the extension problem and our proof of The Extension Theorem. Several of the points noted in this section and their proofs were observed in conversation with I. M. Singer—it is a pleasure to record here our gratitude for his stimulating comments and criticism at the final stage of this work.

With regard to the statement of The Extension Theorem, the hypothesis $b_1 \subset b_2'$ is the weakest possible in this direction, since $b_1 \subset b_1'$ and $b_2 \subset b_2'$. As-suming the existence of the mapping ϕ as described, one can expect no more than $b_1 \subset b_2$, for the mapping ϕ will take normal states on $\mathfrak{A}_2^-$ into normal states on $\mathfrak{A}_1^-$ but need not take a vector state of $\mathfrak{A}_2^-$ into a vector state of $\mathfrak{A}_1^-$ (depending on the situation of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ relative to their commutants $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$).

The remaining chapters of this work will make it amply clear that countable-decomposability of the algebras in question rather than the more familiar as-sumption of separability of the underlying Hilbert spaces (which implies count-able-decomposability) is the appropriate core situation with which to deal. Aside from the felicity of proof the assumption of countable-decomposability permits us and its appropriateness in dealing with the more general case, it allows us to draw conclusions at once about important situations not covered by the separability assumption. Indeed, a factor of type II_1 is automatically countably-decomposable and so falls under the simpler case.

Concerning the proof of The Extension Theorem, one may wonder at the con-sistent use of monotone sequences and the technical difficulties (identification of $\mathfrak{A}_{1s}^m$ and $E_0\mathfrak{A}_{1s}^m E_0$) attendant upon their use. A more direct approach would seem to be the following. Let (A_n) be a sequence of operators in S_1^+ tending weakly to 0, so that (f_n) in $\mathfrak{L}$, with $\phi_1(f_n) = A_n$, tends to 0 in L_1-norm for each $\rho\phi_1$, ρ a normal state of $\mathfrak{A}_1$. Unfortunately we cannot conclude from this that (f_n) tends to 0 almost everywhere relative to $\rho\phi_1$, for then the set S on which (f_n) does not converge to 0 would lie in $\mathfrak{N}_{\phi_1}$. It would follow then that (f_n) tends to 0 relative to the L_1-norm induced by $\omega\phi_1$ (where $\omega = \omega_z\phi$ of the proof), i.e., $(\omega(A_n))$ tends to 0 and the proof would be complete. Even the fact that some subsequence of (f_n) tends to 0 almost everywhere relative to $\rho\phi_1$ does not help, for the subsequence varies with ρ, and it may not be possible to choose one sub-sequence effective for each (directed sequences offer no solution). Now such a choice is possible for a countable family of ρ's, and it may be thought that, at least in the separable case, a countable, dense family of ρ's will suffice together with a continuity argument. This is not the case, for taking state extensions of $\rho\phi_1$ (as opposed to linear functional extensions) from $\mathfrak{L}$ to $C(X)$ loses touch with density properties (as can be illustrated by example). Again, the general nature of the state extensions rule out the possibility of dealing with a measure on X

with respect to which all the measures in question are absolutely continuous (even the extensions of a single state may not form a "bounded" family in the sense of absolute continuity). With (f_n) monotone (or almost everywhere monotone relative to $\mathfrak{N}_{\phi_1}$), (f_n) converges to some Borel measurable function f almost everywhere relative to $\mathfrak{N}_{\phi_1}$, and the sequence becomes amenable to the techniques available from our hypothesis. Critical use of this amenability is really made at only one point of the proof—establishing the inductive step: $\bar{L}^\mu$ maps onto $\mathfrak{A}_{1s}^\mu$—however, the proof being inductive, this feature of monotonicity actually pervades the entire proof. Our payment for the amenability of monotone sequences is the technical difficulty with $\mathfrak{A}_{1s}^m$.

Simplifying the conditions of The Extension Theorem, we can state:

COROLLARY 2.3.1. *If ϕ is a *-homomorphism of the operator algebra $\mathfrak{A}_1$ acting on the separable Hilbert space $\mathfrak{K}_1$ onto the operator algebra $\mathfrak{A}_2$ acting on the Hilbert space $\mathfrak{K}_2$ then ϕ has a *-homomorphic extension mapping $\mathfrak{A}_1^-$ into $\mathfrak{A}_2^-$ and weakly continuous on the unit sphere of $\mathfrak{A}_1^-$ if and only if $\mathfrak{N}_\phi$ contains the canonical null ideal $\mathfrak{N}(\mathfrak{A}_1)$ of ι, the canonical isomorphism of the representing function system $(\mathfrak{L}, X)$ of $\mathfrak{A}_1$ onto $\mathfrak{A}_1$.*

PROOF. Take ϕ_1 as ι and ϕ_2 as $\phi\iota$ in The Extension Theorem, and observe that our present ϕ is the ϕ of that theorem. Now ϕ being weakly continuous on S_1^- and a *-homomorphism of $\mathfrak{A}_1$ into $\mathfrak{A}_2$, it follows at once, from a limit argument, that ϕ is a *-homomorphism of $\mathfrak{A}_1^-$ into $\mathfrak{A}_2^-$. The converse is immediate.

In Example 5.2.1, we shall present an instance of an isomorphic mapping between abelian C^*-algebras which, while it has an extension (weakly continuous on the unit sphere) to the weak closure, does not have an isomorphic extension, so that the inverse of the original isomorphism does not admit an extension to the weak closure. Thus, even under the most stringent conditions, extension is not generally possible, and some added conditions are necessary such as the null ideal hypothesis. It is a triviality that the so-called "weighted spectrum", i.e., the family of normal states of $\mathfrak{A}_1^-$ brought back to $\mathfrak{L}$ via ϕ_1, is an extension invariant, from the first few lines of the proof of The Extension Theorem, and, indeed, the substance of the entire proof is the reduction of the null set hypothesis to the "weighted spectrum" condition.

From another viewpoint, the null set hypothesis and The Extension Theorem may be thought of as a "Galois-like" statement and situation. Identifying regular Borel measures on X with their integration processes on $\mathfrak{B}$, we have a family $\mathfrak{B}'$ of positive linear functionals on $\mathfrak{B}$ continuous with respect to some topology (defined by regularity). Given a cone $\mathfrak{K}'$ in $\mathfrak{B}'$, we can ask for its positive kernel $\mathfrak{K}$ in $\mathfrak{B}$ and the annihilator of $\mathfrak{K}$ in $\mathfrak{B}'$. In the countably-decomposable case, the substance of the proof of The Extension Theorem is that, with $\mathfrak{K}'$ the regular integrations in $\mathfrak{B}'$ induced by all positive extensions from $\mathfrak{L}$ to $C(X)$ of all $\rho\phi_1$, ρ a normal state of $\mathfrak{A}_1^-$, this annihilator is precisely $\mathfrak{K}'$. The duality property indicated is not valid in general, and seems, in the case we consider, to depend upon all of the special structure involved.

In the next section we shall discuss the question of representations of general

function systems. For the present, however, disregarding kernel, i.e., dealing with isomorphisms, we note that the representing function system (on the pure state space $\mathcal{P}$) is minimal, and the representation on the full state space $\mathcal{S}$ is maximal. Indeed, with $(\mathcal{L}, X)$ a function system such that $\mathcal{L}$ separates the points of X and ϕ an order-isomorphism of $\mathcal{L}$ onto the operator algebra $\mathfrak{A}$, each point p induces a state ρ_p of $\mathfrak{A}$ defined by $\rho_p(A) = \phi^{-1}(A)(p)$. The mapping $p \to \rho_p$ is one to one, since $\mathcal{L}$ separates points of X, and continuous. In fact, an arbitrary subbasic open neighborhood of ρ_{p_0} in $\mathcal{S}$ is given by an operator A in $\mathfrak{A}$ and consists of all states ρ such that $|\rho(A) - \rho_{p_0}(A)| < 1$. Now $\phi^{-1}(A)$ is continuous, so that the set of points p in X such that

$$1 > |\phi^{-1}(A)(p) - \phi^{-1}(A)(p_0)| = |\rho_p(A) - \rho_{p_0}(A)|$$

is open. Thus the mapping $p \to \rho_p$ is a homeomorphism of X into $\mathcal{S}$, since this mapping is one to one and continuous, and X is compact. In this sense, $\mathcal{S}$ is maximal. We think now of X imbedded in $\mathcal{S}$ by the process just described. Restricting the functions in $\mathcal{L}$ to $\mathcal{P}$, the closure of the set of pure states of $\mathfrak{A}$, is an order-isomorphism. Via this restriction mapping, then, each pure state ρ of $\mathfrak{A}$ is a pure state of $\mathcal{L}$, acting on X, which has a pure state extension to $C(X)$. This extension corresponds to a point p of X. Now p and ρ as points of $\mathcal{S}$ do not separate $\mathcal{L}$, so that $p = \rho$, and each pure state of $\mathfrak{A}$ lies in X. Since X is compact in $\mathcal{S}$, X is closed and contains $\mathcal{P}$, so that $\mathcal{P}$ is minimal. In particular, if the algebra $\mathfrak{A}$ is concretely represented upon the Hilbert space $\mathcal{K}$ and $\mathcal{V}$ denotes the closure in $\mathcal{S}$ of the family of vector states of $\mathfrak{A}$ then $\mathcal{V}$ contains $\mathcal{P}$, since $\mathfrak{A}$ is represented upon $\mathcal{V}$ in an order-isomorphic fashion. If $\mathfrak{A}$ acts irreducibly upon $\mathcal{K}$ then, as is noted in [43], each vector state of $\mathfrak{A}$ is a pure state, so that $\mathcal{V}$ coincides with $\mathcal{P}$.

If $(\mathcal{L}, X)$ is a function system and ϕ_1, ϕ_2 are order-isomorphisms of $\mathcal{L}$ as concrete operator systems $(\mathfrak{A}_1, \mathcal{K}_1)$, $(\mathfrak{A}_2, \mathcal{K}_2)$, respectively, such that each normal (vector) state of $\mathfrak{A}_1$, $\mathfrak{A}_2$ induces a state of $\mathcal{L}$ corresponding to a point of X, it is a simple matter to determine whether ϕ_1 and ϕ_2 admit the "extension" ϕ. In fact, suppose $\mathfrak{N}_{\phi_1}$ (or $\mathfrak{N}'_{\phi_1}$) is contained in $\mathfrak{N}_{\phi_2}$ (or $\mathfrak{N}'_{\phi_2}$). If p is the point of X corresponding to a normal (vector) state of $\mathfrak{A}_1^-$, then the state $f \to f(p)$ of $C(X)$ induces a regular Borel measure on X which assigns to the set $\{p\}$ whose only member is p measure 1, so that $\{p\}$ is not in $\mathfrak{N}_{\phi_1}(\mathfrak{N}'_{\phi_1})$. Moreover, if $\{p\}$ is not in $\mathfrak{N}_{\phi_1}(\mathfrak{N}'_{\phi_1})$, then, according to Lemma 2.2.2, the state $g \to g(p)$ of $\mathcal{L}$ is induced by a normal (vector) state of $\mathfrak{A}_1^-$. Thus the normal (vector) states of $\mathfrak{A}_1^-$ are precisely those corresponding to points p of X such that $\{p\}$ does not lie in $\mathfrak{N}_{\phi_1}(\mathfrak{N}'_{\phi_1})$. Of course, the same considerations apply to $\mathfrak{A}_2^-$, so that a normal (vector) state corresponds to a point p such that $\{p\}$ does not lie in $\mathfrak{N}_{\phi_2}(\mathfrak{N}'_{\phi_2})$ and therefore, by hypothesis, not in $\mathfrak{N}_{\phi_1}(\mathfrak{N}'_{\phi_1})$; whence to a normal (vector) state of $\mathfrak{A}_1^-$. The extension result follows for this situation. In particular, if $(\mathcal{L}, X)$ is a function system with X the full state space of $\mathcal{L}$, and ϕ_1, ϕ_2 are order-isomorphisms of $\mathcal{L}$ as operator systems, the extension result follows from the simple remarks just made. If $(\mathcal{L}, X)$ is a function system and ϕ_1, ϕ_2 two order-

isomorphisms of $\mathfrak{L}$ as concrete operator systems $(\mathfrak{A}_1, \mathfrak{K}_1)$, $(\mathfrak{A}_2, \mathfrak{K}_2)$ such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ act irreducibly upon $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, then, by the foregoing, X contains the pure state space of $\mathfrak{L}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ which is the vector state space (i.e., w^*-closure of the set of vector states) of $\mathfrak{A}_1$, $\mathfrak{A}_2$, and, again, the extension result follows. Thus, dealing only with essentials, we have the extension result easily if X is large enough (for example, if it is the state space of $\mathfrak{L}$) which is automatically the case when the representations in question are irreducible.

2.4. Remarks on applications of the extension theorem

Although our later applications of The Extension Theorem will refer to representations of C^*-algebras, we wish to point out the methods of and reasons for applying this theorem to the more general situation it encompasses. In the first place, we have discussed certain types of order-representations of function systems other than the representing function systems of C^*-algebras. While each C^*-algebra has a function system canonically associated with it—its representing function system—it may be difficult, when dealing with specific C^*-algebras or classes of C^*-algebras, to determine the pure state space and the representing linear space of functions on it. However, it may be possible to find some other compact Hausdorff space and a linear space of functions on it, more evident from the description of the algebra and order-isomorphic with it. For this reason, it is important for us to have a theory applicable to representations of function systems more general than those canonically associated with C^*-algebras.

Secondly, we have discussed the Jordan algebras of self-adjoint elements in a C^*-algebra and C^*-homomorphisms of these as well as C^*-algebras and their $*$-homomorphisms. This generality merits attention by virtue of the fact that the representing function system of a C^*-algebra, through which we shall investigate the multiplicity structure of its representations, characterizes the C^*-algebra algebraically only up to C^*-isomorphisms, i.e., characterizes only the Jordan algebra of self-adjoint elements in the C^*-algebra (see [17]).

To obtain the more general form of The Unitary Invariants Theorems from the results of Chapter IV, we proceed as follows. The definition of multiplicity function of a representation of a C^*-algebra (Definition 4.4.1) applies directly to an order-representation of a function system as an operator algebra. The Unitary Invariants Theorems apply to order-homomorphisms of function systems, and give conditions for their "semi-unitary equivalence"—this being the direct sum of a unitary equivalence and the composition of a conjugate unitary equivalence followed by the $*$-operation. The proofs proceed in precisely the same way as the proofs of The Unitary Invariants Theorems. The equality of the multiplicity functions, now, establishes a C^*-isomorphism between appropriate parts of the rings in question. According to [16], a C^*-isomorphism of a ring is the direct sum of a $*$-isomorphism and a $*$-anti-isomorphism. The material presented in Chapter IV handles the portion on which the mapping is a

*-isomorphism. The portion on which the mapping is a *-anti-isomorphism is dealt with in the same manner, making use of the fact that a *-anti-isomorphism between rings which have, together with their commutants, joint generating vectors is implemented by a conjugate unitary transformation followed by the *-operation, in place of The Unitary Implementation Theorem. It may be of value to sketch a proof of this variant of The Unitary Implementation Theorem at this point in view of the facts that the proofs of both results are nearly identical and that we make vital use of these theorems.

Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be rings of operators with x' and y joint generating vectors for $\mathfrak{R}_1$, $\mathfrak{R}_1'$ and $\mathfrak{R}_2$, $\mathfrak{R}_2'$, respectively, and let ϕ be a *-anti-isomorphism of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$. The state $A \to (\phi(A)y, y)$ of $\mathfrak{R}_1$ is a normal state and therefore corresponds to a vector state, $A \to (Ax'', x'')$, since $\mathfrak{R}_1$ has the separating vector x' (by a proof close to that in which we concluded the continuity of ω' from its countable-additivity on $\mathfrak{A}_{1s}^m$ in the countably-decomposable case of The Extension Theorem, see [3]). It is obvious that x'' is separating for $\mathfrak{R}_1$, so that $[\mathfrak{R}_1'x''] = [\mathfrak{R}_1'x']$; whence $[\mathfrak{R}_1x''] \sim [\mathfrak{R}_1x']$, by [28; Lemma 9.3.3]. Let V' be a partial isometry in $\mathfrak{R}_1'$ mapping $[\mathfrak{R}_1x'']$ onto $[\mathfrak{R}_1x']$, the entire Hilbert space, and let x be $V'x''$. It is easily seen that x is a generating vector for $\mathfrak{R}_1$, and, since $(Ax, x) = (AV'x'', V'x'') = (Ax'', x'') = (\phi(A)y, y)$, x is separating for $\mathfrak{R}_1$. The mapping J taking Ax onto $\phi(A)^*y$ is densely-defined, conjugate-linear and norm-preserving, in view of the foregoing equality, and therefore has a unique conjugate unitary extension. It follows easily that $(JAJ^{-1})^* = \phi(A)$, as we wished to show.

Our intent in stating the results of Chapter IV in terms of representations of C^*-algebras was to avoid a tedious and confusing duplication of hypotheses and conclusions and to deal directly with the more familiar constructs of the subject. We stress the fact, however, that the theory is intended to cover the more general situation just discussed, and we envisage applications in this more general form.

CHAPTER III. SOME RING OF OPERATORS PRELIMINARIES

We gather together various general results concerning rings of operators which we shall need in the succeeding chapters. These results deal with properties of projections in rings of operators, and the chapter is divided into sections containing results centered about particular properties. The first section deals with central carriers and separating properties; the second develops a dimension theory for projections in a finite ring of operators; and the third deals with cyclicity and countable-decomposability.

3.1. Central carriers

If A is an operator in a ring of operators $\mathfrak{R}$ and $\{P_\alpha\}$ is a family of central projections in $\mathfrak{R}$ such that $P_\alpha A = 0$ then $PA = 0$, where P is the union of $\{P_\alpha\}$. Indeed Ax is orthogonal to the range of P_α for each α, hence to the linear span of their ranges, and thus to the range of P. We define the central carrier of A to be $I - P$ where P_α runs through all central projections in $\mathfrak{R}$ such that $P_\alpha A = 0$

and we observe, in view of the previous remark, that $C_A A = A$, where C_A denotes the central carrier of A. It is clear that C_A may also be defined as the intersection of all central projections Q with the property $QA = A$.

The following lemma is an extension of [28; Theorem III, p. 140] to rings of operators. The proof of [28] with trivial modifications yields this result. We give a shortened version of this proof more in the spirit of our treatment.

LEMMA 3.1.1. *If $\Re$ is a ring of operators with commutant $\Re'$; B_{ij}, $i, j = 1, \cdots, n$ are operators in $\Re$; and B'_{ij}, $i, j, = 1, \cdots, n$ are operators in $\Re'$, then $\sum_k B_{ik} B'_{kj} = 0$, $i, j, = 1, \cdots, n$ if and only if there exist central operators A_{ij}, $i, j = 1, \cdots, n$ such that $\sum_k B_{ik} A_{kj} = 0$, $i, j = 1, \cdots, n$, and $\sum_k A_{ik} B'_{kj} = B'_{ij}$, $i, j = 1, \cdots, n$. In particular, $BB' = 0$ with B in $\Re$, B' in $\Re'$ if and only if $C_B C_{B'} = 0$.*

PROOF. The stated condition is sufficient in order that $\sum_k B_{ik} B'_{kj} = 0$, for

$$0 = \sum_h \sum_k B_{ik} A_{kh} B'_{hj} = \sum_k B_{ik} \sum_h A_{kh} B'_{hj} = \sum_k B_{ik} B'_{kj} .$$

Suppose now that $\sum_k B_{ik} B'_{kj} = 0$, $i, j = 1, \cdots, n$. Let $\Re'_n$ be the $n \times n$ matrix ring over $\Re'$; B the matrix (B_{ij}), $P'_n = (A_{ij})$ be the union of all projections in $\Re'_n$ which are annihilated under left multiplication by B, with A_{ij} in $\Re'$. We show that each A_{ij} lies in the center of $\Re'$. To do this, we must show that P'_n commutes with each diagonal matrix Q'_n all of whose entries are Q', some projection in $\Re'$. Clearly $BQ'_n P'_n = 0$, so that the projection on the closure of the range of $Q'_n P'_n$ is annihilated under left multiplication by B and, hence, is contained in P'_n. Thus $P'_n Q'_n P'_n = Q'_n P'_n$, so that $P'_n Q'_n = (P'_n Q'_n P'_n)^* = Q'_n P'_n$. Hence A_{ij} is in the center of $\Re'$.

The matrix $B' = (B'_{ij})$ satisfies $BB' = 0$, by hypothesis, and, again, $P'_n B' = B'$ or $\sum_k A_{ik} B'_{kj} = B'_{ij}$, $i, j = 1, \cdots, n$. Moreover, $BP'_n = 0$, so that $\sum_k B_{ik} A_{kj} = 0$, $i, j = 1, \cdots, n$.

Finally, if $C_B C_{B'} = 0$, then $0 = BC_B C_{B'} B' = BB'$; and if $BB' = 0$, then, from the above, there is a central projection P in $\Re$ such that $PB = 0$, $PB' = B'$, whence $PC_B = 0$ and $C_{B'} \leq P$. Thus $C_B C_{B'} = 0$, and the proof is complete.

COROLLARY 3.1.2. *If $\Re$ is a ring of operators with commutant $\Re'$ and F' is a projection in $\Re'$, the kernels of the restriction mappings, $A \to AF'$ and $A \to AC_{F'}$, of $\Re$ upon $\Re F'$ and $\Re$ upon $\Re C_{F'}$, respectively, are identical.*

PROOF. According to Lemma 3.1.1, $AF' = 0$ if and only if $C_A C_{F'} = 0$, which in turn is true if and only if $AC_{F'} = 0$, from which our assertion follows.

LEMMA 3.1.3. *If ϕ is an isomorphism of the ring of operators $\Re_1$ onto the ring of operators $\Re_2$ and F'_1, F'_2 are projections in their commutants $\Re'_1$, $\Re'_2$, respectively, then the mapping ψ of $\Re_1 F'_1$ onto $\Re_2 F'_2$, defined by $\psi(AF'_1) = \phi(A)F'_2$ is an isomorphism if and only if $\phi(C_{F'_1}) = C_{F'_2}$.*

PROOF. If ψ is an isomorphism of $\Re_1 F'_1$ onto $\Re_2 F'_2$ then $\psi(C_{F'_1} F'_1) = \phi(C_{F'_1})F'_2 = \psi(F'_1) = F'_2$, so that $\phi(C_{F'_1}) \geq C_{F'_2}$. Applying this to ψ^{-1}, $\phi^{-1}(C_{F'_2}) \geq C_{F'_1}$, so that $C_{F'_2} \geq \phi(C_{F'_1})$, and $\phi(C_{F'_1}) = C_{F'_2}$. On the other hand, if $\phi(C_{F'_1}) = C_{F'_2}$, and $\psi(AF'_1) = \phi(A)F'_2 = 0$, then, from Lemma 3.1.1, $0 = C_{\phi(A)} C_{F'_2} = \phi(C_A)\phi(C_{F'_1}) = \phi(C_A C_{F'_1})$ so that $0 = C_A C_{F'_1}$, and $AF'_1 = 0$. It follows that ψ is an isomorphism of $\Re_1 F'_1$ onto $\Re_2 F'_2$.

We remark that if $\mathfrak{R}$ is a ring of operators and F' is a projection in its commutant $\mathfrak{R}'$, then $\mathfrak{C}F'$ is the center of $\mathfrak{R}F'$, where $\mathfrak{C}$ is the center of $\mathfrak{R}$. Indeed, it follows from Lemma 3.1.3, that $\mathfrak{R}C_{F'}$ and $\mathfrak{R}F'$ are isomorphic with $\mathfrak{C}C_{F'}$ mapping onto $\mathfrak{C}F'$. However, it is trivial that $\mathfrak{C}C_{F'}$ is the center of $\mathfrak{R}C_{F'}$, so that $\mathfrak{C}F'$ is the center of $\mathfrak{R}F'$ as asserted. Now, by [28; Lemma 11.3.2], if E is a projection in $\mathfrak{R}$, $E\mathfrak{R}E$ is a ring of operators with commutant $\mathfrak{R}'E$ when viewed as acting on $E(\mathcal{K})$, so that the center of $E\mathfrak{R}E$ is $\mathfrak{C}E$. Thus, if E is abelian in $\mathfrak{R}$, $E\mathfrak{R}E = \mathfrak{C}E$, and if F is a projection in $\mathfrak{R}$ contained in E then $F = AE$ with A in $\mathfrak{C}$. It follows at once that A may be chosen as a projection P (in fact, $C_F A$) in $\mathfrak{C}$. We may put the result just demonstrated in another form (for which we shall have later use): an abelian projection is minimal in a class of projections having the same central carrier.

3.2. Dimension theory in finite rings of operators

We develop a dimension theory for projections in a finite ring of operators without recourse to the trace function [6] on such rings. Our treatment of dimension does not follow the treatment of [28] for factors and provides an alternative method of developing the dimension function in factors.

DEFINITION 3.2.1. *A projection E, in a finite ring of operators $\mathfrak{R}$, which is the sum of b, orthogonal, equivalent copies of a projection F in $\mathfrak{R}$, c, orthogonal equivalent copies of which have sum a central projection in $\mathfrak{R}$, is said to be a rational projection in $\mathfrak{R}$ with dimension b/c. (We define 0 to be rational with dimension 0.) If G is an arbitrary projection in $\mathfrak{R}$, the supremum $u(G)$ of the dimensions of all rational projections contained in G is called the upper dimension of G. The infimum $L(G)$ of $u(PG)$ as P ranges over the non-zero central projections in $\mathfrak{R}$ is called the lower dimension of G.*

We note some elementary consequences of the above definition.

REMARK 3.2.2. If E is a rational projection in $\mathfrak{R}$ with dimensions a/b and c/d, then $a/b = c/d$. In fact, observe first that if m/n is an integer, and, if the central projection P is the sum of n, orthogonal, equivalent copies of F, then F is the sum of m/n copies of G if and only if P is the sum of m copies of G. This follows easily from The Comparison Lemma and finiteness of $\mathfrak{R}$. Now, if $[b, d]$ is the least common multiple of b and d, there is a projection G in $\mathfrak{R}$ such that P, the central carrier of E, is the sum of $[b, d]$ copies of G. Thus E is the sum of $a[b, d]/b$ and $c[b, d]/d$ copies of G. The finiteness of E tells us that $a[b, d]/b = c[b, d]/d$, which yields our assertion.

REMARK 3.2.3. The projection E is 0 if and only if $u(E) = 0$. In fact, if E is 0 then $u(E) = 0$ as an immediate consequence of Definition 3.2.1. On the other hand, if $u(E) = 0$, and F_m is a rational projection of dimension $1/m$, with $C_{F_m} = C_E$, then $E \prec F_m$, for otherwise, by The Comparison Lemma, E contains a rational projection of dimension $1/m$. Thus, from their definition and minimal property, the central carrier of E contains no abelian projections, so that there are rational projections F_{2^n} of dimension $1/2^n$ with central carrier C_E. From the preceding remark, $F_{2^n} > F_{2^n+1}$, so that we may choose the se-

quence $\{F_{2^n}\}$ descending and each containing E. The intersection of the F_{2^n} has an infinite number of copies, and is therefore 0, by the finiteness of $\mathfrak{R}$. Thus $E = 0$.

LEMMA 3.2.4. *If E is a projection in the finite ring of operators $\mathfrak{R}$ and $\{Q_\alpha\}$ is a family of central projections in $\mathfrak{R}$ with union Q, then $u(QE) = \sup_\alpha u(Q_\alpha E)$. For each positive real number r, there is a unique central projection Q_r such that $u(Q_r E) \leq r$ and Q_r contains each central projection P such that $u(PE) \leq r$. Letting $Q_0 = I - C_E$ and $Q_t = 0$ for $t < 0$, $\{Q_r\}$ is a resolution of the identity.*

PROOF. From Definition 3.2.1, it is clear that $\sup_\alpha u(Q_\alpha E) \leq u(QE)$, since each rational projection in $Q_\alpha E$ is contained in QE. On the other hand, if F is a non-zero rational projection contained in QE, then $C_F \leq Q$, so that $C_F Q_{\alpha'} \neq 0$ for some α'. Thus $Q_{\alpha'} F (\neq 0)$ is a rational projection contained in $Q_{\alpha'} E$ with the same dimension as F, so that $u(QE) \leq \sup_\alpha u(Q_\alpha E)$.

The second assertion of the lemma is obvious now if we take as Q_r the union of all central projections P such that $u(PE) \leq r$. Clearly $Q_r \leq Q_t$ if $r \leq t$, and $Q_t = I$ if $t \geq 1$. Moreover, $\bigcap_{r>0} Q_r = I - C_E$, by Remark 3.2.3, above. Finally, $\bigcap_{r>t} Q_r = Q_t$, for clearly $Q_t \leq \bigcap_{r>t} Q_r$, and, in addition, $u[(\bigcap_{r>t} Q_r)E] \leq t$, so that $\bigcap_{r>t} Q_r \leq Q_t$; and $\{Q_r\}$ is a resolution of the identity.

We shall call Q_r, constructed in the above lemma, the central portion of E for upper dimension r. We call the operator $D(E)$ with resolution $\{Q_r\}$ the dimension of E, and observe, as an immediate consequence of the nature of its resolution, that $D(E)$ is bounded, positive, and central.

LEMMA 3.2.5. *If E and F are orthogonal projections in the finite ring of operators $\mathfrak{R}$, then $u(E + F) \geq u(E) + L(F).)$*

PROOF. Suppose, first, that E and F are rational projections of dimensions a/b and c/b, respectively, such that $C_E C_F \neq 0$. According to Lemma 3.2.4,

$$u(E + F)$$
$$= \max \{u[(C_E - C_E C_F)(E + F)], u[(C_F - C_E C_F)(E + F)], u[(C_E C_F(E + F)]\}.$$

Now $C_E C_F E$ and $C_E C_F F$ are the sums of a and c, orthogonal, equivalent copies, respectively, of a projection, b copies of which have sum $C_E C_F$, so that $C_E C_F(E + F)$ is rational with dimension $(a + c)/b$. But

$$u[(C_E - C_E C_F)(E + F)] = u[(C_E - C_E C_F)E] \leq a/b \leq (a + c)/b$$

and $u[(C_F - C_E C_F)(E + F)] \leq (a + c)/b$, so that $u(E + F) = (a + c)/b$. Clearly, it was no restriction to assume that the dimensions of E and F have the same denominator.

With E and F arbitrary orthogonal projections in $\mathfrak{R}$, choose a rational projection G with dimension $a/b > u(E) - \varepsilon$ in E, where ε is a preassigned, positive number, and choose a rational projection G' in $C_G F$ with dimension $c/d > u(C_G F) - \varepsilon$. Then $G + G'$ is contained in $E + F$, so that, by the foregoing,

$$u(E + F) \geq u(G + G') = a/b + c/d \geq u(E) + u(C_G F) - 2\varepsilon$$

$$\geq u(E) + L(F) - 2\varepsilon,$$

this computation holding even if $C_G F = 0$ (in which case $G' = 0$). Since this inequality holds for each positive ε, we conclude the desired inequality.

LEMMA 3.2.6. *Two projections E and F in a finite ring of operators $\mathfrak{R}$ are equivalent if and only if $u(PE) = u(PF)$ for each central projection P in $\mathfrak{R}$.*

PROOF. If $E \sim F$ then $PE \sim PF$ for each central projection P in $\mathfrak{R}$, so that $u(PE) = u(PF)$, since equivalent rational projections have the same dimension. (Recall that, in finite ring, equivalences between projections can be implemented by a unitary operator in the ring.)

If E and F are not equivalent, there is a central projection P such that, say, $PE \prec PF$, and, by the first paragraph of this proof, we may assume that $PE < PF$. Let G be a non-zero rational projection of dimension $d(> 0)$ contained in $PF - PE$. If we denote the upper and lower dimension functions of the ring $\mathfrak{R}Q$, with Q a central projection in $\mathfrak{R}$, by u_Q and L_Q, respectively, then

$$u(C_G PF) \geqq u_{C_G}(G + C_G PE) \geqq u_{C_G}(C_G PE) + L_{C_G}(G) = u(C_G PE) + d,$$

so that $u(C_G PF) > u(C_G PE)$.

LEMMA 3.2.7. *If E is a projection in the finite ring of operators $\mathfrak{R}$ and X is the pure state space of the center $\mathfrak{C}$ of $\mathfrak{R}$, then*

$$u(PE) = \sup \{D(E)(x) : P(x) = 1, x \text{ in } X\}$$

$$L(E) = \inf \{D(E)(x) : x \text{ in } X\}.$$

PROOF. As we have done in the statement of this lemma, we employ the same symbol for a central operator and its representing function in $C(X)$. Now Q_t is the characteristic function of the complement of the closure of the set of points in X where $D(E)$ exceeds t, with $\{Q_t\}$ the spectral resolution of $D(E)$. Moreover the definition of Q_t is such that if Q is a non-zero central projection in $I - Q_t$ then $u(QE) > t$. Thus if $D(E)$ exceeds t at some point x such that $P(x) = 1$, then there is a non-zero central projection Q in $P(I - Q_t)$, so that $t < u(QE) \leqq u(PE)$, and $u(PE) \geqq \sup \{D(E)(x) : P(x) = 1\}$. On the other hand, if $t > \sup \{D(E)(x) : P(x) = 1\}$ then $P \leqq Q_t$, so that $u(PE) \leqq t$. Thus $u(PE) = \sup \{D(E)(x) : P(x) = 1\}$, from which, by continuity of $D(E)$, extreme disconnectedness of X, [47], and the definition of $L(E)$, it follows that $L(E) = \inf \{D(E)(x) : x \text{ in } X\}$.

THEOREM 3.2.8. (The Dimension Theorem). *If $\mathfrak{R}$ is a finite ring of operators acting on the Hilbert space $\mathfrak{K}$, the dimension function D on $\mathfrak{R}$ has the following properties:*

(a) $D(E) \geqq 0$ *and* $D(E) = 0$ *if and only if* $E = 0$.

(b) $D(PE) = PD(E)$, *with P in the center $\mathfrak{C}$ of $\mathfrak{R}$.*

(c) $D(F) \lesseqgtr D(E)$ *if and only if* $E \gtrsim\!\!\!\!< F$.

(d) $D(\sum E_n) = \sum D(E_n)$, *with $\{E_n\}$ a countable, orthogonal family of projections in $\mathfrak{R}$, where $\sum D(E_n)$ is understood as the supremum of the finite, partial sums in the complete, bounded lattice sense in $\mathfrak{C}$, the center of $\mathfrak{R}$.*

(e) *With $\Re$ of type II_1, the range of D is the set of all positive operators in the unit sphere of $\mathcal{C}$. If $\Re$ is of type I_m, the range of D is the set of operators of the form $\sum_{j=0}^{m} (j/m)P_j$, with $\{P_j\}$ orthogonal central projections.*

PROOF. Ad (a): Lemma 3.2.7 and Remark 3.2.3 show that if $E \neq 0$ then $D(E) > 0$, since $u(E) > 0$.

Ad (b): By Lemma 3.2.7, the supremum of $D(PE)$ on $X - S$ is $u[(I - P)PE] = u(0) = 0$, where X is the pure state space of $\mathcal{C}$ and S is the set of points of X at which P is 1. Moreover, if $P \geqq Q$ then

$$\sup \{(D(PE))(x):Q(x) = 1\} = u(QE) = \sup \{(D(E))(x):Q(x) = 1\}$$

$$= \sup \{(PD(E))(x):Q(x) = 1\}.$$

Since $D(PE)$ and $PD(E)$ are continuous and have the same supremum on each clopen subset of X, and since X is extremely disconnected, $D(PE) = PD(E)$.

Ad (c): If E is equivalent to F then PE is equivalent to PF for each central projection P, so that $u(PE) = u(PF)$, by Lemma 3.2.6. As in the proof of (b), Lemma 3.2.7 and the continuity of $D(E)$ and $D(F)$, yield $D(E) = D(F)$. If $D(E) = D(F)$ and $\{P_r\}$, $\{Q_r\}$ are the spectral resolutions of $D(E)$ and $D(F)$, respectively, then $P_r = Q_r$, for each r, so that $u(PE) = u(PF)$, for each central projection P. In fact, if $u(PE) = r$ then $P \leqq P_r$ and $P \nleqq P_t$ for $t < r$, so that $u(PF) = r$. Thus, by Lemma 3.2.6, $E \sim F$.

If $F \prec E$ then F is equivalent to a subprojection of E which, by the foregoing has the same dimension as F. We can assume, therefore, that $F < E$. In this case $r \geqq u(P_rE) \geqq u(P_rF)$, so that $P_r \leqq Q_r$. Since $\{P_r\}$, $\{Q_r\}$ are the spectral resolutions of $D(E)$, $D(F)$, respectively, $D(F) \leqq D(E)$. However, if $D(F) = D(E)$ then $F \sim E$, contrary to our assumption. Thus $D(F) < D(E)$.

If $D(F) < D(E)$ then E is not equivalent to F and if $F \nprec E$ there is a central projection P such that $PE \prec PF$. Thus, from (b) and the above, $PD(E) = D(PE) < D(PF) = PD(F)$, contrary to assumption. Thus $F \prec E$.

Ad (d): We show, first, that if E and F are mutually orthogonal projections in $\Re$ then $u(E) + u(F) \geqq u(E + F)$. According to Lemma 3.2.4, it suffices to establish this in the I_m and II_1 cases separately. In the I_m case, however, the desired inequality is a simple consequence of the fact that the projections in question can be expressed as orthogonal sums of abelian projections and that abelian projections have upper dimension $1/m$. We assume, henceforth, that $\Re$ is of type II_1. We shall have occasion to consider the ring $\Re P$ and projections PE, PF, where P is some central projection constructed during the argument, in place of $\Re$, E and F. This change will be effected by saying "restrict attention to P" and then dropping P from our notation, i.e., retaining the old notation.

Let G be a rational projection in $E + F$ with dimension d. We shall establish the desired inequality by showing that $d \leqq u(E) + u(F)$. Note that, under restriction to a central projection P such that $PC_G \neq 0$, PG is rational with dimension d and $u(PE) \leqq u(E)$, $u(PF) \leqq u(F)$; so that establishing the inequality in the restricted situation establishes it in general. To avoid the possi-

bility that some central projection constructed will be orthogonal to C_G, we begin by restricting attention to C_G. If $C_E C_F = 0$, then Lemma 3.2.4 establishes our inequality. We assume that $C_E C_F \neq 0$ and restrict attention to $C_E C_F$.

Let $\varepsilon > 0$ be assigned, and let G' be a rational projection of dimension $1/n$ where n is so chosen that nd is an integer and $1/n < \varepsilon$. Restrict attention to $C_{G'}$. Then G is an nd fold copy of G'. Suppose E admits p copies of G' with remainder E', so that $u(E) \geq p/n$. Since $G' \precsim E'$, there is, by The Comparison Lemma, a non-zero central projection P such that $QE' < QG'$ for each non-zero central projection $Q \leq P$. Restrict attention to P. Similarly, if F admits q copies of G' with remainder F', then $u(F) \geq q/n$, and, again, we can find a non-zero central projection R such that $QF' < QG'$ for each non-zero central projection $Q \leq R$. Restrict attention to R. Since $E' < G'$, $F' < G'$, $E'F' = 0$, and $E' + F'$ is finite, $E' + F'$ admits at most one copy of G' with remainder G''. By restriction, we can arrange, once again, that $G'' < G'$. Thus, under the indicated partitioning, $E + F$ admits not more than $p + q + 1$ copies of G' with remainder $G''(< G')$. If nd were to exceed $p + q + 1$ then $E + F$ would not be finite, for we could map the $p + q + 1$ copies of G' onto the same number of copies of G' among the nd copies of G' in G, and map G'' properly into one of the remaining copies. Thus $p + q + 1 \geq nd$, so that $u(E) + u(F) + 2\varepsilon \geq (p + q + 2)/n \geq d$. This being true for each positive ε, we conclude that $u(E) + u(F) \geq d$, and, finally, that $u(E) + u(F) \geq u(E + F)$.

From Lemma 3.2.7 and the last inequality, we have that the supremum of $D(E)$ added to the supremum of $D(F)$ over any clopen (closed and open) subset of X is not less than the supremum of $D(E + F)$ over this set. The continuity of $D(E)$, $D(F)$, $D(E + F)$, and the extreme disconnectedness of X yield $D(E) + D(F) \geq D(E + F)$. Employing the inequality $u(E + F) \geq u(E) + L(F)$ of Lemma 3.2.5 and Lemma 3.2.7 in this manner, we conclude that $D(E + F) \geq D(E) + D(F)$, so that $D(E + F) = D(E) + D(F)$. Of course, we now have the corresponding result for any finite number of projections.

Clearly, $\sum_{n=1}^{\infty} D(E_n) \leq D(\sum_{n=1}^{\infty} E_n)$, from the definition of $\sum D(E_n)$ and the additivity of D on finite, orthogonal sets of projections in $\mathfrak{R}$. Suppose, however, that $\sum D(E_n) < D(\sum E_n)$. In this case, we can find a clopen set S in X and a positive, rational number b/c such that $P \sum D(E_n) \leq (b/c)P < PD(\sum E_n)$, where P is the characteristic function (central projection) of the set S, and such that P is the sum of c orthogonal copies of some projection E in $\mathfrak{R}$. Then, by (b) and finite additivity of D, we have $D(E) = (1/c)P$. If F denotes the sum of b copies of E, then $D(F) = (b/c)P \geq P \sum D(E_n)$. By virtue of (c) and finite additivity of D, we can find an orthogonal family of projections $\{F_n\}$ in $\mathfrak{R}$ contained in F such that $F_n \sim PE_n$, so that $\sum PE_n \sim \sum F_n \leq F$. Thus $D(F) = (b/c)P \geq D(\sum PE_n) = PD(\sum E_n)$, contrary to the choice of P. Hence $D(\sum E_n) = \sum D(E_n)$.

Ad (e): From the argument above, we see that if a/b is any non-negative, rational number not exceeding 1, in case $\mathfrak{R}$ is of type II_1, and with $b = m$ if $\mathfrak{R}$ is of type I_m, and P is a central projection, then $(a/b)P$ is in the range of D.

Now if $\{P_n\}$ is an orthogonal family of central projections, $\{a_n/b_n\}$ are rational numbers of the above type, and $\{E_n\}$ is a collection of projections in $\mathfrak{R}$ such that $D(E_n) = (a_n/b_n)P_n$, then $D(\sum P_n E_n) = \sum D(P_n E_n) = \sum(a_n/b_n)P_n$, by (d), and thus $\sum (a_n/b_n)P_n$ is in the range of D.

Now, from The Spectral Theorem, each positive operator A in the unit sphere of $\mathfrak{C}$ is the uniform limit of a monotone increasing sequence $\{A_n\}$ of finite, linear combinations, with non-negative, rational coefficients not exceeding 1, of mutually-orthogonal, central projections (spectral projections for A). With $\mathfrak{R}$ of type II_1, each $A_n = D(E_n)$ for some projection E_n in $\mathfrak{R}$. Using (c), we can arrange the E_n so that $E_1 \leq E_2 \leq \cdots$. Define $F_1 = E_1$ and $F_n = E_n - E_{n-1}$. Then $\{F_n\}$ is an orthogonal family of projections, and $D(\sum_{n=1}^{k} F_n) = \sum_{n=1}^{k} D(F_n) = A_k$, so that $\sum_{n=1}^{\infty} F_n(= E)$, the union of $\{E_n\}$, has dimension $D(E) = D(\sum F_n) = \sum D(F_n) = \lim A_n = A$, by (d). If $\mathfrak{R}$ is of type I_m, the range of D consists of operators $\sum_{j=0}^{m} (j/m)P_j$, with $\{P_j\}$ an orthogonal family of central projections in $\mathfrak{R}$.

3.3. Cyclicity, separation and countable-decomposability

We begin by noting that $\mathfrak{R}F'$ is countably-decomposable if $\mathfrak{R}$ is, with F' a projection in the commutant of $\mathfrak{R}$. Indeed, with $\mathfrak{R}$ countably-decomposable, $C_{F'}$ is countably-decomposable, so that $\mathfrak{R}C_{F'}$ is countably-decomposable. However $\mathfrak{R}C_{F'}$ is isomorphic to $\mathfrak{R}F'$, by Lemma 3.1.3, and isomorphisms clearly carry countably-decomposable projections onto countably-decomposable projections.

LEMMA 3.3.1. *A central projection P in a ring of operators $\mathfrak{R}$ is the central carrier of a cyclic projection in $\mathfrak{R}$ if and only if P is countably-decomposable relative to the center $\mathfrak{C}$ of $\mathfrak{R}$. A cyclic projection in $\mathfrak{R}$ is countably-decomposable. Projections with the same generating vector (in $\mathfrak{R}$ and $\mathfrak{R}'$) have the same central carrier.*

PROOF. If E is a cyclic projection in $\mathfrak{R}$ with generating vector x and $P = C_E$, and if $\{P_\alpha\}$, $\{E_\beta\}$ are orthogonal families of projections in $\mathfrak{C}$ and $\mathfrak{R}$, respectively contained in P and E, respectively, then $\sum_\alpha \| P_\alpha x \|^2$ and $\sum_\beta \| E_\beta x \|^2$ are each not greater than $\| x \|^2$ and, hence, finite. Thus at most a countable number of terms of each sum are non-zero. However, if $E_\beta x = 0$ then $\{0\} = [\mathfrak{R}'E_\beta x] = [E_\beta \mathfrak{R}'x] = E_\beta E = E_\beta$, and if $P_\alpha x = 0$ then, similarly, $P_\alpha E = 0$, so that $0 = P_\alpha P = P_\alpha$, from Lemma 3.1.1. Thus $\{P_\alpha\}$, $\{E_\beta\}$ have at most a countable number of non-zero members, and P and E are countably-decomposable.

Suppose now that P is countably-decomposable, and let $\{P_n\}$ be a (countable) set of projections cyclic under $\mathfrak{C}'$ with sum P, each P_n with a unit generating vector x_n. Let $x = \sum_n x_n/n$. Then $[\mathfrak{C}'x] = P$, for $[\mathfrak{C}'x]$ contains $[\mathfrak{C}'P_n x] = [\mathfrak{C}'x_n] = P_n$, for each n. Letting $[\mathfrak{R}'x] = E$, we have that $E \leq P$, since $\mathfrak{R}'$ is contained in $\mathfrak{C}'$, i.e., $PE = E$. Moreover, if Q is in $\mathfrak{C}$ and $QE = E$, then $Qx = x$, so that $[\mathfrak{C}'x] = [\mathfrak{C}'Qx] = [Q\mathfrak{C}'x]$, whence $P = QP$. Thus $P = C_E$, with E cyclic in $\mathfrak{R}$. From this argument, we see that P is also the central carrier of $[\mathfrak{R}x]$, which establishes the last assertion of the lemma.

We shall call a vector x "a separating vector" for the ring of operators $\mathfrak{R}$,

when $Ax = 0$ for an operator A in $\mathfrak{R}$ implies that $A = 0$. Clearly such a vector is a generating vector for I under $\mathfrak{R}'$, for $I - [\mathfrak{R}'x]$ lies in $\mathfrak{R}$, annihilates x, and is therefore 0. On the other hand, if $[\mathfrak{R}'x] = I$ and $Ax = 0$, with A in $\mathfrak{R}$, then $[A\mathfrak{R}'x] = 0$, so that $A = 0$, and x is separating for $\mathfrak{R}$. Combining this with the first statement in the above lemma, we see that an abelian ring of operators $\mathfrak{R}$ which is countably-decomposable possesses a separating vector. In fact, I is the central carrier of a cyclic projection in $\mathfrak{R}$, which must be I itself, since $\mathfrak{R}$ is abelian. From the foregoing, it follows that each generating vector for I under $\mathfrak{R}'$ is separating for $\mathfrak{R}$. This fact is basic to the classical abelian, multiplicity theory, and appears in each account of this theory in a slightly altered form. The version just noted appears in [32]. The first statement of Lemma 3.3.1 is the non-commutative extension of this result.

We note for future reference that an abelian projection E which has a countably-decomposable central carrier is cyclic. In fact, by Lemma 3.3.1, C_E is the central carrier of some cyclic projection F, and, since E is an abelian projection with the same central carrier as F, we have $E \precsim F$, and E is cyclic.

LEMMA 3.3.2. *If $\mathfrak{R}$ is a ring of operators acting on a Hilbert space $\mathcal{H}$, $\{Q_k\}$ is a countable family of orthogonal, central projections in $\mathfrak{R}$, and $\{E_k'\}$ is a family of projections cyclic under $\mathfrak{R}$ such that $E_k' \leqq Q_k$, then $E' = \sum_k E_k'$ is cyclic under $\mathfrak{R}$.*

PROOF. Let x_k be a generating unit vector for E_k', $k = 1, 2, \cdots$, and let $x = \sum_{n=1}^{\infty} x_k/k$. Then $[\mathfrak{R}x]$ contains $[\mathfrak{R}Q_k x] = [\mathfrak{R}x_k] = E_k'$, for each k, so that $[\mathfrak{R}x]$ contains E'. On the other hand, $E'x = x$, so that E' contains $[\mathfrak{R}x]$. Thus $E' = [\mathfrak{R}x]$, and E' is cyclic as we wished to prove.

LEMMA 3.3.3. *If E and F are projections in a ring of operators $\mathfrak{R}$ such that E is countably-decomposable, F is purely-infinite, and $C_E \leqq C_F$, then $E \precsim F$. Two purely-infinite, countably-decomposable projections in $\mathfrak{R}$ are equivalent if and only if they have the same central carrier.*

PROOF. Assume that $E \not\precsim F$. Employing The Comparison Lemma and restricting $\mathfrak{R}$ to the central projection so obtained, we may assume that $QF < QE$ for each non-zero central projection Q in $\mathfrak{R}$. Since $C_E \leqq C_F$, F does not become 0 under this restriction, and F is infinite. Thus, there is a partial isometry in $\mathfrak{R}$ mapping F upon a proper subprojection F_1', F_1' upon a proper sub-projection F_2', and so forth. Clearly $F - F_1'$, $F_1' - F_2'$, $\cdots$ are orthogonal, non-zero, equivalent projections contained in F. Thus F contains an orthogonal, countably-infinite family $\{F_n\}$ of equivalent, non-zero projections. Let $\{E_n\}$ be a maximal, orthogonal family of projections contained in E, equivalent to F_n. By maximality, $E_1 \not\precsim E - \sum E_n$, so that, by The Comparison Lemma, there is a non-zero central projection Q such that $Q(E - \sum E_n) < QE_1 \sim QF_1$. Since $QE_1 \sim QF_2$, $QE_2 \sim QF_3$, $\cdots$ and $Q(E - \sum E_n) + \sum QE_n = QE$, we see that $QE \precsim QF$, contradicting $QF < QE$. Thus $E \precsim F$.

If E and F are both purely-infinite and countably-decomposable, and $C_E = C_F$, then, from the foregoing, $E \precsim F$ and $F \precsim E$, so that $E \sim F$. In general, if E and F are equivalent projections in $\mathfrak{R}$, there is a partial isometry V in $\mathfrak{R}$ such that $V^*V = E$ and $VV^* = F$; so that if $PF = F$ then $PV = PVV^*V =$

$VV^*V = V$, and $PV^*V = V^*PV = V^*V$, with P a central projection in $\mathfrak{R}$. Thus $QE = E$ implies $QF = F$, with Q a central projection in $\mathfrak{R}$, and $C_E = C_F$.

LEMMA 3.3.4. *If E and E' are projections lying in a ring of operators $\mathfrak{R}$ and its commutant $\mathfrak{R}'$, respectively, and E and E' have a joint generating vector (under $\mathfrak{R}'$ and $\mathfrak{R}$, respectively), then the finiteness of one of E or E' implies the finiteness of the other. A projection which is cyclic under a finite ring of operators is finite.*

PROOF. Suppose E is finite and E' is infinite. By restricting $\mathfrak{R}$ and $\mathfrak{R}'$ to a suitable central projection, we can assume that E' is purely-infinite. If QE is abelian, for some non-zero, central projection Q, then QE' is abelian, by [28; Lemma 9.3.3], contradicting the pure-infiniteness of E'. Thus E can be expressed as the sum of two orthogonal projections F and G in $\mathfrak{R}$ with central carrier C_E. Let x and y be generators for F and G, respectively, and let $F' = [\mathfrak{R}x]$, $G' = [\mathfrak{R}y]$. If F', say, were infinite, then there would exist a central projection Q such that QF' is purely-infinite (with central carrier $Q \leqq C_{E'} = C_E$, by Lemma 3.3.1) and countably-decomposable, so that QF' and QE' would be equivalent, by Lemma 3.3.3. It would follow from [28; Lemma 9.3.3] that QE and QF are equivalent. However, $QE = QF + QG$ and $QG \neq 0$, since $C_G = C_E$, so that QF is properly less than QE, contradicting the finiteness of E. Thus F' and G' are finite, so that, since E' is purely-infinite, there are two orthogonal projections F'' and G'' equivalent to F' and G', respectively, contained in E'. If V' is a partial isometry in $\mathfrak{R}'$ mapping F' upon F'', then

$$F = [\mathfrak{R}'x] \geqq [\mathfrak{R}'V'x] \geqq [\mathfrak{R}'V'^*V'x] = [\mathfrak{R}'x] = F,$$

and $[\mathfrak{R}V'x] = V'[\mathfrak{R}x] = F''$. Thus F, F'' and G, G'' have joint generating vectors ($V'x$, as shown, in the case of F, F'') z and w, respectively. We have

$$E = F + G \geqq [\mathfrak{R}'(z + w)] \geqq [\mathfrak{R}'F''(z + w)] + [\mathfrak{R}'G''(z + w)]$$

$$= [\mathfrak{R}'z] + [\mathfrak{R}'w] = F + G = E,$$

and

$$F'' + G'' \geqq [\mathfrak{R}(z + w)] \geqq [\mathfrak{R}F(z + w)] + [RG(z + w)]$$

$$= [\mathfrak{R}z] + [\mathfrak{R}w] = F'' + G''.$$

From [28; Lemma 9.3.3], the finite projection $F'' + G''$ is equivalent to the infinite projection E', a contradiction. Thus E' is finite.

If E' in $\mathfrak{R}'$ is cyclic under the finite ring of operators $\mathfrak{R}$ with generating vector x, then $E = [\mathfrak{R}'x]$ is finite, so that E' is finite.

In connection with the first assertion of the following lemma, see also [42; Lemma 1.1].

LEMMA 3.3.5. *If the projection E in the ring of operators $\mathfrak{R}$ is finite and C_E is countably-decomposable relative to the center $\mathfrak{C}$ of $\mathfrak{R}$ then E is countably-decomposable relative to $\mathfrak{R}$. If, in addition, $\mathfrak{R}'$ is purely-infinite then E is cyclic.*

PROOF. Let $\{E_\alpha\}$ be an orthogonal family of projections in $\mathfrak{R}$ contained in E. From the comments following Lemma 3.3.1, we can choose a separating unit vector x for $\mathfrak{C}E$ (which is isomorphic to the countably-decomposable ring $\mathfrak{C}C_E$).

Then, if $(D(E_\alpha)x, x)$ were non-zero for an uncountable set of α, some countable subcollection E_1, E_2, $\cdots$ would be such that $\infty = \sum_n (D(E_n)x, x) = (D(\sum_n E_n)x, x) \leqq (D(E)x, x) = 1$, where D is the dimension function on $E\Re E$, a finite ring. Thus $D(E_\alpha)x = 0$ for all but a countable number of α. But $D(E_\alpha)$ lies in $\mathfrak{C}E$, so that, by choice of x, $D(E_\alpha) = 0$ and $E_\alpha = 0$ for all but a countable number of α and E is countably-decomposable.

If R' is purely-infinite, let $\{E_n\}$ be a maximal orthogonal family of cyclic projections in $\Re$ contained in E with x_n a generating vector for E_n. By maximality, $\sum_n E_n = E$. Let $E'_n = [\Re x_n]$, so that E'_n is finite by Lemma 3.3.4. Since R' is purely-infinite, we can choose an orthogonal family $\{E''_n\}$ in $\Re'$ with $E''_m \sim E'_m$. As in Lemma 3.3.4, there exists a unit generating vector y_n for E''_n and E_n. Then $y = \sum y_n/n$ is in the range of E and $[\Re'y] \geqq [\Re'E''_n y] = [\Re'y_n] = E_n$, so that $[\Re'y] = E$; and E is cyclic.

We shall say that a cyclic projection E in a ring of operators $\Re$ is maximal cyclic when $E \prec F$ implies F is not cyclic, and we shall say that E is absolutely, maximal cyclic when $E < F$ implies F is not cyclic. Of course $C_E = I$, for such an E and the center of $\Re$ is countably-decomposable. Since equivalence of projections preserves cyclicity, an absolutely, maximal, cyclic projection is maximal cyclic, though the converse is not true. Indeed if E is absolutely, maximal cyclic and $E \prec F$ then $E \sim E_1 < F$, and $I - E$ and $F - E_1$ have equivalent, non-zero subprojections E_0 and F_0, respectively. Then $E + E_0 \sim E_1 + F_0$ and $E_1 + F_0$ is cyclic if F is, contradicting the absolute, maximal cyclicity of E. Thus F is not cyclic and E is maximal cyclic. Clearly, equivalence preserves maximal cyclicity, though not absolute, maximal cyclicity. Thus, if I is cyclic it is certainly absolutely, maximal cyclic and any subprojection equivalent to I is therefore maximal cyclic without being absolutely, maximal cyclic. If E is maximal cyclic and P is a central projection in $\Re$ then PE is maximal cyclic in $\Re P$, for with F cyclic in $\Re P$ and $F > PE$, $F + (I - P)E > E$ and $F + (I - P)E$ is cyclic, by Lemma 3.3.2, contradicting the maximal cyclicity of E. Concluding these remarks, we note that all maximal, cyclic projections in $\Re$ are equivalent. In fact, if E and F are maximal cyclic and P is a central projection such that $PE \precsim PF$ then $PE \sim PF$, by maximal cyclicity of PE in $\Re P$, so that $E \sim F$, by symmetry and The Comparison Lemma.

Adapting the argument of [10; Lemma 1.2.8], we prove, along with Lemma 1.2.8. of [10] itself:

LEMMA 3.3.6. *If $\Re$ and its commutant $\Re'$ are purely-infinite rings of operators with a countably-decomposable center $\mathfrak{C}$, then $\Re$ contains a purely-infinite, cyclic projection with central carrier I. The class of maximal cyclic projections in $\Re$ is precisely the class of purely-infinite, countably-decomposable projections with central carrier I. If $\Re$ is countably-decomposable, there is a central projection cyclic under $\Re$ whose orthogonal complement is cyclic under $\Re'$.*

PROOF. Let $\{x_\alpha\}$ be a collection of unit vectors in the Hilbert space $\mathfrak{IC}$, upon which $\Re$ and $\Re'$ act, maximal with respect to the property that $\{E_\alpha\}$, $\{E'_\alpha\}$ are orthogonal families of projections in $\Re$ and $\Re'$, respectively, E_α, E'_α cyclic under $\Re'$, $\Re$, respectively, with generating vector x_α, and $\{E_\alpha\}$ mutually equiv-

alent projections with central carrier I. Any countable subfamily $\{E_n\}$ of $\{E_\alpha\}$ has sum E which is cyclic under $\mathfrak{R}'$ with generating vector $x = \sum_n x_n/n$, for $[\mathfrak{R}'x]$ contains $[\mathfrak{R}'E_n'x] = [\mathfrak{R}'x_n] = E_n$, and x is in the range of E. (The family $\{E_\alpha\}$ is non-null, by Lemma 3.3.1.)

If the collection $\{E_n\}$ is infinite then E is purely-infinite with central carrier I, since the E_n are equivalent with central carrier I. We may assume, therefore, that $\{E_n\}$ is a finite set, $E_1, \cdots, E_k$. We shall establish, first, the existence of some purely-infinite, cyclic projection or, what amounts to the same thing— by restricting to a suitable non-zero, central projection, the existence of some infinite, cyclic projection. Thus, we may assume, for this purpose, that E_1 is finite, whence E, being a finite sum of finite projections, is finite. Since $E' = E_1' + \cdots + E_k'$ has the same generator as E, we have by Lemma 3.3.4 that E' is finite. Thus $I - E$ and $I - E'$ are purely-infinite with central carrier I, so that, by Lemma 3.1.1, $(I - E)(I - E') \neq 0$, and, in fact, the mappings $A \to A(I - E')$, $A' \to A'(I - E)$ of $\mathfrak{R}$ onto $\mathfrak{R}(I - E')$ and $\mathfrak{R}'$ onto $\mathfrak{R}'(I - E)$, respectively, are *-isomorphisms. Thus $(I - E)(I - E')$ is purely-infinite and has central carrier $I - E'$ relative to $\mathfrak{R}(I - E')$, so that $(I - E)(I - E') \succsim E_1(I - E')$, according to Lemma 3.3.3. (Recall that E_1 is cyclic in $\mathfrak{R}$, so that $E_1(I - E')$ is countably-decomposable in $\mathfrak{R}(I - E')$.) Let F_{k+1} be a subprojection of $(I - E)(I - E')$ equivalent to $E_1(I - E')$. Then F_{k+1} is finite in

$$\mathfrak{R}(I - E')$$

and hence cyclic under the purely-infinite ring $(I - E')\mathfrak{R}'(I - E')$, according to Lemma 3.3.5. Let x_{k+1} be a generating vector for F_{k+1}, and let

$$E_{k+1} = [\mathfrak{R}'x_{k+1}].$$

Then $(I - E')E_{k+1} = [(I - E')\mathfrak{R}'(I - E')x_{k+1}] = F_{k+1}$, so that E_{k+1} is equivalent to E_1, the mapping $A \to A(I - E')$ being an isomorphism of $\mathfrak{R}$ upon $\mathfrak{R}(I - E')$, and E_{k+1} is orthogonal to $E_1, \cdots, E_k$. Moreover, $[\mathfrak{R}x_{k+1}] = E_{k+1}'$ is orthogonal to E', since $E'x_{k+1} = 0$.

The existence of the vector x_{k+1} contradicts the maximal property of

$$\{x_1, \cdots, x_k\},$$

so that E_1 is infinite, and we have established the existence of a purely-infinite, cyclic projection in $\mathfrak{R}$. Now, let $\{F_\alpha\}$ be a collection of purely-infinite, cyclic projections in $\mathfrak{R}$ maximal with respect to the property that $\{C_{F_\alpha}\}$ is an orthogonal family. Since $\mathfrak{C}$ is countably-decomposable, there are at most a countable number of F_α. By Lemma 3.3.2, $F = \sum_\alpha F_\alpha$ is cyclic. Clearly F is purely-infinite and has central carrier $P = \sum C_{F_\alpha}$. However, $\mathfrak{R}(I - P)$ and $\mathfrak{R}'(I - P)$ are either purely-infinite or (0), (in case $I = P$). If they are purely-infinite, then $\mathfrak{R}(I - P)$ contains a purely-infinite, cyclic projection (with central carrier in $I - P$, of course). This would contradict the maximal property of $\{F_\alpha\}$, so that $I = P$, and F is the desired purely-infinite, cyclic projection with central carrier I.

Since each cyclic projection in $\mathfrak{R}$ is countably-decomposable, it follows from

Lemma 3.3.3, that each such projection E in $\mathfrak{R}$ satisfies $E \precsim F$, where F is the purely-infinite projection with $C_F = I$, just constructed. Thus F is maximal cyclic in $\mathfrak{R}$, and only projections equivalent to F are maximal cyclic. From Lemma 3.3.3, again, this class is the family of purely-infinite, cyclic projections with central carrier I.

We assume now only that $\mathfrak{R}$ is countably-decomposable, and, returning to the beginning of the proof, we construct the family $\{x_\alpha\}$ as we did, dropping the condition that the projections in $\{E_\alpha\}$ be equivalent. Since $\mathfrak{R}$ is countably-decomposable, $\{E_\alpha\}$ is at most countable; we relable it as $\{E_n\}$, and note, as before, that E and E' are cyclic under $\mathfrak{R}'$ and $\mathfrak{R}$, respectively with generating vector x (constructed as before). If $(I - E)(I - E') \neq 0$, a non-zero vector y in the range of $(I - E)(I - E')$ adjoined to the set $\{x_n\}$ contradicts the maximal property of this set, for $[\mathfrak{R}'y]$ and $[\mathfrak{R}y]$ are non-zero and orthogonal to E and E', respectively. Thus $(I - E)(I - E') = 0$, so that $C_{I-E}C_{I-E'} = 0$. Now $I - C_{I-E} \leqq E$ so that $I - C_{I-E}$ is cyclic under $\mathfrak{R}$. Similarly, $I - C_{I-E'}$ is cyclic under $\mathfrak{R}'$. But $C_{I-E} \leqq I - C_{I-E'}$, so that C_{I-E} is cyclic under $\mathfrak{R}'$, and the proof is complete.

LEMMA 3.3.7. *Each cyclic projection E in a ring of operators $\mathfrak{R}$ with countably-decomposable center $\mathfrak{C}$ is contained in a maximal, cyclic projection.*

PROOF. Clearly, the sum of maximal, cyclic projections in $\mathfrak{R}P$ and $\mathfrak{R}Q$, with P and Q orthogonal, central projections, is maximal cyclic in $\mathfrak{R}(P + Q)$, so that it suffices to establish the lemma in the cases in which $\mathfrak{R}$ is finite and purely-infinite.

We note, first, that if $\mathfrak{R}$ contains a maximal, cyclic projection F our lemma follows. In fact, QF is maximal cyclic in $\mathfrak{R}Q$, with Q a central projection in $\mathfrak{R}$. Thus $E \precsim F$, for otherwise there is a central projection Q such that $QF \prec QE$, contradicting the maximal cyclicity of QF in $\mathfrak{R}Q$. Let R be a central projection such that $QE \prec QF$, for each non-zero, central projection Q in R, and

$$(I - R)E \sim (I - R)F.$$

Let G be a subprojection of RF such that $RE \sim G$. Then $RF - G \precsim R - RE$, for otherwise there is a central projection Q in R such that

$$Q - QE \prec QF - QG,$$

and, since $QE \sim QG$, we would have Q equivalent to a proper subprojection of QF, so that Q and QF are purely-infinite. Then QE must be finite, for if not PE is purely-infinite for some central projection $P \leqq QC_E \leqq R$, so that, by Lemma 3.3.3, since PF is countably-decomposable, $PF \precsim PE$, contrary to the choice of R. Thus $Q - QE$ is purely-infinite and, again, by Lemma 3.3.3, since $QF - QG$ is countably-decomposable, $QF - QG \precsim Q - QE$, contrary to the choice of Q. Now, with $RF - G \precsim R - RE$ and $RE \sim G$, there is a projection E_0 in R, containing RE and equivalent to RF. Thus $E_0 + (I - R)E$ is equivalent to F, and is therefore the desired maximal, cyclic projection in $\mathfrak{R}$ containing E.

If $\mathfrak{R}$ and $\mathfrak{R}'$ are purely-infinite, it follows from Lemma 3.3.6 that $\mathfrak{R}$ contains a purely-infinite cyclic projection with central carrier I, and that this projection is maximal cyclic in $\mathfrak{R}$. We assume that one of $\mathfrak{R}$, $\mathfrak{R}'$ is finite, say $\mathfrak{R}$, and establish the existence of absolutely, maximal, cyclic projections in $\mathfrak{R}$ and $\mathfrak{R}'$. In fact, since $\mathfrak{C}$ is countably-decomposable, it follows from Lemma 3.3.5 that $\mathfrak{R}$ is countably-decomposable and from Lemma 3.3.6 that there is a central projection P in $\mathfrak{R}$ such that $P = [\mathfrak{R}'y]$ and $I - P = [\mathfrak{R}x]$. If $F = [\mathfrak{R}'x]$ then F is absolutely, maximal cyclic in $\mathfrak{R}(I - P)$, for if $F < [\mathfrak{R}'(I - P)z]$ then, by finiteness of $\mathfrak{R}$, $F \prec [\mathfrak{R}'(I - P)z]$, an impossibility, since then, $I - P \prec [\mathfrak{R}(I - P)z]$, by [28; Lemma 9.3.3]. Thus $F + P$ is absolutely, maximal cyclic in $\mathfrak{R}$. In addition, $[\mathfrak{R}y]$ is absolutely, maximal cyclic in $\mathfrak{R}'P$. In fact, $[\mathfrak{R}y]$ is finite, by Lemma 3.3.4, so that if $[\mathfrak{R}y] < [\mathfrak{R}Pz]$ then $[\mathfrak{R}y] \prec [\mathfrak{R}Pz]$, and $P \prec [\mathfrak{R}'Pz]$, as before, and this is impossible. Thus $I - P + [\mathfrak{R}y]$ is absolutely, maximal cyclic in $\mathfrak{R}'$, and the proof is complete.

We observe some simple consequences of Lemma 3.3.7. If ϕ is an isomorphism of one ring of operators $\mathfrak{R}_1$ with a countably-decomposable center onto another such ring $\mathfrak{R}_2$, then ϕ preserves equivalence of projections, and if, in addition, ϕ and ϕ^{-1} preserve cyclicity of projections, then ϕ (and ϕ^{-1}) preserve maximal cyclicity and absolute, maximal cyclicity. On the other hand, if ϕ alone is assumed to preserve maximal cyclicity, then ϕ and ϕ^{-1} preserve cyclicity, so that ϕ^{-1} preserves maximal cyclicity. In fact, if E is cyclic projection in $\mathfrak{R}_1$, then, by Lemma 3.3.7, E is contained in a maximal, cyclic projection, so that $\phi(E)$ is contained in the image of this maximal, cyclic projection, and $\phi(E)$ is cyclic. If $\phi(G)$ is cyclic, choose a maximal, cyclic projection F in $\mathfrak{R}_1$. Then $G \precsim F$, for otherwise there is a central projection P such that $PF \prec PG$, so that

$$\phi(P)\phi(F) \prec \phi(P)\phi(G),$$

contradicting the maximal cyclicity of $\phi(P)\phi(F)$ in $\mathfrak{R}_2\phi(P)$. (Recall that

$$\phi(P)\phi(G)$$

is cyclic in $\mathfrak{R}_2\phi(P)$.) Thus G is cyclic.

THEOREM 3.3.8. (The Coupling Theorem). *If $\mathfrak{R}$ and its commutant $\mathfrak{R}'$ are finite rings of operators with dimension functions D and D', respectively, and if $E = [\mathfrak{R}'x]$, $E' = [\mathfrak{R}x]$, $F = [\mathfrak{R}'y]$, $F' = [\mathfrak{R}y]$, then $D(E)D'(F') = D'(E')D(F)$.*

PROOF. In view of (b) of The Dimension Theorem, in order to establish the desired equality, it suffices to show that $D(P_nE)D'(P_nF') = D'(P_nE')D(P_nF)$ for each projection P_n of a set $\{P_n\}$ of central projections with union C_EC_F. (From Lemma 3.3.1, $C_E = C_{E'}$ and $C_F = C_{F'}$). We take P_n to be the characteristic function of the closure of the set of points in X, the pure state space of the center $\mathfrak{C}$ of $\mathfrak{R}$, at which each of $D(E)$, $D'(E')$, $D(F)$, and $D'(F')$ exceeds $1/n$. We restrict attention to $\mathfrak{R}P_n$ and $\mathfrak{R}'P_n$. Thus $D(E)$, $D'(E')$, $D(F)$, and $D'(F')$ are invertible.

Clearly, it suffices to establish $D(Q_\alpha E)D'(Q_\alpha F') = D'(Q_\alpha E')D(Q_\alpha F)$ for each projection Q_α in an orthogonal family $\{Q_\alpha\}$ of central projections with sum I,

so that we may deal with cases where $\mathfrak{R}$ is type II_1 and I_m, separately. Denote by C and C' the images under D and D', respectively, of absolutely, maximal, cyclic projections in $\mathfrak{R}$ and $\mathfrak{R}'$, respectively, so that $D(E) \leqq C$, $D'(E') \leqq C'$, and C, C' are invertible. Let ψ be the mapping defined by $\psi(D([\mathfrak{R}'z])) = D'([\mathfrak{R}z])$ of the set of operators $\mathfrak{C}_1$ in the range of D between 0 and C onto the set of operators $\mathfrak{C}_2$ in the range of D' between 0 and C'. Note that [28; Lemma 9.3.3] and the properties of the dimension function listed in The Dimension Theorem assure us that each element of $\mathfrak{C}_1$ is representable in the form $D([R'z])$, that the image of such an element under ψ is independent of the representation chosen, and that ψ is a one to one mapping of $\mathfrak{C}_1$ onto $\mathfrak{C}_2$ which is order-preserving with an order-preserving inverse.

If $D(G) \leqq C$, and $D'(G') = \psi[D(G)]$, then there exists a vector z such that $[\mathfrak{R}'z] = G$ and $[\mathfrak{R}z] = G'$. In fact, by definition of ψ there is a vector w such that $[\mathfrak{R}'w] = G$ and $D'([\mathfrak{R}w]) = \psi[D(G)] = D'(G')$. Thus, by (c) of The Dimension Theorem, $[\mathfrak{R}w] \sim G'$, and the existence of the vector z follows as in Lemma 3.3.4. Now, if A, B and $A + B$ lie in $\mathfrak{C}_1$ and $\psi(A) + \psi(B)$ lies in $\mathfrak{C}_2$, then

$$\psi(A + B) = \psi(A) + \psi(B).$$

In fact, by (e) of The Dimension Theorem, we can choose orthogonal pairs of projections M, M' and N, N' each in $\mathfrak{R}$, $\mathfrak{R}'$, respectively, such that $D(M) = A$, $D(N) = B$, $D'(M') = \psi(A)$, and $D'(N') = \psi(B)$. By the foregoing remarks, we can find vectors z, w such that $[\mathfrak{R}'z] = M$, $[\mathfrak{R}z] = M'$, $[\mathfrak{R}'w] = N$, and

$$[\mathfrak{R}w] = N'.$$

Letting $v = z + w$, we have

$$M + N \geqq [\mathfrak{R}'(z + w)] \geqq [\mathfrak{R}'M'(z + w)] + [\mathfrak{R}'N'(z + w)] = M + N,$$

so that $[\mathfrak{R}'v] = M + N$, and, from this argument, $[\mathfrak{R}v] = M' + N'$. Thus

$$\psi(A) + \psi(B) = \psi[D(M)] + \psi[D(N)] = D'(M') + D'(N')$$
$$= D'(M' + N') = \psi[D(M + N)] = \psi(A + B),$$

and ψ is additive where defined. The homogenity of ψ with respect to positive scalars follows on rational scalars from the additivity and on real scalars, since it is order-preserving, when ψ is defined on the operators in question.

Thus ψ has a linear, order-isomorphic extension mapping the linear space spanned by $\mathfrak{C}_1$ onto the linear space spanned by $\mathfrak{C}_2$. We denote this extension by ψ again, and note that ψ is defined on each central projection P and has P as an image element both when $\mathfrak{R}$ is of type II_1 and $\mathfrak{R}$ is of type I_m, since C and C' are invertible and by the nature of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in the type II_1 and I_m cases (as indicated in (e) of The Dimension Theorem). Let $C_0 = \psi(I)$, so that $C_0 \geqq C'$ and C_0 is invertible. Define the mapping τ by $\tau(A) = C_0^{-1}\psi(A)$, so that τ is a linear, order-isomorphism with $\tau(I) = I$. By definition of ψ and τ, and by Lemma 3.3.1, τ preserves central carriers. Since

$$I = \tau(P + I - P) = \tau(P) + \tau(I - P),$$

the preceding comments yield $\tau(P) = P$, for each central projection P, whence τ is the identity mapping. Thus

$$D(P_n E) = \tau[D(P_n E)] = C_0^{-1} \psi[D(P_n E)] = C_0^{-1} D'(P_n E'),$$

and $D(P_n F) = C_0^{-1} D'(P_n F')$, so that $D(P_n E) D'(P_n F') = D'(P_n E') D(P_n F)$, as we wished to show.

CHAPTER IV. UNITARY INVARIANTS

The present chapter is devoted to the developments centering about The Unitary Invariants Theorem. The first three sections are concerned with a preliminary spatial investigation of a ring of operators and its commutant. The division into these sections is based on some of the possible combinations of finiteness and pure-infiniteness of the ring and its commutant. Each of the combinations of these sections requires its characteristic techniques. The final section of this chapter welds the previous considerations into our main theorem, The Unitary Invariants Theorem.

4.1. Rings with infinite commutants

Discrete, finite and infinite, cardinal techniques dominate this section, the basic tool being a systematic study of the "coupling character" of central projections as described in the following:

DEFINITION 4.1.1. *A central projection P in a ring of operators $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ is said to have coupling character b (relative to $\mathcal{R}$) when each countably-decomposable, non-zero, central subprojection Q of P is the sum of b orthogonal, equivalent projections in $\mathcal{R}$, each cyclic under $\mathcal{R}'$ and Q cannot be expressed as a sum of fewer than b such projections. The ring $\mathcal{R}$ is said to have coupling character b when I has coupling character b.*

We observe that the above definition assumes within itself the "purity" of the coupling character of a central projection which has a coupling character. That is, any non-zero, central subprojection of a central projection with coupling character b has coupling character b.

LEMMA 4.1.2. *If $\mathcal{R}$ is a ring of operators acting on the Hilbert space $\mathcal{H}$ and $\{Q_\alpha\}$, α in S, is a family of central projections in $\mathcal{R}$ having coupling character b, then their union Q has coupling character b.*

PROOF. We may assume that the subscript family S is well-ordered. In order to show that Q has coupling character b, we must show that each countably-decomposable, central, subprojection P of Q is the sum of b, orthogonal, equivalent, cyclic projections and is not a sum of fewer than b such projections. However, since all the projections involved commute, P is the union of $\{PQ_\alpha\}$; and each PQ_α has coupling character b. It suffices therefore to deal with the case where the center $\mathcal{C}$ of $\mathcal{R}$ is countably-decomposable and to establish, in this case, that Q is a sum of b, orthogonal, equivalent, cyclic projections in $\mathcal{R}$ and is not a sum of fewer than b such projections.

We establish this result first for a monotone-increasing family of central projections $\{P_\alpha\}$, indexed by elements of S, and, in the process, establish this

fact for a countable orthogonal family of central projections. Let G_α be

$$P_\alpha - \cup_{\beta<\alpha}P_\beta .$$

Since the P_α are well-ordered by inclusion, Lemma 1.2 tells us that the G_α are orthogonal and $\cup_\alpha G_\alpha = \cup_\alpha P_\alpha = P$. Since $\mathfrak{C}$ is countably-decomposable, all but a countable number of G_α are 0. Relable the non-zero ones as G_1, G_2, $\cdots$. It follows from the fact that each G_α is contained in P_α, that G_α is 0 or has coupling character b. Let $\{E_{k\gamma}\}$ be a family of b, orthogonal, equivalent, cyclic projections in $\mathfrak{R}$ with sum (over γ) equal to G_k. Then, with $E_\gamma = \sum_k E_{k\gamma}$, we see that $\sum_k G_k(= P)$ is the sum of the b, orthogonal, equivalent projections $\{E_\gamma\}$. That each E_γ is cyclic, follows from its definition and Lemma 3.3.2. If P were the sum of b', orthogonal, equivalent, cyclic projections $\{F_\gamma\}$, then G_k would be the sum of the b', orthogonal, equivalent cyclic projections $\{G_kF_\gamma\}$; and, since G_k has coupling character b, we conclude that $b \leqq b'$. Thus P has coupling character b.

We now make the specific choice $\cup_{\beta\leqq\alpha}Q_\beta$ for P_α. Clearly the first P_α, being the first Q_α, has coupling character b. If each P_α, $\alpha < \beta$, has coupling character b, then, by the above, $\cup_{\alpha<\beta}P_\alpha$, $Q_\beta - Q_\beta(\cup_{\alpha<\beta}P_\alpha)$, and

$$Q_\beta - Q_\beta(\cup_{\alpha<\beta}P_\alpha) + \cup_{\alpha<\beta}P_\alpha(= P_\beta)$$

all have coupling character b. It follows, by transfinite induction, that each P_β has coupling character b, so that, from the above, $\cup_\alpha P_\alpha = \cup_\alpha Q_\alpha = Q$ has coupling character b.

LEMMA 4.1.3. *If $\mathfrak{R}$ is a ring of operators acting on a Hilbert space $\mathfrak{IC}$ of dimension d, we can associate with $\mathfrak{R}$ an orthogonal family of central projections $\{P_b\}$, indexed by the cardinal numbers b not exceeding d, such that P_b is either 0 or has coupling character b and such that $\sum_{b\leqq d} P_b = I$. If $\{Q_b\}$ is a family of central projections in $\mathfrak{R}$ such that Q_b is either 0 or has coupling character b and the union of $\{Q_b\}$ is I, then $Q_b = P_b$, for all $b \leqq d$.*

PROOF. With b a cardinal number not exceeding d, let P_b be the union of all central projections in $\mathfrak{R}$ with coupling character b. (Let P_b be 0 if there are no such central projections.) According to Lemma 4.1.2, P_b is either 0 or has coupling character b. If b and c are distinct cardinal numbers not exceeding d, then $P_bP_c = 0$, since, otherwise, P_bP_c would have coupling character b and c. Let $P = I - \sum_{b\leqq d} P_b$. If P is not 0, the class C of all cardinal numbers c with the property that some non-zero, central subprojection Q of P is the sum of c, orthogonal, equivalent, cyclic projections in $\mathfrak{R}$, is not empty. In fact, if P has a non-zero, abelian subprojection E, let $\{E_\alpha\}$ be a maximal orthogonal family of projections in $\mathfrak{R}$ equivalent to E; so that the E_α are necessarily cyclic and contained in P. (We may assume C_E is countably-decomposable.) Since E is not equivalent to a subprojection of $P - \sum_\alpha E_\alpha$, by maximality of $\{E_\alpha\}$, there is a non-zero, central projection Q in P such that $Q'(P - \sum_\alpha E_\alpha) \prec Q'E$, for each central Q' in Q. However, with E abelian, this can occur only if $Q = \sum_\alpha QE_\alpha$, so that Q is a sum of orthogonal, equivalent, cyclic projections in $\mathfrak{R}$. We assume, henceforth, that P contains no abelian subprojection.

If P contains a non-zero, purely-infinite, central subprojection Q, let F be a non-zero, cyclic projection in Q and let $\{F_\alpha\}$ be a maximal orthogonal family of projections in $\mathfrak{R}$ equivalent to F. As above, we find a non-zero, central projection R in C_F such that $R(Q - \sum_\alpha F_\alpha) \prec RF$. If $\{F_\alpha\}$ is a finite family and F_1 one of its elements, then RF_1 is purely-infinite, and the purely-infinite projection R is the sum of $R(Q - \sum_\alpha F_\alpha)(\prec RF_1)$ and a finite number of copies of RF_1. Thus $R(Q - \sum_\alpha F_\alpha) + RF_1$, being the sum of two countably-decomposable projections, one purely-infinite, is itself countably-decomposable and purely-infinite, and, hence, equivalent to each of the RF_α, by Lemma 3.3.3. Thus R is the sum of the orthogonal family of equivalent, cyclic projections, $\{R(Q - \sum_\alpha F_\alpha) + RF_1, RF_\alpha\}_{\alpha \neq 1}$. On the other hand, if $\{F_\alpha\}$ is an infinite family, let $F_1, F_2, \cdots$ be a countable subfamily, and let E_n be a subprojection of RF_n equivalent to $R(Q - \sum_\alpha F_\alpha)$. Let G_1 denote $R(Q - \sum_\alpha F_\alpha) + RF_1 - E_1$ and G_n be $E_{n-1} + RF_n - E_n$, for $n = 2, 3, \cdots$. Then $\{G_n, RF_\alpha\}_{n=1,2,\cdots;\alpha \neq 1,2,\cdots}$ is a family of orthogonal, cyclic projections equivalent to RF and having sum R. We note especially, for use in the lemma which follows, that we have just proved that if P is a purely-infinite, central projection and F a non-zero cyclic projection in P, then P contains a non-zero, central subprojection R which is the sum of an orthogonal family of projections equivalent to RF.

Finally, with P finite and having no abelian subprojections, and with $\{G_\alpha\}$ a maximal, orthogonal family of projections in $\mathfrak{R}$ equivalent to a cyclic subprojection, G, of P, we can again find a non-zero, central projection Q in C_G such that $Q(P - \sum_\alpha G_\alpha) \prec QG$. Since P is finite, $\{G_\alpha\}$ has a finite number of elements, say n. Employing Theorem 3.2.8, we find that

$$(n + 1)D(QG) \geqq nD(QG) + D[Q(P - \sum_\alpha G_\alpha)] = Q,$$

so that $D(QG) \geqq Q/n + 1$. Theorem 3.2.8 tells us that there is a projection E in P with dimension $Q/n + 1$, that $E \prec QG$—so that E is cyclic, and that Q is the sum of $n + 1$ orthogonal projections equivalent to E.

Thus, in any event, the class C is non-empty, as asserted. Let b be the least cardinal number in C, and let Q be a non-zero, central subprojection of P which is expressible as the sum of b, orthogonal, equivalent, cyclic projections. Clearly, from the minimal nature of b, Q has coupling character b. However, Q is orthogonal to P_b, contrary to the construction of P_b. Thus P is 0, and $\{P_b\}$ is the desired coupling character decomposition of $\mathfrak{R}$.

If $\{Q_b\}$ is a family of central projections with the properties noted in the statement of this lemma, then Q_b is contained in P_b, by construction of P_b. Since the P_b are mutually orthogonal and the Q_b have union I, we conclude that $Q_b = P_b$.

We shall refer to P_b as "the central portion of $\mathfrak{R}$ with coupling character b", and to $\{P_b\}$ as "the coupling character decomposition of $\mathfrak{R}$".

LEMMA 4.1.4. *If $\mathfrak{R}$ is a purely-infinite ring of operators with a countably-decomposable center and coupling character b, then I is the sum of b, orthogonal, maximal, cyclic projections in $\mathfrak{R}$. If $\mathfrak{R}'$ is purely infinite then I is the sum of b, orthogonal, purely-infinite, cyclic projections, and if $\mathfrak{R}'$ is finite then I is the sum*

of b, *orthogonal projections* $\{E_\alpha\}$ *with generating vectors* $\{y_\alpha\}$ *such that* $[\Re y_\alpha] = I$, *for each* α.

PROOF. The last statement of the lemma follows from the first and the nature of the maximal, cyclic projections in $\Re$ when $\Re'$ is as described, by Lemmas 3.3.6, 3.3.5 and [28; Lemma 9.3.3].

To prove the first statement, we note that if b is finite, then I is the sum of a finite number of orthogonal, equivalent, cyclic projections. Since I is purely-infinite in $\Re$, each of these projections is purely-infinite in $\Re$, and, hence, by Lemma 3.3.6, maximal cyclic in $\Re$.

We assume that b is infinite. Let F be a maximal, cyclic projection in $\Re$, the existence of which is guaranteed by Lemma 3.3.7. Let $\{Q_\alpha\}$ be a maximal, orthogonal family of central projections such that Q_α is the sum of an orthogonal family of cyclic projections each equivalent to $Q_\alpha F$. We observe that $\sum_\alpha Q_\alpha = I$. In fact, if $Q = I - \sum_\alpha Q_\alpha \neq 0$, then Q is purely-infinite and the comment in the proof of Lemma 4.1.3 applies, showing that Q has a non-zero, central subprojection R which is the sum of an orthogonal family of projections each equivalent to RF, contradicting the maximality of $\{Q_\alpha\}$.

We show now that each Q_α is the sum of b, orthogonal, cyclic projections equivalent to $Q_\alpha F$. Write R for Q_α and let $\{F_\gamma\}$ be a family of orthogonal, cyclic projections equivalent to RF with sum R. By hypothesis R is the sum of b, orthogonal, equivalent, cyclic projections E_β. Let x_β be a generator for E_β, and let S_β be the family of those γ's for which $F_\gamma x_\beta \neq 0$. Since

$$\sum_\gamma \| F_\gamma x_\beta \|^2 = \| x_\beta \|^2$$

is finite, there are at most a countable number of elements in S_β. On the other hand, if $F_\gamma x_\beta = 0$ then $F_\gamma E_\beta = 0$, and since $0 \neq F_\gamma \leq R$ and $\sum_\beta E_\beta = R$, each γ lies in some S_β. Thus, this standard "invariance of dimension argument" for Hilbert spaces tells us that the cardinality of $\{F_\gamma\}$ does not exceed $\aleph_0 b = b$. By definition of "coupling character", however, the cardinality of $\{F_\gamma\}$ is not less than b, from which it follows that R is the sum of b, orthogonal, cyclic projections equivalent to RF.

Let $Q_\alpha = \sum_\gamma F_{\gamma\alpha}$, with $\{F_{\gamma\alpha}\}$ an orthogonal family consisting of b, cyclic projections equivalent to $Q_\alpha F$, and let $F_\gamma = \sum_\alpha F_{\gamma\alpha}$. Then

$$\sum_\gamma F_\gamma = \sum_\alpha Q_\alpha = I,$$

and F_γ is equivalent to F, which concludes the proof.

LEMMA 4.1.5. *If* $\Re$ *is a ring of operators with a countably-decomposable center,* $\{P_b\}$ *its coupling character decomposition, and* S *the set of ordinal numbers less than the initial ordinal whose cardinal is not less than each cardinal* c *with the property that* P_c *is non-zero, then* $\Re$ *possesses an orthogonal family of cyclic projections* $\{E_\alpha\}$, *indexed by elements of* S, *such that* $\sum_\alpha E_\alpha = I$ *and* $C_{E_\beta} \leq C_{E_\alpha}$ *if* $\alpha \leq \beta$. *If* $\Re$ *and* $\Re'$ *are purely-infinite, then each* E_α *can be chosen so as to be purely-infinite.*

PROOF. Let $\{E_{b\alpha}\}$ be an orthogonal family of cyclic projections, indexed by elements of S, equivalent for α less than the initial ordinal with cardinal b, zero

for other ordinals, and having sum P_b. Since the center of $\mathfrak{R}$ is countably-de-composable, there are at most a countable number of non-zero P_b; so that, by Lemma 3.3.2, $E_\alpha = \sum_b E_{b\alpha}$ is cyclic. Clearly, $\sum_\alpha E_\alpha = \sum_b P_b = I$, and the family $\{E_\alpha\}$ is orthogonal. We can explicitly determine the central carrier of E_α from the fact that $C_{E_{b\alpha}}$ is P_b, provided $E_{b\alpha}$ is not 0. In fact, if the cardinal of α is c then $C_{E_\alpha} = \sum_{c<b} P_b$, from which $C_{E_\beta} \leqq C_{E_\alpha}$ if $\alpha \leqq \beta$. The last state-ment of the lemma follows from the fact that the non-zero $E_{b\alpha}$ can be chosen purely-infinite, according to Lemma 4.1.4, with $\mathfrak{R}$ and $\mathfrak{R}'$ purely-infinite.

We refer to $\{E_\alpha\}$ as a descending carrier decomposition of $\mathfrak{R}$.

LEMMA 4.1.6. *A *-isomorphism ϕ between two rings of operators $\mathfrak{R}_1$ and $\mathfrak{R}_2$ acting on Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, each of which has a generating vector, is unitarily implemented if and only if ϕ preserves maximal cyclicity of projections.*

PROOF. Let x be a generating vector for I under $\mathfrak{R}_1$; let $E_1 = [\mathfrak{R}_1'x]$, so that E_1 is a maximal cyclic projection in $\mathfrak{R}_1$ and, hence, $\phi(E_1)$ is maximal cyclic in $\mathfrak{R}_2$; and let y be a generating vector for $\phi(E_1)$ under $\mathfrak{R}_2'$, so that $[\mathfrak{R}_2y]$ is maxi-mal cyclic in $\mathfrak{R}_2'$, hence equivalent to I, and, as in Lemma 3.3.4, y can be chosen such that $[\mathfrak{R}_2y] = \mathfrak{K}_2$. Choose, by Lemma 4.1.5, a descending carrier decom-position $\{E_\alpha\}$ for $(I - E_1)\mathfrak{R}_1(I - E_1)$ acting on $(I - E_1)(\mathfrak{K}_1)$ (any cyclic de-composition will do). Clearly, then, $\{E_1, E_\alpha\}$ is a descending carrier decomposi-tion for $\mathfrak{R}_1$, and, since E_1 is maximal cyclic, $E_\alpha \precsim E_1$ for each α. Note, in this connection, the countable-decomposability of the centers of $\mathfrak{R}_1$ and $\mathfrak{R}_2$, since $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ possess the separating vectors x and y, respectively, by Lemma 3.3.1. The same situation holds relative to $\mathfrak{R}_2$ and the projections $\phi(E_1)$, $\phi(E_\alpha)$. Let V_α be a partial isometry in $\mathfrak{R}_1$ with initial space G_α in E_1 and final space E_α. Take $V_1 = E_1$. Now $E_1\mathfrak{R}_1E_1$, $\mathfrak{R}_1'E_1$ and $\phi(E_1)\mathfrak{R}_2\phi(E_1)$, $\mathfrak{R}_2'\phi(E_1)$ have joint generating vectors x and y, respectively. Thus, by The Unitary Implementation Theorem, the isomorphism ϕ restricted to $E_1\mathfrak{R}_1E_1$ is implemented by a unitary transformation U_1 of $E_1(\mathfrak{K}_1)$ onto $\phi(E_1)(\mathfrak{K}_2)$. Let $U_\alpha = \phi(V_\alpha)U_1V_\alpha^*$, so that U_α is a unitary transformation of E_α onto $\phi(E_\alpha)$; and let U be the unitary trans-formation of $\mathfrak{K}_1$ onto $\mathfrak{K}_2$ defined as U_α on each E_α. Note that

$$U_1 = \phi(E_1)U_1E_1 = U_1,$$

so that there is no conflict between the notation U_α and the original choice of U_1. Because of this original choice, U implements ϕ on $E_1\mathfrak{R}_1E_1$. Moreover,

$$UV_\alpha U^{-1} = UV_\alpha U_1^{-1} = U_\alpha V_\alpha U_1^{-1} = \phi(V_\alpha)U_1V_\alpha^*V_\alpha U_1^{-1}$$

$$= \phi(V_\alpha)\phi(G_\alpha) = \phi(V_\alpha G_\alpha) = \phi(V_\alpha).$$

Thus U implements ϕ restricted to the subring of $\mathfrak{R}_1$ generated by $E_1\mathfrak{R}_1E_1$ and the V_α. However, this ring is all of $\mathfrak{R}_1$, for $A = (\sum_\alpha E_\alpha)A(\sum_\alpha E_\alpha)$ and we have to show merely that $E_\alpha A E_{\alpha'}$ is in the generated subring. But,

$$E_\alpha A E_{\alpha'} = V_\alpha V_\alpha^* A V_{\alpha'} V_{\alpha'}^* = V_\alpha E_1 V_\alpha^* A V_{\alpha'} E_1 V_{\alpha'}^*.$$

It follows that U implements ϕ on $\mathfrak{R}_1$, and the proof is complete.

LEMMA 4.1.7. *If ϕ is a *-isomorphism of one ring of operators $\mathcal{R}_1$ onto another $\mathcal{R}_2$ and their respective commutants $\mathcal{R}_1'$, $\mathcal{R}_2'$ are each purely-infinite with coupling character b, then ϕ is induced by a unitary transformation of $\mathcal{K}_1$, the Hilbert space upon which $\mathcal{R}_1$ acts, onto $\mathcal{K}_2$, the Hilbert space upon which $\mathcal{R}_2$ acts.*

PROOF. Note first that, since ϕ maps $\mathcal{C}_1$, the center of $\mathcal{R}_1$, isomorphically upon $\mathcal{C}_2$, the center of $\mathcal{R}_2$, ϕ carries a maximal orthogonal family of countably-decomposable projections in $\mathcal{C}_1$ onto such a family in $\mathcal{C}_2$. Since $\mathcal{R}_1'$ and $\mathcal{R}_2'$ restricted to countably-decomposable central projections are purely-infinite with coupling character b, it suffices to deal with the case where $\mathcal{C}_1$ and $\mathcal{C}_2$ are countably-decomposable. We assume this throughout the remainder of the proof.

Let P be the central projection in $\mathcal{R}_1$ such that $\mathcal{R}_1 P$ is finite and $\mathcal{R}_1(I - P)$ is purely-infinite. According to Lemma 4.1.4, P is the sum of b, orthogonal, equivalent, cyclic projections $\{E_\alpha'\}$ in $\mathcal{R}_1' P$ such that if x_α is a generator for E_α' then $[\mathcal{R}_1' P x_\alpha] = P$; and $I - P$ is the sum of b, orthogonal, equivalent, purely-infinite, cyclic projections $\{F_\alpha'\}$ in $\mathcal{R}_1'(I - P)$. Now $\phi(P)$ plays the same role with respect to $\mathcal{R}_2$ as P does with respect to $\mathcal{R}_1$, so that we can find orthogonal families of projections $\{M_\alpha'\}$ and $\{N_\alpha'\}$ in $\mathcal{R}_2' \phi(P)$ and $\mathcal{R}_2'(I - \phi(P))$, respectively, with properties corresponding to those of $\{E_\alpha'\}$ and $\{F_\alpha'\}$, respectively.

From Lemma 3.1.3, we see that the mapping taking $A_1 P E_\alpha' = A_1 E_\alpha'$ in

$$\mathcal{R}_1 P E_\alpha' = \mathcal{R}_1 E_\alpha'$$

onto $\phi(A_1)\phi(P)M_\alpha' = \phi(A_1)M_\alpha'$ in $\mathcal{R}_2\phi(P)M_\alpha' = \mathcal{R}_2 M_\alpha'$ is an isomorphism. Now if x_α is a generator for E_α', we see, from the properties of E_α', that x_α is a joint cyclic vector for $\mathcal{R}_1 E_\alpha'$ and $E_\alpha' \mathcal{R}_1' E_\alpha'$ acting on $E_\alpha'(\mathcal{K}_1)$. Similarly $\mathcal{R}_2 M_\alpha'$ and

$$M_\alpha' \mathcal{R}_2' M_\alpha'$$

possess a joint cyclic vector. Thus, according to The Unitary Implementation Theorem, the mapping of $\mathcal{R}_1 E_\alpha'$ onto $\mathcal{R}_2 M_\alpha'$, described above, is implemented by a unitary transformation U_α of $E_\alpha'(\mathcal{K}_1)$ onto $M_\alpha'(\mathcal{K}_2)$. The unitary transformation of $P(\mathcal{K}_1)$ onto $\phi(P)(\mathcal{K}_2)$ defined as U_α on each $E_\alpha'(\mathcal{K}_1)$, clearly implements ϕ restricted to $\mathcal{R}_1 P$ (onto $\mathcal{R}_2\phi(P)$).

It remains to show that ϕ restricted to $\mathcal{R}_1(I - P)$ is implemented by a unitary transformation. There is, of course, no loss in generality in assuming that $P = 0$, in view of the above. We do so in the interest of simpler notation, so that $\mathcal{R}_1$, $\mathcal{R}_1'$, $\mathcal{R}_2$ and $\mathcal{R}_2'$ are assumed to be purely-infinite, $\sum_\alpha F_\alpha' = I$, and $\sum_\alpha N_\alpha' = I$. As above, the map of $\mathcal{R}_1 F_\alpha'$ onto $\mathcal{R}_2 N_\alpha'$ which carries $A_1 F_\alpha'$ on $\phi(A_1)N_\alpha'$ is an isomorphism and it suffices to show that these maps are implemented by unitary transformations of $F_\alpha'(\mathcal{K}_1)$ onto $N_\alpha'(\mathcal{K}_2)$ We may therefore assume that $F_\alpha' = I$ and $N_\alpha' = I$. (Observe that, since F_α' and N_α' are purely-infinite, $\mathcal{R}_1 F_\alpha'$, $F_\alpha' \mathcal{R}_1' F_\alpha'$, $\mathcal{R}_2 N_\alpha'$ and $N_\alpha' \mathcal{R}_2' N_\alpha'$ are purely-infinite.) Thus $\mathcal{R}_1$, $\mathcal{R}_1'$, $\mathcal{R}_2$ and $\mathcal{R}_2'$ are purely-infinite, and $\mathcal{R}_1$, $\mathcal{R}_2$ have generating vectors. It follows, from Lemma 3.3.6, that the maximal cyclic projections in $\mathcal{R}_1$ and $\mathcal{R}_2$ are precisely the countably-decomposable, purely-infinite projections in these rings. Thus

ϕ preserves maximal cyclicity and is unitarily implemented, according to Lemma 4.1.6, which completes the proof.

4.2. Finite rings with finite commutants

The connectedness of the real line is basic to the analysis of the present section, which draws heavily upon the dimension theory for finite rings developed in §3.2. The following definition plays a role analogous to Definition 4.1.1, for this continuous situation.

DEFINITION 4.2.1. *We define the upper coupling number of a countably-decomposable, central projection P in a finite ring of operators $\Re$ with finite commutant $\Re'$ to be $1/L(E')$, if $E' \neq P$, where E' is a maximal, cyclic projection in $\Re'P$, $L(E')$ is taken relative to $\Re'P$, and $1/L(E')$ is understood to be ∞ if $L(E') = 0$; and we define the upper coupling number of P to be $u(E)$, if $E' = P$, where E is a maximal, cyclic projection in $\Re P$. For an arbitrary central projection Q in $\Re$ we define the upper coupling number to be the supremum of the upper coupling numbers of the countably-decomposable, central projections contained in Q. We call the upper coupling number of I the upper coupling number of $\Re$, $\Re'$ and denote it by $(\Re, \Re')$.*

Regarding the above definition, note that $(\Re Q, \Re'Q) \leqq (\Re, \Re')$, since

$$L(QE') \geqq L(E')$$

and $u(QE) \leqq u(E)$. From which it follows that no ambiguity arises in determining the upper coupling number of a countably-decomposable central projection directly or by taking a supremum. Note also that the upper coupling number as defined is independent of the maximal cyclic projections chosen since all maximal cyclic projections in a ring with countably-decomposable center are equivalent.

We have used the term "coupling character" in §4.1 and "upper coupling number" in the above definition. The terminology suggests a relation between the two concepts which does, indeed, exist. The discreteness of the situation studied in §4.1 allows us to consider only "pure" coupling characters while the continuousness of the present situation requires us to consider central portions of rings corresponding to "intervals of coupling numbers" (in analogy with the relation between the studies of the spectral decompositions of operators with discrete, pure-point spectrum and those with continuous spectrum). We could define "coupling number" in the present case, e.g., by the purity of the upper coupling number. Having done this, it would follow easily from the dimension theory of §3.2 that a projection with a certain coupling number, in the present sense, had this number as coupling character.

LEMMA 4.2.2. *If $\{P_\alpha\}$ is a family of central projections, with union P, in a finite ring of operators $\Re$ with finite commutant $\Re'$ and each P_α has an upper coupling number not exceeding r then the upper coupling number of P does not exceed r. There is a family $\{Q_r\}$ of central projections in $\Re$, with union I, and such that Q_r has upper coupling number not exceeding r (r finite) and contains each central*

projection with upper coupling number not exceeding r. *If* $\{R_r\}$ *is a family of central projections such that* R_r *has the properties listed for* Q_r *then* $R_r = Q_r$.

PROOF. To establish the first statement of this lemma, we must show that $(\Re QP, \Re'QP) \leqq r$ for each countably-decomposable, central projection Q in $\Re$. All projections in question commuting, we have that PQ is the union of $P_\alpha Q$. It suffices, therefore, to assume that P is countably-decomposable (and hence all P_α are countably-decomposable) and to show that $(\Re P, \Re'P) \leqq r$. For notational simplification, we may just as well assume that $P = I$.

If $r = \infty$, there is nothing to prove, so that we may assume that r is finite. Let E and E' be maximal, cyclic projections in $\Re$ and $\Re'$, respectively. Note that since $\bigcup_\alpha P_\alpha = I$, $X = (\bigcup_\alpha S_\alpha)^-$, where X is the pure state space of the center of $\Re$ and $S_\alpha = \{x : x \text{ in } X, P_\alpha(x) = 1\}$. In fact, each S_α being open and X being extremely disconnected, $X - (\bigcup_\alpha S_\alpha)^-$ is a clopen subset of X whose characteristic function corresponds to a central projection orthogonal to each P_α and hence to I, [47]. If $E' = I$ then

$$(\Re, \Re') = u(E) = \sup \{D(E)(x) : x \text{ in } X\},$$

and $P_\alpha E' = P_\alpha$, so that $r \geqq (\Re P_\alpha, \Re' P_\alpha) = \sup \{D(E)(x) : x \text{ in } S_\alpha\}$, by Lemma 3.2.7. Thus the closed set, $\{x : D(E)(x) \leqq r\}$, contains $\bigcup_\alpha S_\alpha$ and is therefore X; whence $(\Re, \Re') \leqq r$. If $E' \neq I$ then

$$(\Re, \Re') = 1/L(E') = 1/\inf \{D'(E')(x) : x \text{ in } X\} = \sup \{1/D'(E')(x) : x \text{ in } X\}.$$

We wish to show that $r \geqq (\Re, \Re')$, or that $D'(E') \geqq 1/r$. From the above, it suffices, of course, to show that $D'(E')(x) \geqq 1/r$ for x in S_α, for all α. We observe that, for some α', $P_{\alpha'}E' \neq P_{\alpha'}$, since $E' \neq I$ and $\bigcup_\alpha P_\alpha = I$. Thus

$$r \geqq (\Re P_{\alpha'}, \Re' P_{\alpha'}) = 1/L(P_{\alpha'}E') \geqq 1$$

and $1 \geqq 1/r$. Now, if x is in such an $S_{\alpha'}$, from the remarks above,

$$r \geqq 1/D'(E')(x) \qquad \text{or} \qquad D'(E')(x) \geqq 1/r.$$

On the other hand, if x is in S_α and $P_\alpha E' = P_\alpha$ then

$$D'(E')(x) = D'(P_\alpha E')(x) = P_\alpha(x) = 1 \geqq 1/r.$$

Thus, in any event, $(\Re, \Re') \leqq r$, and the upper coupling number of P does not exceed r.

For Q_r we take the union of all central projections in $\Re$ with upper coupling number not exceeding r. From the foregoing, Q_r clearly has the stated properties. Moreover, if R_r is another central projection with the properties stated for Q_r then Q_r contains R_r and R_r contains Q_r so that $Q_r = R_r$.

Finally, we show that $\bigcup_{r<\infty} Q_r = I$. Indeed, if this is not the case, then, with $Q = I - \bigcup_r Q_r$, there is no central subprojection of Q with finite upper coupling number. Thus, if F' is a maximal cyclic projection in $\Re'Q$, we have, from Definition 4.2.1, that $L(RF') = 0$, relative to $\Re'R$, for each non-zero, central subprojection R of Q. Using Lemma 3.2.7, it follows that $D'(F')$ takes the value 0

in each non-empty clopen subset of the pure state space of the center of $\mathfrak{R}Q$. The continuity of $D'(F')$ and the extreme disconnectedness of this pure state space imply that $D'(F') = 0$, whence $F' = 0$. Thus $Q = 0$, and the proof is complete.

We shall refer to the projection Q_r, constructed above, as the central portion of $\mathfrak{R}$, $\mathfrak{R}'$ for upper coupling number r.

LEMMA 4.2.3. *A projection E in a finite ring $\mathfrak{R}$ with finite commutant $\mathfrak{R}'$ is cyclic if and only if C_E is countably-decomposable and $u(PE) \leqq (\mathfrak{R}P, \mathfrak{R}'P)$ for each non-zero central projection P in $\mathfrak{R}$.*

PROOF. If E is cyclic then, according to Lemma 3.3.1, C_E is countably-decomposable. In any case, $u(PE) = u(PC_E E)$ and

$$(\mathfrak{R}PC_E, \mathfrak{R}'PC_E) \leqq (\mathfrak{R}P, \mathfrak{R}'P),$$

so that, working relative to $\mathfrak{R}C_E$ and $\mathfrak{R}'C_E$, we see that it suffices to deal with the case where $\mathfrak{R}$ has a countably-decomposable center. We make this assumption.

Let F and F' be maximal cyclic projections in $\mathfrak{R}$ and $\mathfrak{R}'$, respectively. If E is cyclic then $E \precsim F$, so that $u(PE) \leqq u(PF) \leqq (\mathfrak{R}P, \mathfrak{R}'P)$. If $(\mathfrak{R}, \mathfrak{R}') < 1$ then $F' = I$, so that $u(F) = (\mathfrak{R}, \mathfrak{R}')$; and, if $(\mathfrak{R}, \mathfrak{R}') \geqq 1$, then, either $F' = I$, and $1 = (\mathfrak{R}, \mathfrak{R}') = u(F)$, or else $F' \neq I$, in which case there is, by Lemmas 3.3.5 and 3.3.6, a non-zero central projection Q such that $QF = Q$; so that

$$1 \geqq u(F) \geqq u(Q) = 1.$$

Applying this last comment to the case where $\mathfrak{R}P$, $\mathfrak{R}'P$, PF, and PF' replace $\mathfrak{R}$, $\mathfrak{R}'$, F, and F', respectively, we see that the hypothesis $u(PE) \leqq (\mathfrak{R}P, \mathfrak{R}'P)$ entails $u(PE) \leqq u(PF)$; for if $(\mathfrak{R}P, \mathfrak{R}'P) < 1$ then

$$u(PF) = (\mathfrak{R}P, \mathfrak{R}'P) \geqq u(PE),$$

and if $(\mathfrak{R}P, \mathfrak{R}'P) \geqq 1$ then $1 = u(PF) \geqq u(PE)$. The continuity of $D(E)$ and $D(F)$ on the pure state space X of the center of $\mathfrak{R}$, the extreme disconnectedness of X, and Lemma 3.2.7, show us now that $D(E) \leqq D(F)$, whence $E \precsim F$, by (c) of Theorem 3.2.8, and E is cyclic.

LEMMA 4.2.4. *If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are finite rings of operators with finite commutants $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$, respectively, acting on Hilbert spaces $\mathfrak{IC}_1$ and $\mathfrak{IC}_2$, respectively, and if ϕ_0 is a *-isomorphism of the subalgebra $\bigcup_{r<\infty}\mathfrak{R}_1 P_r$ of $\mathfrak{R}_1$ onto the subalgebra*

$$\bigcup_{r<\infty}\mathfrak{R}_2 Q_r,$$

*where P_r and Q_r are the central portions of $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively, for upper coupling number r, which carries P_r onto Q_r, then ϕ_0 has a *-isomorphic extension mapping $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ which is unitarily implemented.*

PROOF. We define the extension ϕ of ϕ_0 by means of the limit: $\phi(A) = $ strong limit$_{r\to\infty} \phi_0(AP_r)$ for A in $\mathfrak{R}_1$. We must show, of course, that this limit exists. This will have been established when we have shown that $\lim_{r\to\infty} \phi_0(AP_r)x$ exists for each vector x in $\mathfrak{IC}_2$. Let $\varepsilon > 0$ be given. According to Lemma 4.2.2,

Q_r approaches I strongly, so that we can choose t such that

$$\| (I - Q_t)x \| \leq \varepsilon/2 \| A \| .$$

If r and r' are greater than t, then

$$\| \phi_0(AP_r)x - \phi_0(AP_{r'})x \| \leq \| \phi_0[A(P_r - P_{r'})](x - Q_t x)\|$$
$$+ \| \phi_0[A(P_r - P_{r'})]Q_t x \| \leq 2 \| A \| \varepsilon/2 \| A \|$$
$$+ \| \phi_0[A(P_r - P_{r'})P_t]x \| = \varepsilon.$$

Thus $\phi_0(AP_r)x$ is Cauchy and has a limit. Since $\mathfrak{R}_2$ is strongly closed, $\phi(A)$ lies in $\mathfrak{R}_2$.

Applying the foregoing with ϕ_0^{-1} in place of ϕ_0, we see that $\phi_0^{-1}(BQ_t)$ has a strong limit, which we shall denote by $\psi(B)$, for each B in $\mathfrak{R}_2$. Now $\psi(B)P_r =$ strong limit$_{t \to \infty} \phi_0^{-1}(BQ_t)\phi_0^{-1}(Q_r) = \phi_0^{-1}(BQ_r)$, so that $\phi_0(\psi(B)P_r) = BQ_r$ and $\phi(\psi(B)) = B$. Thus ϕ maps $\mathfrak{R}_1$ onto $\mathfrak{R}_2$. Clearly, ϕ restricted to $\mathfrak{R}_1 P_r$ is ϕ_0. It is a simple matter to show that ϕ is a *-isomorphism. We verify that

$$\phi(AB) = \phi(A)\phi(B)$$

and that $A = 0$ if $\phi(A) = 0$. In fact,

$$\phi(AB) = \text{strong limit } \phi_0(ABP_r) = \text{strong limit } \phi_0(AP_r)\phi_0(BP_r)$$
$$= \text{strong limit } \phi_0(AP_r) \text{ strong limit } \phi_0(BP_r) = \phi(A)\phi(B).$$

If $\phi(A) = 0$ then $0 = \phi(A)\phi(P_r) = \phi(AP_r) = \phi_0(AP_r)$, so that $AP_r = 0$, for all r. Since P_r tends strongly to I, $A = 0$.

As a consequence of the fact that $\phi(P_r) = Q_r$, we have that P and $\phi(P)$ have the same upper coupling number, for each central projection P in $\mathfrak{R}_1$. Indeed, if P has upper coupling number r, with r finite, then $P \leq P_r$, so that

$$\phi(P) \leq Q_r .$$

Thus the upper coupling number of $\phi(P)$, t, does not exceed r. Applying this to $P = \phi^{-1}\phi(P)$, we see that $r \leq t$, so that $r = t$. Thus, if the upper coupling number of P or $\phi(P)$ is ∞ then the other must be ∞. Since ϕ preserves the upper dimension of projections and the upper coupling number of central projections, it follows from Lemma 4.2.3 that ϕ and ϕ^{-1} map cyclic projections onto cyclic projections.

Restricting ϕ and the rings $\mathfrak{R}_1$, $\mathfrak{R}_1'$ to a countably-decomposable, central projection does not alter the hypotheses, whence we may assume that the centers of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are countably-decomposable and, hence, by Lemma 3.3.5, that $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_1'$, and $\mathfrak{R}_2'$ are themselves countably-decomposable. This being assumed, it follows from the definition of "upper coupling number of a central projection" that P_1 and Q_1 are cyclic under $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively. Moreover, $\phi(P_1) = Q_1$, so that ϕ maps $\mathfrak{R}_1 P_1$ isomorphically upon $\mathfrak{R}_2 Q_1$. Clearly, this restriction of ϕ inherits the property of preserving maximal cyclicity. Thus, by Lemma 4.1.6, the restriction of ϕ to $\mathfrak{R}_1 P_1$ is unitarily implemented. It remains then to show that the restriction of ϕ to $\mathfrak{R}_1(I - P_1)$ is unitarily implemented,

or, what amounts to the same thing, to prove our result under the assumption that P_1 and Q_1 are zero. We make this assumption and observe that, in consequence, I is cyclic under $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$, for otherwise, by Lemma 3.3.6, there is a central projection P which is non-zero and cyclic under $\mathfrak{R}_1$. From Definition 4.2.1, then, P would have upper coupling number not exceeding 1 and be contained in P_1.

We note next that if E and E' are cyclic projections in $\mathfrak{R}_1$ and $\mathfrak{R}_1'$, respectively, which have a joint generating vector, and if G' is a projection in $\mathfrak{R}_2'$, then $\phi(E)$ and G' have a joint generating vector if and only if

$$\phi[D_1'(E')] = D_2'(G').$$

Indeed, if there is a joint generating vector for $\phi(E)$ and G', and if F_1', F_2' are maximal cyclic projections in $\mathfrak{R}_1'$, $\mathfrak{R}_2'$, respectively, so that the pairs I, F_1' and I, F_2' have joint generating vectors under $\mathfrak{R}_1$, $\mathfrak{R}_1'$ and $\mathfrak{R}_2$, $\mathfrak{R}_2'$, respectively, then by The Coupling Theorem (3.3.8),

$$D_1(E)D_1'(F_1') = D_1'(E')D_1(I) = D_1'(E')$$

and

$$D_2(\phi(E))D_2'(F_2') = D_2'(G')D_2(I) = D_2'(G').$$

Thus

$$\phi(D_1'(E')) = \phi(D_1(E))\phi(D_1'(F_1')) = D_2(\phi(E))\phi(D_1'(F_1')),$$

and our task is to prove that $D_2'(F_2') = \phi(D_1'(F_1'))$. Now, if P is a non-zero, central projection in $\mathfrak{R}_1$, the assumption that P_1 and Q_1 are zero implies, as in the foregoing paragraph, that $PF_1' \neq P$ and that $\phi(P)F_2' \neq \phi(P)$, so that, relative to $\mathfrak{R}_1'P$ and $\mathfrak{R}_2'\phi(P)$, we have

$$(\mathfrak{R}_1P, \mathfrak{R}_1'P) = 1/L(PF_1') = 1/L(\phi(P)F_2') = (\mathfrak{R}_2\phi(P), \mathfrak{R}_2'\phi(P)).$$

Thus, viewed as functions on the pure state space of the center of $\mathfrak{R}_2$, $\phi(D_1'(F_1'))$ and $D_2'(F_2')$ have the same infimum over each clopen set in this space, by Lemma 3.2.7. Hence, by the continuity of $\phi(D_1'(F_1'))$ and $D_2'(F_2')$, and the extreme disconnectedness of this pure state space, $\phi(D_1'(F_1')) = D_2'(F_2')$, and $\phi(D_1'(E')) = D_2'(G')$ as we wished to show.

If we assume that $\phi(D_1'(E')) = D_2'(G')$ and that y is a generating vector for $\phi(E)$, and $G'' = [\mathfrak{R}_2 y]$, then, from the preceding paragraph,

$$D_2'(G'') = \phi[D_1'(E')] = D_2'(G').$$

Thus G' and G'' are equivalent in $\mathfrak{R}_2'$. Let V' be a partial isometry in $\mathfrak{R}_2'$ with initial space G'' and final space G', and let $y' = V'y$. Then

$$\phi(E) = [\mathfrak{R}_2'y] = [\mathfrak{R}_2'G''y] = [\mathfrak{R}_2'V'^*V'y] \subset [\mathfrak{R}_2'V'y] = [\mathfrak{R}_2'y'] \subset [\mathfrak{R}_2'y] = \phi(E),$$

so that $[\mathfrak{R}_2'y'] = \phi(E)$; and

$$[\mathfrak{R}_2 y'] = [\mathfrak{R}_2 V'y] = V'[\mathfrak{R}_2 y] = G',$$

which establishes the existence of a joint generating vector for $\phi(E)$ and G'.

Since $\mathcal{R}_1'$ is countably-decomposable, we can choose a sequence $\{E_i'\}$ of mutually-orthogonal, cyclic projections with sum I in $\mathcal{R}_1'$. Let y_i be a generating vector for E_i', and suppose that we have chosen projections $F_1', \cdots, F_{n-1}'$ in $\mathcal{R}_2'$ which are mutually-orthogonal and such that $D_2'(F_i') = \phi(D_1'(E_i'))$, $i = 1, \cdots, n-1$. In this case,

$$D_2'(I - \textstyle\sum_{i=1}^{n-1} F_i') = \phi(D_1'(I - \textstyle\sum_{i=1}^{n-1} E_i')) \geqq \phi(D_1'(E_n')),$$

so that $I - \sum_{i=1}^{n-1} F_i'$ contains a projection F_n' with $D_2'(F_n') = \phi(D_1'(E_n'))$. This follows from (e) of Theorem 3.2.8 in the type II case, and in the "mixed" type II, type I case, we observe that our information on the relation of the "couplings" of $\mathcal{R}_1$, $\mathcal{R}_1'$ and $\mathcal{R}_2$, $\mathcal{R}_2'$, and the fact that ϕ is an isomorphism between the centers of $\mathcal{R}_1$ and $\mathcal{R}_2$, assures us that $\phi(D_1'(E_n'))$ is of the appropriate form to be the dimension of some projection in $\mathcal{R}_2'$, which projection can then be chosen in $I - \sum_{i=1}^{n-1} F_i'$. Note that the above argument establishes the existence of F_1'. Thus we can find an orthogonal family of projections $\{F_i'\}$, $i = 1, 2, \cdots$ in $\mathcal{R}_2'$ such that $D_2'(F_i') = \phi(D_1'(E_i'))$, so that

$$D_2'(\textstyle\sum_{i=1}^{\infty} F_i') = \textstyle\sum_{i=1}^{\infty} D_2'(F_i') = \textstyle\sum_{i=1}^{\infty} \phi(D_1'(E_i')) = \phi[\textstyle\sum_{i=1}^{\infty} D_1'(E_i')]$$

$$= \phi[D_1'(\textstyle\sum_{i=1}^{\infty} E_i')] = \phi(D_1'(I)) = \phi(I) = I,$$

from which $\sum_{i=1}^{\infty} F_i' = I$.

If we let $E_i = [\mathcal{R}_1' y_i]$, the criterion we established above assures us that $\phi(E_i)$ and F_i' have a joint generating vector. Let PE_i' be a central projection in $\mathcal{R}_1 E_i'$, with P central in $\mathcal{R}_1$, so that PE_i' is a cyclic central projection in $E_i' \mathcal{R}_1 E_i'$. Thus the upper coupling number of PE_i' is $u(PE_i E_i')$, since $PE_i E_i'$ is a maximal cyclic projection in $\mathcal{R}_1 PE_i'$. Similarly, the upper coupling number of $\phi(P)F_i'$ in $\mathcal{R}_2 F_i'$ is $u(\phi(P)\phi(E_i)F_i') = u(\phi(PE_i)F_i')$. Now $\phi(C_{E_i'}) = \phi(C_{E_i}) = C_{\phi(E_i)} = C_{F_i'}$, so that, by Lemma 3.1.3, the mapping ϕ_i defined by $\phi_i(AE_i') = \phi(A)F_i'$ is an isomorphism of $\mathcal{R}_1 E_i'$ onto $\mathcal{R}_2 F_i'$. Thus $u(PE_i E_i') = u(\phi(PE_i)F_i')$, and ϕ_i preserves the upper coupling numbers of central projections and hence, as before, maximal cyclicity of projections. Thus, by Lemma 4.1.6, each ϕ_i is implemented by a unitary transformation U_i of $E_i'(\mathcal{H}_1)$ onto $F_i'(\mathcal{H}_2)$. The unitary transformation of $\mathcal{H}_1$ onto $\mathcal{H}_2$ defined as U_i on each $E_i'(\mathcal{H}_1)$ implements ϕ, and the proof is complete.

4.3. Infinite rings with finite commutants

The case now under consideration presents a new type of difficulty and requires, for its final description, constructs which we have not as yet encountered. The essentially new feature of the present situation is the failure of a certain matching or normalization to take place algebraically or under simple numerical restrictions. The basic manageable situation is that of an isomorphism between rings which have, together with their commutants, joint generating vectors. In each case, of course, we must keep track of what the isomorphism does to those elements of the center which are distinguished by the rôle they play rela-

tive to the commutant. Once this is done, our problem becomes that of reducing the factor case to the joint generating vector situation.

If we are dealing with a factor having an infinite commutant, we effect this reduction by taking a maximal, orthogonal family of cyclic projections in the commutant (having the smallest cardinality for such families). If the factor itself is finite then the commutant possesses a generating vector and the reduction is effected. If the factor is infinite, one must continue with a choice of a maximal, orthogonal family of maximal, cyclic projections in the factor. Now, as Lemma 3.3.6 tells us, the maximal, cyclic projections in an infinite factor whose commutant is infinite are precisely the countably-decomposable infinite projections. Since countable-decomposability and infiniteness are preserved by *-isomorphisms, the family of maximal, cyclic projections chosen is mapped into another such family. Thus, in any event, the infinite commutant situation requires only the minimal cardinality of the orthogonal, cyclic family in the commutant as invariant, to guarantee that the isomorphism between the original factors can be unitarily implemented.

If we are dealing with finite factors having finite commutants, the dimension of the maximal, cyclic projections (either in the factors or their commutants—whichever projections happen to differ from I) must be the same for the isomorphism to be unitarily implemented. If the maximal, cyclic projections in the commutants happen to differ from I, then we are in the case analogous to that of a factor with an infinite commutant, and the dimension of the maximal, cyclic projection plays the same role as the coupling character. If the maximal, cyclic projections in the factors themselves differ from I, then the situation is analogous to the case of an infinite factor with a finite commutant. However, with both the factor and its commutant finite, the maximal, cyclic projections in the factor can be singled out by means of one number (their common dimension). This number, then, serves as the invariant which determines when the isomorphism is unitarily implemented.

In the case of infinite factors with finite commutants, however, the situation is considerably changed. The first reduction is not necessary, since the infinite factors have generating vectors. On the other hand, the second reduction, which necessitates the choosing of a maximal family of maximal, cyclic projections, leads to the difficulty that we can no longer describe the maximal, cyclic projections algebraically or in terms of invariantly associated numerical constants, and, hence, cannot guarantee, in a simple manner, that the isomorphism between the infinite factors carries maximal, cyclic projections onto maximal, cyclic projections, or even that an automorphism of such a factor preserves maximal cyclicity. Indeed, it is shown in [18] that this need not be the case. In a certain sense, the present case, while the most uncomfortable, from a technical viewpoint, is the most natural. Here, the maximal, cyclic projections do not enjoy the special algebraic and numerical privileges which they do in the other cases.

We note that in the case of an infinite factor of type I with a finite com-

mutant, the maximal, cyclic projections can be algebraically characterized as those projections which are the sum of a certain fixed number (depending on the commutant) of orthogonal, minimal projections, so that, in reality, the characteristic difficulties of the present case lie in the situation of a purely-infinite ring of type II with a finite commutant. We shall, however, defer, until the end of the next section, the separate investigation of the type I case.

In the case of the purely-infinite ring of type II with finite commutant, one must, in effect, take the entire equivalence class of maximal, cyclic projections as invariant. With regard to representations, however, this class reflects itself in a family of ideals of Borel sets in the pure state space and, so, becomes an invariant of much the same type as those at which we arrive in the other cases. We proceed to this investigation.

DEFINITION 4.3.1. *If ϕ is a representation of an operator system as a concrete operator system $(\mathfrak{A}, \mathfrak{IC})$, we call a null ideal band $\{\mathfrak{N}_{\phi|E_\alpha}\}$ a characteristic null ideal band of ϕ when each E_α is maximal cyclic in $\mathfrak{A}^-C_{E_\alpha}$.*

We remark that, with $\mathfrak{A}^-$ purely infinite, characteristic null ideal bands exist. In fact, let $\{P_\gamma\}$ be an orthogonal family of countably-decomposable central projections in $\mathfrak{A}^-$ having sum I, and, with the aid of Lemma 3.3.7, choose an orthogonal family $\{E_{\beta\gamma}\}$ of maximal cyclic projections in $\mathfrak{A}^-P_\gamma$ with sum P_γ, for each γ. Then $\{\mathfrak{N}_{\phi|E_{\beta\gamma}}:\beta,\gamma\}$ is a characteristic null ideal band of ϕ.

THEOREM 4.3.2. *Two representations ϕ_1, ϕ_2 of a C^*-algebra $\mathfrak{A}$ as concrete operator algebras $(\mathfrak{A}_1, \mathfrak{IC}_1)$, $(\mathfrak{A}_2, \mathfrak{IC}_2)$ such that $\mathfrak{A}_1^-$, $\mathfrak{A}_2^-$ are purely-infinite and $\mathfrak{A}_1'$, $\mathfrak{A}_2'$ are finite, are unitarily equivalent if they have a common characteristic null ideal band and only if the set of characteristic null ideal bands of ϕ_1 coincides with the set of those of ϕ_2.*

PROOF. Since the characteristic null ideal bands of ϕ_1 and ϕ_2 are defined in terms of ϕ_1, ϕ_2, $\mathfrak{IC}_1$ and $\mathfrak{IC}_2$, it is clear that the sets of such bands for ϕ_1 and ϕ_2 coincide if ϕ_1 and ϕ_2 are unitarily equivalent.

Suppose now that the characteristic null ideal bands $\{\mathfrak{N}_{\phi_1|E_\alpha}\}$ and $\{\mathfrak{N}_{\phi_2|F_\alpha}\}$ are identical. Then, from The Extension Theorem, there is a *-isomorphism ϕ of $\mathfrak{A}_1^-$ onto $\mathfrak{A}_2^-$, taking E_α onto F_α, for each α, such that $\phi\phi_1 = \phi_2$ and weakly-bicontinuous on the unit spheres of $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$. Let $\{P_\beta\}$ be a maximal, orthogonal family of central projections in $\mathfrak{A}_1^-$ such that each P_β is contained in some C_{E_α}, and note that, by Lemma 3.3.1, C_{E_α}, and hence P_β, is countably-decomposable relative to the center of $\mathfrak{A}_1^-$, since E_α is cyclic. Moreover, $E_\alpha P_\beta$ is maximal cyclic in $\mathfrak{A}_1^-P_\beta$, as is $\phi(E_\alpha)\phi(P_\beta) = F_\alpha Q_\beta$ in $\mathfrak{A}_1^-Q_\beta$, by the remarks preceding Lemma 3.3.6, since E_α and F_α are maximal cyclic in $\mathfrak{A}_1^-C_{E_\alpha}$ and $\mathfrak{A}_2^-C_{F_\alpha}$, respectively, and $P_\beta \leqq C_{E_\alpha}$, $Q_\beta \leqq C_{F_\alpha}$. We assert that $\sum_\beta P_\beta = I$. Indeed, if $P = \sum_\beta P_\beta$ then $(I - P)C_{E_\alpha} = 0$, for each α, by maximality of $\{P_\beta\}$, so that

$$(I - P)E_\alpha = 0,$$

and $(I - P)(\sum_\alpha E_\alpha) = I - P = 0$, establishing our assertion.

Of course, if we show that $\phi \mid P_\beta$ is implemented by a unitary transformation, for each β, then ϕ is implemented by a unitary transformation which carries

ϕ_1 onto ϕ_2. Our problem, then, is reduced to showing that an isomorphism ϕ of a purely-infinite ring of operators $\mathfrak{A}_1^-$ with finite commutant $\mathfrak{A}_1'$ and countably-decomposable center onto another such ring $\mathfrak{A}_2^-$, which carries some maximal, cyclic projection in $\mathfrak{A}_1^-$ onto a maximal, cyclic projection in $\mathfrak{A}_2^-$, is implemented by a unitary transformation. Now maximal, cyclic projections are equivalent, by the remarks preceding Lemma 3.3.6, and ϕ, ϕ^{-1} preserve equivalence of projections, so that ϕ and ϕ^{-1} carry maximal, cyclic projections onto maximal, cyclic projections. The hypotheses of Lemma 4.1.6 are fulfilled, and ϕ is implemented by a unitary transformation, once we note that $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$ have generating vectors, in view of Lemma 3.3.5. Now if U is a unitary transformation of $\mathcal{K}_1$ onto $\mathcal{K}_2$ which implements ϕ then $U\phi_1(A)U^{-1} = \phi[\phi_1(A)] = \phi_2(A)$, and the proof is complete.

4.4. The multiplicity function and Unitary Invariants Theorem

We gather together our preceding results to give The Unitary Invariants Theorem. The unitary invariants we obtain for representations of C^*-algebras are most conveniently listed with the aid of a "multiplicity function" which we describe in the following

DEFINITION 4.4.1. *Let $\mathfrak{A}$ be a C^*-algebra, ϕ a representation of $\mathfrak{A}$ as an algebra $\mathfrak{A}_0$ of operators on the Hilbert space $\mathcal{K}$, P the maximal purely-infinite, central projection in $\mathfrak{A}_0'$, Q the maximal purely-infinite, central projection in $\mathfrak{A}_0^-(I - P)$, and $R = I - P - Q$. Let P_b be the central portion of $\mathfrak{A}_0'P$ with coupling character b, for each cardinal number b not exceeding the dimension of $\mathcal{K}$. For each positive real number or infinite cardinal b let $Q_b = Q + R_b + \sum_{c \leq b} P_c$, where R_b is the central portion of $\mathfrak{A}_0^-R$, $\mathfrak{A}_0'R$ for upper coupling number b. (We use the obvious convention that $R_b = R$ with b an infinite cardinal.) We associate with ϕ a "multiplicity function," f_ϕ, which assigns to 0 the set of characteristic null ideal bands of $\phi \mid Q$ and to each positive real number or infinite cardinal b the set of null ideal bands of the representation $\phi_b = \phi \mid Q_b$. We shall say that two multiplicity functions for representations of $\mathfrak{A}$ are "equivalent" if their values at each number have a non-null intersection, and that they are "identical" if their values are the same at each number.*

Concerning this definition, we remark that it is appropriate to use the number 0 for the Q portion of the representation, for this is the portion which is infinite with finite commutant and, so, has "coupling" 0.

THEOREM 4.4.2. (Unitary Invariants Theorem). *Two representations ϕ_1, ϕ_2 of a C^*-algebra $\mathfrak{A}$ as algebras of operators $\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively, acting on Hilbert spaces $\mathcal{K}_1$, $\mathcal{K}_2$, respectively, are unitarily equivalent if their associated multiplicity functions are equivalent and only if they are identical.*

PROOF. Since the multiplicity functions of ϕ_1 and ϕ_2 are defined in terms of the Hilbert space constructs arising from ϕ_1, ϕ_2 and the C^*-algebra $\mathfrak{A}$, the unitary equivalence of ϕ_1 and ϕ_2 implies that these multiplicity functions are identical.

We suppose now that ϕ_1 and ϕ_2 have equivalent multiplicity functions. If we employ the notation of Definition 4.4.1 for the projections defined there, as applied to $\mathfrak{A}_1^-$, $\mathfrak{A}_2^-$, $\mathfrak{A}_1'$, $\mathfrak{A}_2'$, using the superscripts 1 and 2 to indicate those projec-

tions (and mappings) related to the representations ϕ_1 and ϕ_2, respectively, then, by The Extension Theorem (2.2.5), there exist isomorphisms $\phi^{(b)}$ of $\mathfrak{A}_1^- Q_b^{(1)}$ onto $\mathfrak{A}_2^- Q_b^{(2)}$ such that $\phi^{(b)}\phi_{1b} = \phi_{2b}$, where ϕ_{1b}, ϕ_{2b} are the representations $\phi_1 \mid Q_b^{(1)}$ and $\phi_2 \mid Q_b^{(2)}$, respectively. In case $b = 0$, $Q_b^{(1)} = Q^{(1)}$ and $Q_b^{(2)} = Q^{(2)}$, whence, in view of the fact that $\phi_1 \mid Q^{(1)}$ and $\phi_2 \mid Q^{(2)}$ have a common characteristic null ideal band and Theorem 4.3.2, not only do we have the isomorphism $\phi^{(0)}$; but we have that it is unitarily implemented, so that ϕ_{10}, ϕ_{20} are unitarily equivalent.

We next observe that, when $b \leq c$, $\phi^{(b)}$ is $\phi^{(c)}$ restricted to $\mathfrak{A}_1^- Q_b^{(1)}$. Indeed, by [21], $Q_b^{(1)}$ is the strong limit of a directed sequence of operators $\phi_1(A_\gamma)$ in the unit sphere of $\mathfrak{A}_1$, so that $Q_b^{(1)}$ is the strong limit of $\phi_1(A_\gamma)Q_b^{(1)} = \phi_{1b}(A_\gamma)$ and

$$Q_b^{(1)} Q_c^{(1)} = Q_b^{(1)}$$

is the strong limit of $\phi_1(A_\gamma)Q_c^{(1)} = \phi_{1c}(A_\gamma)$. Thus $\phi^{(c)}(Q_b^{(1)})$ is the weak limit of $\phi^{(c)}[\phi_{1c}(A_\gamma)] = \phi_{2c}(A_\gamma)$, and $\phi^{(b)}[\phi_{1b}(A_\gamma)] = \phi_{2b}(A_\gamma)$ tends weakly to $\phi^{(b)}(Q_b^{(1)}) = Q_b^{(2)}$. Hence $\phi_{2c}(A_\gamma)Q_b^{(2)} = \phi_2(A_\gamma)Q_c^{(2)}Q_b^{(2)} = \phi_2(A_\gamma)Q_b^{(2)} = \phi_{2b}(A_\gamma)$ tends weakly to $\phi^{(c)}(Q_b^{(1)})Q_b^{(2)}$ and to $Q_b^{(2)}$; whence $\phi^{(c)}(Q_b^{(1)})Q_b^{(2)} = Q_b^{(2)}$, and $Q_b^{(2)} \leq \phi^{(c)}(Q_b^{(1)})$. Applying this result to $(\phi^{(c)})^{-1}$, we have that $Q_b^{(1)} \leq (\phi^{(c)})^{-1}(Q_b^{(2)})$, so that

$$\phi^{(c)}(Q_b^{(1)}) \leq Q_b^{(2)},$$

and, thus, $\phi^{(c)}(Q_b^{(1)}) = Q_b^{(2)}$. Now

$$\phi^{(c)}[\phi_1(A)Q_b^{(1)}] = \phi^{(c)}[\phi_1(A)Q_c^{(1)}Q_b^{(1)}] = \phi_{2c}(A)Q_b^{(2)}$$
$$= \phi_2(A)Q_b^{(2)} = \phi^{(b)}[\phi_1(A)Q_b^{(1)}],$$

so that $\phi^{(c)}$ and $\phi^{(b)}$ agree on a weakly dense subalgebra of $\mathfrak{A}_1^- Q_b^{(1)}$. The weak continuity of $\phi^{(c)}$ and $\phi^{(b)}$ on the unit sphere of $\mathfrak{A}_1^- Q_b^{(1)}$ (and [21]) imply that $\phi^{(c)}$ and $\phi^{(b)}$ agree on $\mathfrak{A}_1^- Q_b^{(1)}$. We can therefore define a mapping ϕ_0 of $\bigcup_b \mathfrak{A}_1^- Q_b^{(1)}$ onto $\bigcup_b \mathfrak{A}_2^- Q_b^{(2)}$ by letting ϕ_0 be $\phi^{(b)}$ on $\mathfrak{A}_1^- Q_b^{(1)}$. Exactly as in the proof of Lemma 4.2.4, it follows that ϕ_0 has a *-isomorphic extension ϕ mapping $\mathfrak{A}_1^-$ onto $\mathfrak{A}_2^-$. (Note that $\bigcup_b Q_b^{(1)} = I$ and $\bigcup_b Q_b^{(2)} = I$.)

We have that ϕ carries $Q^{(1)}(= Q_0^{(1)})$ upon $Q^{(2)}(= Q_0^{(2)})$, since ϕ agrees with $\phi^{(0)}$ where the latter mapping is defined. Thus ϕ restricted to $\mathfrak{A}_1^- Q^{(1)}$ is unitarily implemented. Let us denote by $S^{(1)}$ and $S^{(2)}$ the maximal central projections in $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$, respectively, such that $\mathfrak{A}_1^- S^{(1)}$ and $\mathfrak{A}_2^- S^{(2)}$ are finite and $\mathfrak{A}_1' S^{(1)}$, $\mathfrak{A}_2' S^{(2)}$ are purely infinite. Thus $S^{(1)} + R^{(1)}$ and $S^{(2)} + R^{(2)}$ are the maximal, central, finite projections in $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$, respectively, and $P^{(1)} - S^{(1)}$, $P^{(2)} - S^{(2)}$ are maximal, purely-infinite, central projections in $\mathfrak{A}_1^-(I - Q^{(1)})$ and

$$\mathfrak{A}_2^-(I - Q^{(2)}),$$

respectively. Since ϕ maps $\mathfrak{A}_1^-(I - Q^{(1)})$ isomorphically upon $\mathfrak{A}_2^-(I - Q^{(2)})$, $\phi(S^{(1)} + R^{(1)}) = S^{(2)} + R^{(2)}$ and $\phi(P^{(1)} - S^{(1)}) = P^{(2)} - S^{(2)}$. Now, according to Definition 4.4.1 and Lemma 4.2.2, $\bigcup_{b<\infty} Q_b^{(1)}$ contains $R^{(1)}$ and $\bigcup_{b<\infty} Q_b^{(2)}$ contains $R^{(2)}$. Moreover, by choice of $S^{(1)}$ and $S^{(2)}$, no central subprojection of $S^{(1)}$ or $S^{(2)}$ can have finite coupling character (recall that, by Lemma 3.3.4, each projection cyclic under $\mathfrak{A}_1^- S^{(1)}$ and $\mathfrak{A}_2^- S^{(2)}$ is finite in $\mathfrak{A}_1' S^{(1)}$, $\mathfrak{A}_2' S^{(2)}$, while $S^{(1)}$, $S^{(2)}$

are themselves purely-infinite in $\mathfrak{A}_1'$, $\mathfrak{A}_2'$, respectively); so that $S^{(1)}$ and $S^{(2)}$ are orthogonal to $\bigcup_{b<\infty}Q_b^{(1)}$ and $\bigcup_{b<\infty}Q_b^{(2)}$, respectively. But $\phi(\bigcup_{b<\infty}Q_b^{(1)}) = \bigcup_{b<\infty}Q_b^{(2)}$, so that

$$\phi[(\bigcup_{b<\infty}Q_b^{(1)})(S^{(1)} + R^{(1)})] = \phi(R^{(1)}) = (\bigcup_{b<\infty}Q_b^{(2)})(S^{(2)} + R^{(2)}) = R^{(2)}.$$

Thus $\phi(S^{(1)}) = S^{(2)}$ and, hence, $\phi(P^{(1)}) = P^{(2)}$.

Since $\phi(R^{(1)}) = R^{(2)}$, $\phi(R^{(1)}Q_b^{(1)}) = R^{(2)}Q_b^{(2)}$, i.e., $\phi(R_b^{(1)}) = R_b^{(2)}$. It follows from Lemma 4.2.4 that ϕ restricted to $\mathfrak{A}_1^- R^{(1)}$ is unitarily implemented. Moreover,

$$\phi(P^{(1)}Q_b^{(1)}) = \phi(\sum_{c\le b}P_c^{(1)}) = P^{(2)}Q_b^{(2)} = \sum_{c\le b}P_c^{(2)}.$$

Now, $\sum_{b<d}P_b^{(1)}$ is the union of $\{\sum_{c\le b}P_c^{(1)}\}_{b<d}$, and $\sum_{b<d}P_b^{(2)}$ is the union of $\{\sum_{c\le b}P_c^{(2)}\}$. Thus, since

$$\phi(\sum_{c\le b}P_c^{(1)}) = \sum_{c\le b}P_c^{(2)}, \quad \phi(\sum_{b<d}P_b^{(1)}) = \sum_{b<d}P_b^{(2)}.$$

Of course, then, $\phi(P_b^{(1)}) = P_b^{(2)}$ for each b, and ϕ maps $\mathfrak{A}_1^- P_b^{(1)}$ isomorphically upon $\mathfrak{A}_2^- P_b^{(2)}$. It follows, from Lemma 4.1.7, that ϕ restricted to $\mathfrak{A}_1^- P_b^{(1)}$ is unitarily implemented, so that ϕ restricted to $\mathfrak{A}_1^-(\sum_b P_b^{(1)}) = \mathfrak{A}_1^- P^{(1)}$ is unitarily implemented, and, thus ϕ restricted to $\mathfrak{A}_1^-(Q^{(1)} + R^{(1)} + P^{(1)}) = \mathfrak{A}_1^-$ is unitarily implemented.

The proof is concluded now, as in Theorem 4.3.2, when we observe that $\phi\phi_1 = \phi_2$. Indeed,

$$(\phi[\phi_1(A)])Q_b^{(2)} = \phi[\phi_1(A)Q_b^{(1)}] = \phi[\phi_{1b}(A)] = \phi_{2b}(A) = \phi_2(A)Q_b^{(2)},$$

so that

$$0 = (\phi[\phi_1(A)] - \phi_2(A))(\bigcup_b Q_b^{(2)}) = (\phi[\phi_1(A)] - \phi_2(A))I,$$

and $\phi[\phi_1(A)] = \phi_2(A)$ for each A in $\mathfrak{A}$.

We note that, in the case of a countably-decomposable representation, with ideal bands taken as the permanent null ideal alone corresponding to the various unit projections, the multiplicity function of a representation of a C^*-algebra is monotone-decreasing, i.e., the ideal of Borel sets corresponding to the value b is contained in the ideal corresponding to the value c, if $c \le b$. Indeed, employing the notation of Definition 4.4.1, $Q_c \le Q_b$, so that with $A \ge 0$, A in $\mathfrak{A}$, $\phi(A) \ge 0$ and $\phi(A)(Q_b - Q_c) \ge 0$, whence $\omega(\phi(A)Q_c) \le \omega(\phi(A)Q_b)$, where ω is a normal state of $\mathfrak{A}_0^-$. With $(\mathfrak{L}, X)$ the representing function system of $\mathfrak{A}$ and ι the canonical isomorphism of $\mathfrak{L}$ onto $\mathfrak{A}$ the states ρ and τ of $\mathfrak{L}$ defined by $\rho(f) = \omega(\phi(\iota f)Q_c)$, $\tau(f) = \omega(\phi(\iota f)Q_b)$ stand in the relation $\rho \le \tau$ so that each positive extension of ρ to $C(X)$ is dominated by a positive extension of τ to $C(X)$, and our assertion follows.

We turn to the analysis of an infinite representation with type I commutant in terms other than the characteristic null ideal bands of the representation. this investigation will be carried out in the context of the general situation, and we begin with a modification of the concept of multiplicity function.

DEFINITION 4.4.3. *Let ϕ be a representation of a C^*-algebra $\mathfrak{A}$ as an algebra of operators $\mathfrak{A}_0$ acting on a Hilbert space $\mathcal{K}$, and let Q_0' be the maximal central projection in $\mathfrak{A}_0^-$ such that $\mathfrak{A}_0^- Q_0'$ is of type II_∞ and $\mathfrak{A}_0' Q_0'$ is finite. Adopting the notation of Definition 4.4.1, we define P_b' to be $Q_0' + \sum_{b>1/m} Q_m' + R_b + \sum_{c \le b} P_c$, where Q_m' is the maximal, central projection in $\mathfrak{A}_0^-$ such that $\mathfrak{A}_0^- Q_m'$ is infinite and $\mathfrak{A}_0' Q_m'$ is of type I_m, $m = 1, 2, \cdots$. We associate with ϕ a "separating multiplicity function" f_ϕ' which assigns to 0 the set of characteristic null ideal bands of $\phi \mid Q_0'$ and to each positive real number or infinite cardinal b the set of null ideal bands of the representation $\phi_b' = \phi \mid P_b'$ of $\mathfrak{A}$.*

The use of $\sum_{b>1/m} Q_m'$ in defining P_b' rather than $\sum_{m \le b} Q_m'$, as might have seemed more natural, is a matter of technical necessity. In the proof of The Unitary Invariants Theorem, once the isomorphism ϕ had been constructed, it was possible, on the basis of the construction and the special features of the situation, to show that ϕ preserved the R and P portions of the ring (the Q projection is preserved by assumption, in effect), although they depend upon the commutant as well as the ring in question. In the present case only the Q_0' portion is taken care of automatically, and the Q_m' portions become confused with the P_b portions if care is not exercised. In fact, the multiplicity function defined by using

$$\sum_{m \le b} Q_m'$$

would not constitute a set of unitary invariants for the representation.

THEOREM 4.4.4. (The Second Unitary Invariants Theorem). *Two representations ϕ_1, ϕ_2 of a C^*-algebra $\mathfrak{A}$ as algebras of operators $\mathfrak{A}_1$, $\mathfrak{A}_2$ acting on Hilbert spaces $\mathcal{K}_1$, $\mathcal{K}_2$ are unitarily equivalent if their associated separating multiplicity functions are equivalent and only if they are identical.*

PROOF. The arguments of The Unitary Invariants Theorem apply to show that there is an isomorphism ϕ of $\mathfrak{A}_1^-$ onto $\mathfrak{A}_2^-$ such that $\phi\phi_1 = \phi_2$, that $\phi(R^{(1)}) = R^{(2)}$, $\phi(S^{(1)}) = S^{(2)}$, and $\phi(Q_0'^{(1)}) = Q_0'^{(2)}$. Thus, using the fact that ϕ preserves the purely-infinite part of $\mathfrak{A}_1^-$, with the foregoing,

$$\phi\left(\sum_{b>1/m} Q_m'^{(1)} + \sum_{b \ge c} P_c^{(1)}\right) = \sum_{b>1/m} Q_m'^{(2)} + \sum_{b \ge c} P_c^{(2)}.$$

Choosing b slightly less than 1, we have $\phi(\sum_{m=2}^\infty Q_m'^{(1)}) = \sum_{m=2}^\infty Q_m'^{(2)}$; and with $b = 1$, we have $\phi(\sum_{m=2}^\infty Q_m'^{(1)} + P_1^{(1)}) = \sum_{m=2}^\infty Q_m'^{(2)} + P_1^{(2)}$, so that $\phi(P_1^{(1)}) = P_1^{(2)}$. Now, with b slightly greater than 1, we have $\phi(\sum_{m=1}^\infty Q_m'^{(1)} + P_1^{(1)}) = \sum_{m=1}^\infty Q_m'^{(2)} + P_1^{(2)}$, so that $\phi(\sum_{m=1}^\infty Q_m'^{(1)}) = \sum_{m=1}^\infty Q_m'^{(2)}$. The argument of The Unitary Invariants Theorem now tells us that $\phi(P_b^{(1)}) = P_b^{(2)}$ and $\phi(Q_m'^{(1)}) = Q_m'^{(2)}$, and that ϕ is unitarily implemented when restricted to $\mathfrak{A}_1^-(I - \sum_{m=1}^\infty Q_m'^{(1)})$.

We complete the proof by showing that ϕ restricted to $\mathfrak{A}_1^- Q_m'^{(1)}$ is unitarily implemented. To simplify our notation, we prove that if ϕ is an isomorphism of the ring $\mathcal{R}_1$ onto the ring $\mathcal{R}_2$ and both rings are purely-infinite with commutants of type I_m, then ϕ is unitarily implemented. This proof is carried out by making the usual reduction to the case where $\mathcal{R}_1$ and $\mathcal{R}_2$ have countably-decomposable centers (we assume this done) and then noting that the maximal, cyclic projections in $\mathcal{R}_1$ and $\mathcal{R}_2$ can be characterized as those projections which are expressible

as the sum of m, orthogonal, abelian projections having the identity as central carrier. Let us assume this fact, for the moment, so that ϕ preserves maximal cyclicity. From Lemma 3.3.5, it follows that $\mathfrak{IC}_1$ and $\mathfrak{IC}_2$ have generating vectors under $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively, so that, by Lemma 4.1.6, ϕ is unitarily implemented.

It remains to establish the criterion for maximal cyclicity in a purely-infinite ring $\mathfrak{R}$ with countably-decomposable center and commutant $\mathfrak{R}'$ of type I_m. If we have established that one projection in $\mathfrak{R}$ which is expressible as a sum of m, orthogonal, abelian projections with central carrier I is maximal cyclic then, since all such projections are equivalent, all such projections are maximal cyclic, and since all maximal, cyclic projections are equivalent in $\mathfrak{R}$, all maximal, cyclic projections are so expressible. Let E be a maximal, cyclic projection in $\mathfrak{R}$, and let x be a generating vector for E, so that $[\mathfrak{R}x] = \mathfrak{IC}$, by Lemma 3.3.5, [28; Lemma 9.3.3] and the maximal cyclicity of E. According to Lemma 3.3.4, E is finite, so that $E\mathfrak{R}E$ is finite. Relative to $E(\mathfrak{IC})$, the commutant $\mathfrak{R}'E$ of $E\mathfrak{R}E$ is isomorphic to $\mathfrak{R}'$, by Lemma 3.1.2, since $C_E = I$. Thus $\mathfrak{R}'E$ is of type I_m. Moreover, x is a cyclic vector for both $E\mathfrak{R}E$ and $\mathfrak{R}'E$. From The Coupling Theorem (3.3.8), we see that the dimensions, relative to their respective rings, of cyclic projections in $E\mathfrak{R}E$ and $R'E$ having a joint generating vector are identical. Thus, if F' is an abelian projection in $\mathfrak{R}'E$ (necessarily cyclic) with central carrier E and generating vector y, then, since $\mathfrak{R}'E$ is of type I_m, the dimension of F' is E/m, so that the dimension of $F = [\mathfrak{R}'Ey] = [\mathfrak{R}'y]$, in $\mathfrak{R}$, is E/m. It follows that E is the sum of m copies of F in $E\mathfrak{R}E$, hence in $\mathfrak{R}$. But, from [28; Lemma 9.3.3], F is abelian in $E\mathfrak{R}E$, i.e., $F(E\mathfrak{R}E)F = F\mathfrak{R}F$ is abelian, so that F is abelian in $\mathfrak{R}$. Finally, $C_F = C_E = I$, and the proof is complete.

CHAPTER V. SPECIAL CASES, EXAMPLES, APPLICATIONS AND COMPUTATION

The Unitary Invariants Theorem of the preceding chapter has been stated under the most general conditions. Various simplifications occur when this theorem is applied to special cases. Many of these cases are of sufficient interest to warrant individual attention. In particular, we shall consider the separable, the abelian, the factor cases and the more important subcases and combinations of these cases in the first section of this chapter. The second section is devoted to some examples which clarify and illustrate some of the points and settle some of the questions implicit in the foregoing material. The third section contains a discussion of applications and an (abstract) application—namely, we list unitary invariants determining when two non-normal operators on a Hilbert space are unitarily equivalent. The final section contains results related to the computation of null ideals.

5.1. Special representations

If ϕ is a representation of a C^*-algebra as an operator algebra acting upon a separable Hilbert space, then, of course, all of the higher cardinal considerations of The Unitary Invariants Theorem disappear—all rings in question are count-

ably decomposable, and the domain of the multiplicity function need extend just to $\aleph_0$.

The case of abelian C^*-algebras holds a place of special interest being the first to yield to spectral analysis and being the easiest case in which to view the effects of the various processes we have employed in the spectral decomposition of representations of general C^*-algebras. We shall give a descriptive discussion (without formal proofs) of the result of applying The Unitary Invariants Theorem to a representation of an abelian C^*-algebra, and we shall compare this result with the classical multiplicity decomposition of such a representation.

In point of fact, the earliest work [12, 14] determined the unitary invariants of the action of a self-adjoint operator on a Hilbert space, or, what amounts to the same thing, of a representation of an abelian C^*-algebra as an operator algebra acting on a separable Hilbert space. We begin with a description of this work. If $\mathfrak{A}$ is an abelian C^*-algebra, ϕ a representation of it as an (abelian) operator algebra $\mathfrak{A}_0$ acting upon the separable Hilbert space $\mathfrak{IC}$, one associates with ϕ an orthogonal sequence of projections $\{E'_n\}$ in $\mathfrak{A}'_0$ with sum I, each E'_n cyclic under $\mathfrak{A}_0$ and $(C_{E'_n})$ a decreasing sequence. The descending carrier decomposition of $\mathfrak{A}'_0$ (cf. Lemma 4.1.5) serves as such a decomposition. Now the projections $\{E'_n\}$ are not unique relative to the stated properties, though the ascending chain of null ideals corresponding to the representations $\phi \mid E'_n$ of $\mathfrak{A}$ are; i.e., the ideal chains obtained from two such decompositions are, element by element, identical. On the other hand, the projections $\{Q_b\}$ of Definition 4.4.1 are unique, and it is by passing thru this decomposition that the unitary invariance of the null ideal chain just described is established.

An examination of the effects of these modes of decomposition upon a self-adjoint operator acting on a finite-dimensional space is quite instructive. Let our operator be A, and let its distinct spectral values be $\lambda_1, \cdots, \lambda_n$ with corresponding eigenspaces $E_1, \cdots, E_n$ of dimensions $d_1, d_2, \cdots, d_n$, respectively. A descending carrier decomposition obtained with the aid of $\{E_k\}$ can be described as follows—choose a non-zero vector from each space E_k, and let E'_0 be the space they generate. Choose a non-zero vector, from each of the spaces in which it is possible, orthogonal to the original vectors chosen, and let E'_1 be the space they generate. Continuing in this way, we obtain a descending carrier decomposition $E'_0, \cdots, E'_{d_j-1}$, where $d_j = \max \{d_1, \cdots, d_n\}$. The projection Q_b of the coupling character decomposition is $\sum_{d_i \leq b} E_i$. There is little difficulty now in passing from the descending carrier decomposition to the coupling character decomposition. The projection Q_b is the union of all (central) projections of the ring $\mathfrak{A}$ generated by A, contained in $\sum_{i=0}^{b-1} E'_i$. The description of the construction of a descending carrier decomposition which involves the coupling character decomposition is the substance of the proof of Lemma 4.1.5. Of course, this description does not depend invariantly upon the coupling character decomposition, since this latter decomposition is unique while the descending carrier decomposition is not.

The pure state space X of $\mathfrak{A}$ is the discrete space consisting of n-points

$\{x_1, \cdots, x_n\}$, and the representation ϕ in question takes the function which is ν_i at x_i onto the operator $\sum_{i=1}^{n} \nu_i E_i$. The null ideal of $\phi \mid Q_b$ is the ideal of subsets of $\{x_k : b < d_k\}$ and the null ideal of $\phi \mid E_b'$ is the ideal of subsets of

$$\{x_k : d_k \leqq b\}.$$

Thus the ideals corresponding to E_b' and Q_b are complementary to one another (i.e., each consists of the "Borel" subsets of X which have void intersection with all the subsets of the other)—if we had dealt with $I - Q_b$ instead of Q_b the theories would coincide, in this case. The reduction theory of von Neumann [34] provides a decomposition corresponding to the eigenspace decomposition E_1, $\cdots$, E_n above, in the commutative case; and, in the non-commutative case gives the central decomposition of rings of operators into factors.

In effect then the unitary invariants for the action of a self-adjoint operator upon a separable Hilbert space may be obtained by a two step process: the decomposition of the ring engendered by the operator into distinguished pieces relative to its commutant (multiple copies of maximal abelian algebras, in this case), and the construction of unitary invariants for the situation in which the engendered ring has a single, "simple" constituent in its decompositon, viz., itself. The technique behind The Unitary Invariants Theorem fits into this pattern—The Extension Theorem invariant corresponding to the unitary invariant for a "simple" representation and the ring decomposition occurring as before.

The analysis of the non-separable, single operator (or abelian) situation was carried out by Wecken [49], and later by Plessner-Rohlin [39]. This analysis again is the two-step process we described; however, the ring decomposition step has become much more difficult. In [49], prior to our present advanced state of knowledge of ring decomposition, both steps are taken together and at the expense of the unitary invariants for "simple" representations. Specifically, the so-called "weighted spectrum", which in our terminology is the full set of normal states transferred to the representing function system, is taken as invariant of the "simple" representation rather than the more basic family of permanent null sets of these states. We have noted that the extension problem is a triviality with the full set of normal states taken as invariant. In [39] more modern techniques are introduced into the theory to deal with the non-separable abelian case, but the "weighted spectrum" remains as invariant. The theory of [12, 14] is finally carried over to the non-separable abelian case, with basic invariant the null sets, in [31]. Of course more work must be done with the null set invariant than with the weighted spectrum as invariant since the theory must contain the work of The Extension Theorem (greatly simplified by commutativity, however). In our terminology, Nakano uses as invariant the set of all null ideals corresponding to restrictions of the representation to countably-decomposable projections in the ring of operators generated by the representing operator algebra. This fails to give the single ideal as invariant in the separable case, as we have managed to do by using null ideal bands; though the gap is slight in the com-

mutative case. Halmos [13] manages to avoid the involvements of the extension problem with the null sets used as invariants while employing an invariant somewhat sharper than the full set of normal states.

By a "factor representation" we shall mean a representation ϕ of a C^*-algebra $\mathfrak{A}$ as an algebra of operators $\mathfrak{A}_0$ acting upon some Hilbert space $\mathcal{K}$ in such a way that $\mathfrak{A}_0^-$ is a factor. This class of representations is quite broad including as a special case all irreducible representations (in this case the factor is of type I). Since the only central projections in $\mathfrak{A}_0^-$ are 0 and I, the multiplicity function f_ϕ assigns to 0 the set of characteristic null ideal bands of ϕ and to each other number in its domain the set of null ideal bands of ϕ, if $\mathfrak{A}_0^-$ is infinite and $\mathfrak{A}_0'$ is finite. Otherwise, there is some positive number or infinite cardinal a such that f_ϕ assigns to each $b < a$ the ideal $\mathfrak{B}$ of Borel sets in the pure state space of $\mathfrak{A}$ and to each $c \geq a$ the set of null ideal bands of ϕ. If $\mathfrak{A}_0^-$ is of type I_n and $\mathfrak{A}_0'$ of type I_m, with n and m finite, then, more specifically, $a = m/n$ above and the set of null ideal bands contains $\mathfrak{N}_\phi$ (the ring $\mathfrak{A}_0^-$ being countably-decomposable in this case). If $\mathfrak{A}_0'$ is infinite and has coupling character b then $a = b$ above and the set of null ideal bands of ϕ contains $\mathfrak{N}_\phi$ when $n \leq \aleph_0$, whatever the type of $\mathfrak{A}_0^-$. With n infinite and m finite, we have the situation described for $\mathfrak{A}_0^-$ infinite and $\mathfrak{A}_0'$ finite above, but, in this case, it is more informative to describe the separating multiplicity function f_ϕ'. We have $f_\phi'(b) = \mathfrak{B}$ for $b \leq 1/m$ and $f_\phi'(b)$ is the set of null ideal bands of ϕ for $b > 1/m$, and contains $\mathfrak{N}_\phi$ if $n \leq \aleph_0$. Note that the case $m = 1$ is precisely the case of irreducible representations. With $\mathfrak{A}_0^-$ and $\mathfrak{A}_0'$ of type II_1 having coupling number b, $f_\phi(a) = \mathfrak{B}$ for $a < b$, and $f_\phi(a)$ is the set of null ideal bands of ϕ and contains $\mathfrak{N}_\phi$, for $a \geq b$. In all but the II_∞, II_1 case, the factor representation is determined to within unitary equivalence by a number (possibly an infinite cardinal) and a null ideal band (or $\mathfrak{N}_\phi$ an ideal of Borel sets, in the countably-decomposable case).

5.2. Some examples

To illustrate the role the null ideals play in the problem of map extension (cf. The Extension Theorem of 2.2), we shall begin this section with an example of a *-isomorphism ϕ of one concrete, countably-decomposable abelian C^*-algebra $\mathfrak{A}_1$ upon another $\mathfrak{A}_2$ which admits a (weakly) continuous extension ϕ' mapping $\mathfrak{A}_1^-$ into $\mathfrak{A}_2^-$ with ϕ' a homomorphism but not an isomorphism. Thus this is a situation in which $\mathfrak{N}_\phi$ contains $\mathfrak{N}(\mathfrak{A}_1, \mathcal{K}_1)$ properly. Moreover, ϕ^{-1} does not admit a continuous extension to $\mathfrak{A}_2^-$, for if ϕ'' were such, then $\phi''\phi'$ would be a continuous mapping of $\mathfrak{A}_1^-$ into itself which is the identity on $\mathfrak{A}_1$ and hence the identity on $\mathfrak{A}_1^-$. We would deduce from this that ϕ' was an isomorphism contrary to construction. We proceed to

EXAMPLE 5.2.1. As $\mathcal{K}_1$ we take $[0, 1]$ under Lebesgue measure and as $\mathcal{K}_2$ the open, everywhere-dense subset S of $[0, 1]$ obtained by the Cantor process of taking the centered open quarter of $[0, 1]$, together with the centered open intervals of lengths $\frac{1}{16}$ in the remaining pieces, and so forth. We observe that S has Lebesgue measure $\frac{1}{2}$. For our first C^*-algebra $\mathfrak{A}_1$ we choose the algebra consisting of

those operators T_f defined by $T_f(g) = f \cdot g$ for each g in $L_2([0, 1])$ and f some fixed continuous function on $[0, 1]$. We define $\phi(T_f)$ to be the operator "multiplication by the restriction of f to S on $L_2(S)$, S taken with Lebesgue measure." Clearly $\phi(T_f)$ is a bounded operator on $L_2(S)$ and ϕ is a *-homomorphism of $\mathfrak{A}_1$ onto an (abelian) C^*-algebra $\mathfrak{A}_2$. The mapping ϕ is an isomorphism for if $\phi(T_f) = 0$ then f is 0 almost everywhere on S. Now, f being continuous and S being open, if f is non-zero at some point of S then f is non-zero on some interval in S, contradicting the fact that f is 0 almost everywhere on S. Thus f is 0 on S, and this, together with the everywhere denseness of S and the continuity of f imply that f is 0 on $[0, 1]$, whence $T_f = 0$ and ϕ is an isomorphism. The weak closure $\mathfrak{A}_1^-$ of $\mathfrak{A}_1$ is the algebra consisting of the multiplication operators T_h on $L_2([0, 1])$ where h is an essentially-bounded, measurable function on $[0, 1]$. The isomorphism ϕ has the obvious extension ϕ' to $\mathfrak{A}_1^-$ with the desired continuity and homomorphism properties. That ϕ' has non-zero kernel is apparent; for example, the characteristic function of the complement of S corresponds to a non-zero operator in $\mathfrak{A}_1^-$, since this complement has Lebesgue measure $\frac{1}{2}$, which maps under ϕ into 0.

In The Extension Theorem, we employed the hypothesis that the kernel of a mapping being studied was generated by its positive elements as well as the critical hypothesis that a null ideal band of both mappings coincided. It is natural to inquire whether the kernel of an order representation of one C^*-algebra onto another is necessarily generated by its positive elements, and, at any rate, does not the coincidence of the null ideals of two such mappings entail the coincidence of their kernels? The following simple example of order-representations (states) of an abelian C^*-algebra onto the factor of type I_1 (the complex numbers) illustrates the possibility of the kernel of an order representation (with the property that the inverse image of a positive operator contains a positive operator) not being generated by its positive elements, the null ideals coinciding without the kernels coinciding, without the possibility of map extension (in the sense of The Extension Theorem), and without the unitary (or semi-unitary) equivalence of the mappings (despite the perfect matching of coupling).

EXAMPLE 5.2.2. As our abelian C^*-algebra $\mathfrak{A}$, we choose the algebra of continuous, complex-valued functions on $[0, 1]$. For the map ϕ_1 we choose the state

$$f \rightarrow \int_0^1 f\, dm,$$ with m Lebesgue measure on $[0, 1]$, and for ϕ_2 the state

$$f \rightarrow \left(\tfrac{1}{2}\right) \int_0^{\frac{1}{2}} f\, dm + \left(\tfrac{3}{2}\right) \int_{\frac{1}{2}}^1 f\, dm.$$

Thus our image algebra is the same in both cases, namely, the complex numbers (acting by multiplication on the 1-dimensional Hilbert space of complex numbers). The null ideals $\mathfrak{N}_{\phi_1}$, $\mathfrak{N}_{\phi_2}$ are both the Borel sets in $[0, 1]$ of Lebesgue measure 0. The kernels of both mappings are non-zero, distinct and contain no positive functions. The unitary equivalence of ϕ_1 and ϕ_2 would imply that one is equal to the other.

Our next example contains an instance of a pair of order-isomorphisms of a function system as concrete operator algebras, in fact, abelian algebras, for which the null ideal of one is contained in the null ideal of the other without an extension being possible. As must be expected from The Extension Theorem, the image algebra of the first representation does not have a countably-decomposable weak closure. This example illustrates that the null ideal itself does not serve as an extension invariant in the general case, but that something of the nature of the null ideal band is necessary.

EXAMPLE 5.2.3. Let $\mathfrak{IC}_1$ be a Hilbert space with a complete orthonormal basis $\{x_r\}$ indexed by the real numbers r in the closed interval $[0, 1]$. Let A be the linear transformation on $\mathfrak{IC}_1$ defined by $Ax_r = rx_r$. For $\mathfrak{IC}_2$ choose $L_2(0, 1)$ (under Lebesgue measure) and let B be the operator on $\mathfrak{IC}_2$ defined by $B(f) = xf$. Both A and B have simple spectrum equal to $[0, 1]$. Denote by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ the algebras generated by A and B, respectively, and let ϕ_1 and ϕ_2 be the canonical isomorphisms of the representing function system $C([0, 1])$ onto $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively (so that, if λ denotes the identity transform on $[0, 1]$, $\phi_1(\lambda) = A$ and $\phi_2(\lambda) = B$). The state $f \rightarrow (\phi_1(f)x_r, x_r)$ is a pure state of $C([0, 1])$, since x_r is an eigenvector of A (see [15]); and its value at λ is r, whence, it corresponds to the point r and induces the measure which is 1 on Borel subsets of $[0, 1]$ containing r and 0 on all others. Thus r does not lie in a permanent null set of ϕ_1, and $\mathfrak{N}_{\phi_1} = \{\mathbf{0}\}$. (We see from this that if the null ideal extension result held in general, $\mathfrak{A}_1$ would be a "universal" algebra for representation extensions of $C([0, 1])$; i.e., each representation of $C([0, 1])$ would be factorable through $\mathfrak{A}_1$ by a representation which was extendable to $\mathfrak{A}_1^-$.) Though it is not needed for the following, we remark that $\mathfrak{N}_{\phi_2}$ consists of the Borel subsets of $[0, 1]$ having Lebesgue measure 0. Define ϕ on $\mathfrak{A}_1$ by $\phi\phi_1(f) = \phi_2(f)$, so that ϕ is a *-isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. We show that ϕ does not have an extension to $\mathfrak{A}_1^-$ which is weakly continuous on the unit sphere of $\mathfrak{A}_1^-$. In fact, let f_{nr} be a sequence of functions in $C([0, 1])$ which lie between 0 and 1, and are monotone-decreasing with pointwise limit 1 at r and 0 elsewhere on $[0, 1]$. Then $(\phi_1(f_{nr})x_t, x_t)$ is monotone-decreasing to 1 if $t = r$ and to 0 otherwise, so that $\phi_1(f_{nr})$ tends weakly to E_r, the projection operator on $\mathfrak{IC}_1$ with 1-dimensional range $\{\alpha x_r\}$; while

$$(\phi_2(f_{nr})g, g) = \int_{[0,1]} f_{nr}(\lambda)| g(\lambda) |^2 \, d\lambda \rightarrow 0,$$

since $f_{nr}| g |^2$ is monotone-decreasing to 0 almost-everywhere on $[0, 1]$, so that $\phi_2(f_{nr})$ tends weakly to 0. Thus an extension of ϕ to $\mathfrak{A}_1^-$ which is weakly continuous on the unit sphere of $\mathfrak{A}_1^-$ must map each E_r into 0 and $I = \sum_r E_r$ into 0, so that no such extension exists.

Following Definition 2.2.1, we remarked that the distinction between $\mathfrak{N}_\phi$, the normal state null sets, and $\mathfrak{N}_\phi'$, the vector state null sets, is not vacuous. We discuss an example in which $\mathfrak{N}_\phi$ is properly smaller than $\mathfrak{N}_\phi'$.

EXAMPLE 5.2.4. Let $\mathfrak{B}$ be the algebra of all bounded operators on the Hilbert space $\mathfrak{IC}$, and let X be the full state space of $\mathfrak{B}$ with $\mathfrak{L}$ the representing function

system for $\mathfrak{B}$ on X and ϕ the canonical isomorphism of $\mathfrak{L}$ onto $\mathfrak{B}$. Each point p corresponds to a state ω_p of $\mathfrak{B}$ and ω_p is a normal or vector state of $\mathfrak{B}$ if and only if $\{p\}$ is not in $\mathfrak{N}_\phi$ or $\mathfrak{N}'_\phi$, respectively, by Lemma 2.2.2. Thus, if p corresponds to a normal state of $\mathfrak{B}$ which is not a vector state, for example, half the sum of two vector states corresponding to orthogonal unit vectors, then $\{p\}$ lies in $\mathfrak{N}'_\phi$ but not in $\mathfrak{N}_\phi$.

We present an example, promised in the introduction, of two C^*-algebras which are isomorphic and have unitarily equivalent weak closures but which are not themselves unitarily equivalent. Such an example underscores the contention that a set of unitary invariants for C^*-algebras will necessarily be more detailed than such a set for rings of operators. From the preceding developments we see that there are actually two possible types of example for which we can ask. We can ask for two unitarily-inequivalent representations of a C^*-algebra as concrete C^*-algebras with unitarily-equivalent weak closures or we can ask for two isomorphic, unitarily-inequivalent (concretely-represented) C^*-algebras with unitarily-equivalent weak closures. This second question can be expressed in our representation terminology as follows. We ask for two isomorphisms ϕ_1, ϕ_2 of an abstract C^*-algebra $\mathfrak{A}$ as concrete C^*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively, acting on Hilbert spaces $\mathfrak{K}_1$, $\mathfrak{K}_2$, respectively, such that $\mathfrak{A}_1^-$ is unitarily equivalent to $\mathfrak{A}_2^-$ and such that ϕ_1 is unitarily equivalent to no isomorphic representation of $\mathfrak{A}$ as $\mathfrak{A}_2$. Indeed, having produced an example of the second type, we can let $\mathfrak{A}$ stand for the abstract C^*-algebra whose elements are the operators in one of the C^*-algebras with ϕ_1 as the "identity isomorphism" and ϕ_2 the given algebraic isomorphism, so that, if ϕ_1 is unitarily equivalent to some isomorphic representation of $\mathfrak{A}$ as $\mathfrak{A}_2$ then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unitarily equivalent contrary to assumption. On the other hand, if we have the representations ϕ_1, ϕ_2 described above, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unitarily inequivalent, for a unitary equivalence ϕ of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ gives rise to the isomorphic representation $\phi\phi_1$ of $\mathfrak{A}$ as $\mathfrak{A}_2$ which is unitarily equivalent to ϕ_1, contrary to construction.

Now, if the C^*-algebra $\mathfrak{A}_1$ is unitarily equivalent to $\mathfrak{A}_2$ then it follows from elementary considerations that there is a homeomorphism γ of X, the pure state space of $\mathfrak{A}$, onto itself, which induces a map on $C(X)$ carrying $\mathfrak{L}$, the representing function system for $\mathfrak{A}$, onto itself and such that $\gamma f_{\phi_2} = f_{\phi_1}$. In fact, if ϕ is the unitary equivalence of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ then $\phi_2^{-1}\phi\phi_1$ is an automorphism of $\mathfrak{A}$ which induces the desired homeomorphism γ. On the other hand, if there exists a homeomorphism γ of the type just described then, according to [17], γ induces a C^*-automorphism ϕ of $\mathfrak{A}$ such that $f_{\phi_1} = f_{\phi_1\phi}$, so that ϕ_1 and $\phi_1\phi$ are semi-unitarily equivalent (see §2.4). To get unitarily equivalent algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ rather than semi-unitarily equivalent ones, we can make the algebraic assumption that $\mathfrak{A}$ admits an automorphism inducing a homeomorphism such as γ.

After these preliminaries, we proceed to our example, which consists of two concrete abelian C^*-algebras on separable Hilbert spaces whose weak closures are maximal abelian algebras with pure point spectra (i.e., each is the algebra of all diagonal matrices relative to some orthonormal basis).

EXAMPLE 5.2.5. Let X be the compact Hausdorff space consisting of the points $\{1/n, 0\}_{n=1,2,\ldots}$ in the usual metric topology, and let $\mathfrak{A}$ be the abstract (commutative) C^*-algebra $C(X)$ (i.e., the algebra of convergent sequences). We define two measures on X. The first μ_1 is determined by assigning to the point $1/n$ the measure $6/\pi^2 n^2$ and to the point 0, the measure 0, so that $\mu_1(X) = 1$. Note that each point of X is both a closed and measurable set and that each subset of X is a countable union of single-point sets, so that all subsets of X are Borel and μ_1-measurable. Moreover, μ_1 is clearly a regular Borel measure. The measure μ_2 defined by $\mu_2(1/n) = 6/\pi^2 n^2$ for $n = 2, 3, \cdots$ and $\mu_2(0) = \mu_2(1) = 3/\pi^2$, enjoys these same properties. Let ϕ_1, ϕ_2 be the isomorphic representations of $\mathfrak{A}$ which associate with each continuous function on X the multiplication operators (cf. Example 5.2.1) on $L_2(X, \mu_1)$ and $L_2(X, \mu_2)$ respectively, corresponding to this function. Now $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$ are the algebras of multiplication operators corresponding to essentially-bounded (measurable) functions on (X, μ_1) and (X, μ_2), respectively. Thus $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$ are unitarily equivalent as follows by mapping the natural basis for $L_2(X, \mu_1)$ upon the natural basis for $L_2(X, \mu_2)$, or from the general theorem of Segal [40], since in one case the measure algebra is the Boolean σ-algebra of all subsets of X modulo an ideal consisting of the null set and a set with one point, and in the other, the σ-isomorphic Boolean σ-algebra of subsets of X.

On the other hand, according to our preliminary remarks, we need only show that no homeomorphism of X can carry f_{ϕ_1} onto f_{ϕ_2}, in order to show that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are not unitarily equivalent. Since $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$ are maximal abelian, that is $\mathfrak{A}_1^- = \mathfrak{A}_1'$, $\mathfrak{A}_2^- = \mathfrak{A}_2'$, all are of type I_1 and the discussion of §5.1 shows us that f_{ϕ_1} and f_{ϕ_2} are equal to the ideal of all Borel subsets of X at each point less than 1 and equal to $\mathfrak{N}_{\phi_1}$, $\mathfrak{N}_{\phi_2}$, respectively, at each point not less than 1. (We need not consider ideal bands since $\mathfrak{A}_1^-$ and $\mathfrak{A}_2^-$ are countably-decomposable.) Our task then is to determine $\mathfrak{N}_{\phi_1}$, $\mathfrak{N}_{\phi_2}$ and show that no homeomorphism of X carries $\mathfrak{N}_{\phi_1}$ onto $\mathfrak{N}_{\phi_2}$. We show that $\mathfrak{N}_{\phi_1}$, $\mathfrak{N}_{\phi_2}$ are the ideals of measure 0 sets relative to μ_1, μ_2, respectively, i.e., $\{\emptyset, (0)\}$ and $\{\emptyset\}$. Since the vector states due to the function 1 (in $L_2(X, \mu_1)$ and $L_2(X, \mu_2)$) induces μ_1 and μ_2 integration, respectively, on $C(X)$, the null ideals of these states are precisely $\{\emptyset, (0)\}$ and $\{\emptyset\}$, respectively, so that $\mathfrak{N}_{\phi_1} \subset \{\emptyset, (0)\}$, $\mathfrak{N}_{\phi_2} \subset \{\emptyset\}$. On the other hand, if h_n is the function which is 1 at $1/m$, $m \geq n$; 1 at 0 and 0 elsewhere on X, then $h_n \geq \chi_0$, where χ_0 is the characteristic function of the set (0), and h_n is in $C(X)$. Now $\mathfrak{A}_1^-$ has the separating vector 1, so that each normal state of $\mathfrak{A}_1^-$ is a vector state [3]. Let us take such a state arising from the function f with $L_2(X, \mu_1)$-norm 1 and consider the measure μ which this state induces upon X. From the preceding remarks

$$0 \leq \mu(0) \leq \inf_n\{(\phi_1(h_n)f, f)\} \leq \inf_n \sum_{m=n}^{\infty} |(f(1/m))/m|^2 = 0,$$

since $\sum_{m=1}^{\infty} |(f(1/m))/m|^2 = \pi^2/6$. Thus (0) is in $\mathfrak{N}_{\phi_1}$, and $\mathfrak{N}_{\phi_1} = \{\emptyset, (0)\}$. Clearly $\emptyset$ is in $\mathfrak{N}_{\phi_2}$, so that $\mathfrak{N}_{\phi_2} = \{\emptyset\}$, and, of course, no homeomorphism of X carries $\mathfrak{N}_{\phi_1}$ onto $\mathfrak{N}_{\phi_2}$.

Concerning this example, we may remark that there is no difficulty in establishing the equality of $\mathfrak{N}_\phi$ and $\mathfrak{N}_\phi'$ in the case of representations of commutative

C^*-algebras. This example illustrates a general technique for constructing a class of such examples. We introduce regular Borel measures μ_1, μ_2 on the compact-Hausdorff space X with null sets $\mathfrak{N}_1$, $\mathfrak{N}_2$, respectively, whose measure algebras are isomorphic and such that no homeomorphism of X onto itself carries $\mathfrak{N}_1$ onto $\mathfrak{N}_2$. Then the multiplication representations of $C(X)$ on $L_2(X, \mu_1)$ and $L_2(X, \mu_2)$ are unitarily inequivalent but have unitarily equivalent weak closures.

5.3. Applications

Many investigations in modern physics and modern analysis are concerned with the unitary classification of representations of certain structures such as groups, Lie algebras, commutation relations, etc., by self-adjoint families of operators acting on Hilbert spaces. We hope to have developed a general framework around which to build such investigations, in the preceding chapters. One will expect to take advantage of the special features of the particular structure being studied to locate the appropriate representing function system and associated family of null ideal bands—these invariants being described in the natural parameters of the structure. It is in this area that we envisage the main applications of the theory. In the present section, we describe a certain abstract application of our general theory—viz., the unitary classification of an operator (not necessarily normal) acting upon a Hilbert space.

The question of classifying non-normal operators being a focal point of interest in the subject of operator theory and being a question which is not well-set, it seems worthwhile to begin with some comments concerning the problem. The natural criteria of easy and effective general computability of the invariants involved will not do for determining the acceptability of a purported solution, for the problem is usefully (abstractly) settled for self-adjoint operators while even the spectrum of a particular self-adjoint operator is not easily or effectively computable—much less so, the multiplicity structure on the spectrum as related to its action on the underlying Hilbert space. The solution we present reduces to the classical solution when applied to the case of a self-adjoint operator and has components which are natural extensions of those appearing in that classical solution.

We begin with the trivial observation that the problem of unitary equivalence of non-normal operators is the same as the problem of (simultaneous) unitary equivalence of pairs of self-adjoint operators (simply consider the unique decomposition of the given non-normal operator as a sum of a self-adjoint and skew-adjoint operator). Suppose then that A_1, B_1 and A_2, B_2 are pairs of self-adjoint operators acting on the Hilbert spaces $\mathfrak{IC}_1$ and $\mathfrak{IC}_2$, respectively, and that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the C^*-algebras they generate. If the pair A_1, B_1 is (simultaneously) unitarily equivalent to the pair A_2, B_2, certainly $p(A_1, B_1) = 0$ implies

$$p(A_2, B_2) = 0,$$

with p a (non-commutative) polynomial in two variables, and, since the unitary operator effecting the equivalence carries $p(A_1, B_1)$ onto $p(A_2, B_2)$, the map-

ping ϕ' defined by $\phi'[p(A_1, B_1)] = p(A_2, B_2)$ extends to a *-isomorphism ϕ of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. This can be concisely formulated by requiring that

$$\| p(A_1, B_1) \| = \| p(A_2, B_2) \|,$$

for each polynomial p. The condition stated corresponds to the (algebraic) assumption for single, self-adjoint operators that their spectra be identical. (If A and B are the self-adjoint operators in question, the assumption that their spectra be identical is precisely the assumption that the C^*-algebras they generate are isomorphic via a mapping which carries A onto B.) Taking $\mathfrak{A}$ to be the abstract C^*-algebra whose elements are the elements of $\mathfrak{A}_1$, ϕ_1 to be the "identity" representation of $\mathfrak{A}$ upon $\mathfrak{A}_1$, and ϕ_2 to be $\phi\phi_1$ (where ϕ is the isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ described above), we find that f_{ϕ_1} and f_{ϕ_2} (the assignment of null ideal bands in the pure state space of $\mathfrak{A}$ to non-negative reals and infinite cardinals) are the unitary invariants in question—i.e., the pair A_1, B_1 is (simultaneously) unitarily equivalent to A_2, B_2 if f_{ϕ_1} is equivalent to f_{ϕ_2} and only if f_{ϕ_1} is identical with f_{ϕ_2}. Of course, f_{ϕ_1}, f_{ϕ_2} correspond to the multiplicity functions on the spectra in the case of a single self-adjoint operator.

There is nothing in the preceding discussion which prevents the obvious extension from pairs to arbitrary sets of self-adjoint operators. The major distinction between the unitary classification of self-adjoint and non-normal operators lies in the algebraic portion of the classification. The spectrum is a canonical form for the algebraic system comprised of the C^*-algebra generated by a self-adjoint operator with that operator distinguished in the algebra. We do not have such a canonical form for the corresponding situation in the case of two non-commuting self-adjoint operators.

5.4. Computing null ideals

In this section we deal with matters related to the actual computation of null ideals of representations. The abelian case presents the most favorable circumstances for computing null sets—in the countably-decomposable case, the null sets of a single vector state are those of the representation. We begin by sketching a proof of this classical fact.

To recall—if ϕ is a representation of $C(X)$ as a C^*-algebra of operators $\mathfrak{A}$ acting on the Hilbert space $\mathcal{K}$, then there is a Borel (projection) measure on X,

$$S \rightarrow \mathcal{O}(S),$$ whose range consists of projections in $\mathfrak{A}^-$ such that $\phi(f) = \int_X f \, d\mathcal{O}$

for each f in $C(X)$. (This is a variation of the decomposition theorem for unitary representations of locally compact abelian groups [1, 37, 48] and may be found in the form stated in [24].) If $\mathfrak{A}^-$ is countably-decomposable, then, by Lemma 3.3.1 (cf. [32]), there is a separating vector x for $\mathfrak{A}^-$. The definition of $\mathcal{O}$ is such that the regular Borel measure induced by $\omega_y\phi$ on X is $S \rightarrow (\mathcal{O}(S)y, y)$ for each vector y in $\mathcal{K}$. In particular, and since x is a separating vector for $\mathfrak{A}^-$, it follows that $\mathfrak{N}_x$ coincides with null sets of $\mathcal{O}$. Thus, if S lies in $\mathfrak{N}_x$, $(\mathcal{O}(S)y, y) = 0$ and S lies in $\mathfrak{N}_y$, so that $\mathfrak{N}_x = \mathfrak{N}'_\phi$. With $\mathfrak{A}^-$ countably-decomposable, each normal state is a vector state [3], whence $\mathfrak{N}_\phi = \mathfrak{N}'_\phi$.

Of course one wonders now if anything as fortunate occurs in the general case; i.e., if ϕ is a representation of a function system $(\mathcal{L}, X)$ as a concrete operator system $(\mathfrak{A}, \mathfrak{K})$ such that $\mathfrak{A}^-$ has a separating vector x is it true that $\mathfrak{N}_\phi = \mathfrak{N}_x$. This must be answered negatively, even with $\mathfrak{A}$ commutative, as we shall show. This indicates that the question, although interesting, from a computational viewpoint, is not fairly set; for, in the commutative case, the substance of the preceding paragraph is that it is "approximately" correct that the null sets of some one vector are all that need be computed. The missing condition is the "minimality" of the representing function system; i.e., the question above should be asked with X the pure state space of $\mathfrak{A}$ and ϕ a representation of $\mathfrak{A}$ composed with the canonical isomorphism. In this case and with $\mathfrak{A}$ non-commutative the question must still be answered negatively, as we shall show in the example following our commutative example.

For our examples, we note that if the points p and q in X correspond to normal states ω_p and ω_q; and if $\{p\}$ is not a permanent null set of ω_q, then ω_p is dominated by a positive multiple of ω_q. In any event, $\{p\}$ is not a permanent null set of ω_p. (These facts follow from the comments in the proof of Lemma 2.2.2.) Let $\mathfrak{K}$ be $L_2(0, 1)$ under Lebesgue measure; let $\mathfrak{A}$ be the algebra of multiplications by essentially bounded measurable functions; let X be the full state space of $\mathfrak{A}$, $\mathcal{L}$ its representing function system on X and ϕ the canonical isomorphism of $\mathcal{L}$ onto $\mathfrak{A}$. Let $(f(x) = (2x^{\frac{1}{2}})^{-\frac{1}{2}}$, for $0 < x \leq 1$, so that f is a unit vector in $\mathfrak{K}$. The constant function 1 is a separating vector for $\mathfrak{A}$, however $\mathfrak{N}_f$ does not contain $\mathfrak{N}_1$. Since both $\omega_1\phi$ and $\omega_f\phi$ correspond to points of X, this comment will have been established, according to our initial remark, when we show that $\omega_f \leq K\omega_1$ holds for no K (in which case $\omega_f\phi$, as a point of X, is a null set of $\omega_1\phi$ but not of $\omega_f\phi$). Suppose $K > 0$ is given, and let A be the multiplication operator corresponding to the characteristic function of the interval $[0, \frac{1}{4}K^2]$. Then

$$\omega_f(A) = (Af, f) = \tfrac{1}{2}K > \tfrac{1}{4}K = K\omega_1(A),$$

whence $\omega_f \nleq K\omega_1$, since A is a positive operator in $\mathfrak{A}$.

Our non-commutative example will employ the free group $\mathfrak{G}$ on two generators a and b and deal with the uniform closure $\mathfrak{A}_1$ of the set of operators $\mathfrak{A}_0$ arising from the left-convolution action of functions in $L_1(\mathfrak{G})$ on $L_2(\mathfrak{G})$. Assigning to each operator in $\mathfrak{A}_0$ the supremum of the norms of its images under each *-representation of $\mathfrak{A}_0$, we arrive at a normed *-algebra whose completion $\mathfrak{A}$ is a C^*-algebra with the property that each *-representation of $\mathfrak{A}_0$ is extendable to such a representation of $\mathfrak{A}$ (since each such representation is norm-decreasing by definition of the norm) [38; Theorem 1, p. 49]. In particular, let ϕ be the extension to $\mathfrak{A}$ of the "identity" representation ϕ' of $\mathfrak{A}_0$ as $\mathfrak{A}_0$ acting on $\mathfrak{K}$ ($= L_2(\mathfrak{G})$), so that $\phi(\mathfrak{A}) = \mathfrak{A}_1$, since the image of a *-representation of a C^*-algebra is uniformly closed [44]. According to [50], there is a *-representation ψ' of $\mathfrak{A}_0$ as an algebra of operators $\mathfrak{A}_2$ acting irreducibly on $\mathfrak{K}_2$ such that the family of states of the form $\omega_x\psi'$, x a unit vector in $\mathfrak{K}_2$, of $\mathfrak{A}_0$ is (w^*) dense in the family of vector states of $\mathfrak{A}_0$. Let ψ be the extension of ψ' to $\mathfrak{A}$. Since $\mathfrak{A}_2$ acts irreducibly on $\mathfrak{K}_2$, ω_x is a pure state of $\mathfrak{A}_2$, whence $\omega_x\psi$ is a pure state of $\mathfrak{A}$. With y a unit vector in

$\mathcal{K}$, it follows now that $\omega_y\phi$ is a limit of the pure states of $\mathfrak{A}$ of the form $\omega_x\psi$. In fact, let $\varepsilon > 0$ and $A'_1, \cdots, A'_n$ in $\mathfrak{A}$ be given, and choose $A_1, \cdots, A_n$ in $\phi^{-1}(\mathfrak{A}_0)$ such that $\| A'_i - A_i \| < \varepsilon/3$, $i = 1, \cdots, n$. Let x be a unit vector in $\mathcal{K}_2$ such that $| \omega_x\psi'[\phi(A_i)] - \omega_y[\phi(A_i)] | < \varepsilon/3$, $i = 1, \cdots, n$. Now $\psi'\phi(A_i) = \psi(A_i)$, $i = 1, \cdots, n$; so that

$$| \omega_x\psi(A'_i) - \omega_y\phi(A'_i) | \leqq | \omega_x\psi(A'_i) - \omega_x\psi'[\phi(A_i)] |$$

$$+ | \omega_x\psi'[\phi(A_i)] - \omega_y\phi(A_i) | + | \omega_y\phi(A_i) - \omega_y\phi(A'_i) | < \varepsilon,$$

which establishes our assertion.

We consider ϕ now as a mapping of the representing function system $\mathfrak{L}$ on the pure state space X of $\mathfrak{A}$, and draw from the foregoing computation the fact that $\omega_y\phi$ appears as a point of X. According to our initial remarks, if we locate a separating vector x and a vector y in $\mathcal{K}$ such that no positive multiple of ω_x majorizes ω_y then $\{\omega_y\phi\}$ is a null set of $\omega_x\phi$ but not of $\omega_y\phi$; whence, our example of a "minimal" representation with a separating vector whose null sets are not those of the representation. As x we choose the function which is 1 at the identity of $\mathfrak{G}$ and 0 elsewhere (a generating and separating trace vector for the factor of type II_1, $\mathfrak{A}_1^-$, [30]). Let $\mathfrak{A}$ be the (maximal) abelian subring of $\mathfrak{A}_1^-$ generated by the left-translation operator on $L_2(\mathfrak{G})$ due to a, and let E' be the projection on $[\mathfrak{A}x]$. Then $\mathfrak{A}E'$ restricted to $[\mathfrak{A}x]$ is maximal abelian and unitarily equivalent to the $\mathfrak{A}$ of our preceding commutative example. Indeed, identifying the group element a^n with the integer n, the Fourier-Plancherel transform of the square summable sequences carries $[\mathfrak{A}x]$ unitarily onto L_2 of the circle under Haar-Lebesgue measure and carries $\mathfrak{A}$ onto the multiplication algebra of this measure—which algebra is clearly unitarily equivalent to the algebra $\mathfrak{A}$ of our commutative example). Moreover, this unitary transformation carries x onto the constant function 1, so that there is a unit vector y which maps onto $(2x^{\frac{1}{2}})^{-\frac{1}{2}}$ of that example. It follows at once that no multiple of ω_x majorizes ω_y, since this is true of the corresponding vector states in our commutative example, and our construction is complete.

Expressed in slightly different terms, the desirable situation, from the computational viewpoint, is the one in which equivalent normal states ρ and τ of $\mathfrak{A}^-$ acting on $\mathcal{K}$ have identical families of null sets relative to the representation ϕ of $(\mathfrak{L}, X)$ as $\mathfrak{A}$ (where "equivalence" means that $\rho(A) = 0$, for a positive operator A in $\mathfrak{A}^-$ if and only if $\tau(A) = 0$). We have seen that we cannot expect this in general. Nevertheless, several positive results in this direction hold. In the first, we establish a converse to this.

THEOREM 5.4.1. *If ϕ is an order-homomorphism of the function system $(\mathfrak{L}, X)$ as a concrete operator system $(\mathfrak{A}, \mathcal{K})$ and ρ, τ are normal states of $\mathfrak{A}^-$ such that $\mathfrak{N}_{\rho\phi} = \mathfrak{N}_{\tau\phi}$, then ρ and τ are equivalent on $\mathfrak{A}^-$.*

PROOF. Let G be the complement of the union of all projections in $\mathfrak{A}^-$ which are annihilated by both ρ and τ. If $\{G_\alpha\}$ is an orthogonal family of projections in $\mathfrak{A}^-$ with sum G, then $\rho(G) + \tau(G) = \sum_\alpha (\rho(G_\alpha) + \tau(G_\alpha))$ and $\rho(G_\alpha) =$

$\tau(G_\alpha) = 0$ for all but a countable number of G_α. If G_α is annihilated by ρ and τ, however, it is orthogonal to G (and in G), hence equal to 0. Thus G is countably-decomposable in $\mathfrak{A}^-$.

From The Extension Theorem, it follows that $G\mathfrak{A}_s^m G = G\mathfrak{A}^- G$ and, indeed, that if A' is a positive operator in $\mathfrak{A}^-$ there exists a positive operator S in $\mathfrak{A}_s^m$ such that $GSG = GA'G$ (using the notation of the proof of that theorem). Thus, if we have the equivalence of ρ and τ on $\mathfrak{A}_s^m$, and $0 = \tau(A') = \tau(GA'G) = \tau(GSG) = \tau(S)$, then $0 = \rho(S) = \rho(GSG) = \rho(GA'G) = \rho(A')$; and we have the equivalence of ρ and τ on $\mathfrak{A}^-$. Suppose then that A is a positive operator in $\mathfrak{A}_s^m$ and that $\tau(A) = 0$. From the proof of The Extension Theorem, there is a unique positive $\tilde{f}$ in $\tilde{L}^m$ such that $\tilde{\phi}(\tilde{f}) = A$, and $\tau\phi^0$ has a unique extension $\tau\tilde{\phi}$ from $\tilde{L}$ to $\tilde{L}^m$ (since τ is normal). Thus $\tau\tilde{\phi}(\tilde{f}) = 0$, and if S' is the set of points p in X such that $f(x) > 0$, then S' lies in $\mathfrak{N}_{\tau\phi}$ and hence in $\mathfrak{N}_{\rho\phi}$. It follows that $0 = \rho\tilde{\phi}(\tilde{f}) = \rho(A)$, and the proof is complete.

Our next result indicates that the computation of null sets may be restricted to vectors which generate maximal, cyclic projections.

THEOREM 5.4.2. *If ϕ is an order-representation of the function system $(\mathfrak{L}, X)$ as a concrete operator system $(\mathfrak{A}, \mathfrak{K})$, then each null ideal $\mathfrak{N}_y$ contains a null ideal $\mathfrak{N}_x$ for which $[\mathfrak{A}'x]$ is a maximal, cyclic projection in $\mathfrak{A}^-$, so that $\mathfrak{N}'_\phi$ is the intersection of such null ideals, where $\mathfrak{A}^-$ is assumed to have a countably-decomposable center.*

PROOF. We shall show that there exists a vector x, with $[\mathfrak{A}'x]$ maximal cyclic in $\mathfrak{A}^-$ and such that $B'x = y$ for some B' in $\mathfrak{A}'$. Suppose, for the moment, that we have demonstrated this fact. Then, with f a positive function in $\mathfrak{L}$,

$$(\phi(f)y, y) = (B'^*B'\phi(f)x, x) \leq \| B' \|^2 (\phi(f)x, x),$$

so that $\omega_y\phi \leq \| B' \|^2 \omega_x\phi$. It follows that each extension of $\omega_y\phi$ to $C(X)$ is majorized by some extension of $\| B' \|^2 \omega_x\phi$, which is the $\| B' \|^2$ multiple of an extension of $\omega_x\phi$ (all extensions being positive, of course). Thus the family of null sets of the regular Borel measure induced by the first extension contains the family of null sets corresponding to the second extension, from which it follows that $\mathfrak{N}_x$ is contained in $\mathfrak{N}_y$.

It remains to show that an x and B' of the described type exist. Let $E = [\mathfrak{A}'y]$ and let $[\mathfrak{A}'z]$ be a maximal, cyclic projection containing $[\mathfrak{A}'y]$, the existence of which is guaranteed by Lemma 3.3.7. According to [28; Lemma 9.2.1], there exists a bounded operator A' in, and a closed, densely-defined operator T' affiliated with, the ring $\mathfrak{A}'$ such that $y = A'T'z$. By [28; §4.4, 36; Theorem 7], T' has a polar decomposition, $T' = U'H'$, with U' a partial isometry in $\mathfrak{A}'$ and H' a positive hypermaximal operator affiliated with $\mathfrak{A}'$. We can write $y = A'U'H'z = C'H'z$, with $C' = A'U'$, a bounded operator in $\mathfrak{A}'$. If we can find a hypermaximal operator K', affiliated with $\mathfrak{A}'$, which has a densely-defined (hypermaximal) inverse K'^{-1} with z in its domain, affiliated with $\mathfrak{A}'$ and such that $H'K'$ is bounded on a dense linear manifold, then $y = C'H'K'K'^{-1}z$, with $C'H'K' = B'$ a bounded operator in $\mathfrak{A}'$. Now $K'^{-1}z (= x)$ generates a cyclic

projection F' in $\mathfrak{A}'$ such that $G' \sim F'$, where $G' = [\mathfrak{A}z]$, by [28; Lemma 9.3.1], since $x = K'^{-1}z$ and $z = K'x$. Thus, by [28; Lemma 9.3.3], $[\mathfrak{A}'x] \sim [\mathfrak{A}'z]$, and $[\mathfrak{A}'x]$ is a maximal, cyclic projection in $\mathfrak{A}^-$, so that $y = B'x$ is the desired representation of y.

To construct an operator K' with the properties noted, let f be the function defined on the non-negative real axis as, 1 on the interval $[0, 1]$ and $1/x$ for $x \geqq 1$; and let K' be $f(H')$. If g is the function which is 1 on $[0, 1]$ and x for $x \geqq 1$, then $g(H')$ is inverse to K'. Clearly $H'K'$ is bounded and $g(H')(= K'^{-1})$ has z in its domain, since $g(H') = E_1 + (I - E_1)H'$, and z is in the domain of H', where E_1 is the spectral projection for H' corresponding to the interval $[0, 1]$. The proof is complete.

To aid in the computation of null sets, we may also take advantage of the comment established at the beginning of the preceding proof; viz., if $y = B'x$ with B' in $\mathfrak{A}'$, then only the null sets of x need be taken into account.

While the null sets of equivalent vector states need not agree, a very helpful limitation of the computation necessary, may be supplied in certain cases by our next result.

THEOREM 5.4.3. *If ϕ is an order-homomorphism of the function system $(\mathfrak{L}, X)$ onto the concrete operator system $(\mathfrak{A}, \mathfrak{IC})$ and x and y are unit vectors in $\mathfrak{IC}$ which induce equivalent states ω_x and ω_y, respectively, of $\mathfrak{A}^-$, then $\mathfrak{N}_{\phi|E'} = \mathfrak{N}_{\phi|F'}$ and $\mathfrak{N}'_{\phi|E'} = \mathfrak{N}'_{\phi|F'}$, where $E' = [\mathfrak{A}x]$ and $F' = [\mathfrak{A}y]$.*

PROOF. If $\psi(AE') = AF'$ for each A in $\mathfrak{A}^-$ then ψ is a *-isomorphism of $\mathfrak{A}^-E'$ onto $\mathfrak{A}^-F'$, by Lemma 3.1.3; for $C_{E'} = C_{F'}$, by Lemma 3.3.1, since $[\mathfrak{A}'x] = [\mathfrak{A}'y]$ (an obvious corollary of the equivalence of ω_x and ω_y). Moreover, ψ is weakly bicontinuous on the unit spheres of $\mathfrak{A}^-E'$ and $\mathfrak{A}^-F'$ (since ψ preserves unions of orthogonal sums of projections and, hence, carries completely additive states onto completely additive states), so that $\mathfrak{N}_{\phi|E'} = \mathfrak{N}_{\phi|F'}$, from The Extension Theorem (of course, $\psi\phi \mid E' = \phi \mid F'$, by definition of ψ).

The rings $\mathfrak{A}^-E'$ and $\mathfrak{A}^-F'$ restricted to $E'\mathfrak{IC} = [\mathfrak{A}x]$ and $F'\mathfrak{IC} = [\mathfrak{A}y]$, respectively, have x and y as generating vectors with $E'\mathfrak{A}'E'$ and $F'\mathfrak{A}'F'$ as commutants, respectively. Thus by [28; Lemma 9.3.3], $[E'\mathfrak{A}'x]$ $(= E'[\mathfrak{A}'x])$ and $[F'\mathfrak{A}'y]$ $(= F'[\mathfrak{A}'y])$ are maximal, cyclic projections in $\mathfrak{A}^-E'$ and $\mathfrak{A}^-F'$, respectively. Now ψ preserves equivalence of projections and maps $[E'\mathfrak{A}'x]$ onto $[F'\mathfrak{A}'y]$, by definition, so that ψ preserves maximal cyclicity (since maximal, cyclic projections are equivalent). Lemma 4.1.6 is now applicable, and ψ is unitarily implemented. In particular, $\mathfrak{N}'_{\phi|E'} = \mathfrak{N}'_{\phi|F'}$, and the proof is complete.

We conclude with a derivation of the canonical vector null sets of the algebra of all bounded operators $\mathfrak{B}$ acting on some Hilbert space $\mathfrak{IC}$. From §2.3, we know that the pure state space of $\mathfrak{B}$ and the closure of the set of vector states coincide, since $\mathfrak{B}$ acts irreducibly upon $\mathfrak{IC}$. Let X be the pure state space of $\mathfrak{B}$, $\mathfrak{L}$ its representing function system and ϕ the canonical isomorphism of $\mathfrak{L}$ upon $\mathfrak{B}$. If ρ is a pure state of $\mathfrak{B}$ which does not annihilate the set of completely continuous operators $\mathfrak{C}$ then the representation of $\mathfrak{B}$ induced by ρ is an irreducible *-isomor-

phism, hence weakly bicontinuous on the unit sphere of $\mathfrak{B}$ (as follows from the special nature of $\mathfrak{B}$), and ρ is a vector state. On the other hand, a vector state of $\mathfrak{B}$ which lies in X is not a vector null set of ϕ. Let X_1 be the complement of the set of vector states in X. We assert that $\mathfrak{N}'_\phi$ is the ideal of Borel subsets of X in X_1. In fact if S is such a subset, its measure relative to any state extension of a vector state from $\mathcal{L}$ to $C(X)$ is the supremum of the measures of its closed subsets, by regularity of the measures involved. It suffices, therefore, to deal with the case in which S is closed (hence w^*-compact). If S is not in $\mathfrak{N}'_\phi$, then, according to Lemma 2.2.2, there is a unit vector x such that $\omega(f) = \omega(f\chi_S)$, for each bounded Borel function f, where χ_S is the characteristic function of S and ω is the regular integration process induced by some state extension of $\omega_x\phi$ from $\mathcal{L}$ to $C(X)$. If a state of $\mathfrak{B}$ takes the value 1 at E, the one-dimensional projection with x in its range, then this state is ω_x (since it has the same null space as ω_x). Thus $\phi^{-1}(E)$ takes its maximum α on S and α is less than 1. Of course $\omega[\phi^{-1}(E)] = \omega[\phi^{-1}(E)\chi_S] \leqq \alpha < 1$, while $\omega_x\phi[\phi^{-1}(E)] = \omega_x(E) = 1$. It follows that S lies in $\mathfrak{N}'_\phi$, and $\mathfrak{N}'_\phi$ is precisely the ideal of Borel subsets of X contained in X_1.

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