If $d > 0$ and $\psi' \epsilon C_c^\infty(E^n)$, there exists $T > 0$ such that for $t \geq T$, $\psi' \epsilon C_c^\infty(t^{(v)}(N_d - x))$. We then have $|[J_{x,t}, F(\psi')] - [(2\pi)^{-n/2}/a, F(\psi')]| \leq \epsilon c_7\|F(\psi')\|_{H^1}$. Since $J_{x,t}$ has bounded norm in H^1 as $t \rightarrow \infty$, it follows that, as $t \rightarrow \infty$, $J_{x,t}$ converges weakly to $(2\pi)^{-n/2}(1/a)$ in H'. But then $[J_{x,t}, 1/a] \rightarrow (2\pi)^{-n/2}\|1/a\|^2_{H^1}$, i.e., $t^{1-v}g_{x,t}^{(p)}(I) \rightarrow (2\pi)^{-n} \int d\xi/a(\xi)$. If x_0 were a point outside N, a similar argument would show that $t^{1-v}g_{x_0,t}^{(p)}(x) \rightarrow 0$. By suitable choice of N this is true for any $x_0 \neq x$. Applying a Tauberian theorem of Hardy and Littlewood,[7] we obtain

THEOREM 1. *Let $x, y \epsilon G$. Then, as $\lambda \rightarrow +\infty$, $e_{x,\lambda}(y) = (2\pi)^{-n}w_{A_0}(x)(\delta_{x,y} + 0(1))\lambda^b$, where $b = q^{-1}(\sum_{j=1}^n m_j)$, $w_{A_0}(x) = \int_{A_0(x,\xi)<1} d\xi (\delta_{xy} = 1$ if $x = y$, 0 if $x \neq y)$.*

A_1 is said to be regular if $(A_1 + iI)^{-1}$ is compact. In the regular case $e_{x,\lambda}(y) = \sum_{\lambda_i \leq \lambda} \overline{u_j(x)}\, u_j(y)$ (u_j an orthonormal set of eigenfunctions with eigenvalues λ_j). For the general set of elliptic variational boundary-value problems for which the writer has shown the regularity of solutions on the boundary,[8] we have $t^{1-v}|g_{x,t}^{(p)}(x)| \leq f(x)$, f summable. Interchanging limits by dominated convergence, we have

THEOREM 2. *Let G be a bounded domain of class C^{pq}, A_1 defined with respect to a smooth elliptic boundary-value problem on G.[8] Then $N(z) = \sum_{\lambda_j \leq t} 1 = (1 + 0(1))(2\pi)^{-n}\{\int w_{A_0}(x)\, dx\}t^b$, as $t \rightarrow +\infty$.*

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[4] F. E. Browder, *Compt. rend. Acad. sci.* (Paris), **236**, 2140–2142, 1953.

[5] L. Gårding, *Förh. Kgl. Fysiograf. Sallskap. Lund*, Vol. **21**, No. 11, 1951.

[6] This is the class of operators discussed in F. E. Browder, these PROCEEDINGS, Vol. **42**, 234–236 (February, 1957).

[7] G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.*, **30**, 23–27, 1930.

[8] F. E. Browder, *Communs. Pure and Appl. Math.*, **9**, 351–361, 1956.

IRREDUCIBLE OPERATOR ALGEBRAS

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1. *Introduction.*—One of the classical results of the theory of operator algebras states that each C^*-algebra (operator-bound and adjoint closed algebra of operators on a Hilbert space) has a separating family of representations as irreducible C^*-algebras (those with no *closed* invariant subspaces other than (0) and the entire space)—a corollary of which is the existence of a separating family of strongly continuous, irreducible, unitary representations of a locally compact group.

In this note we shall prove that each irreducible C^*-algebra is algebraically irreducible (has no proper invariant linear manifolds—closed or otherwise) and develop some of its consequences.[1] We shall prove, in fact, the n-fold transitivity of irreducible C^*-algebras.

2. *Transitivity and Irreducibility.*—The main result is contained in

THEOREM 1. *If $\mathfrak{A}$ is a C^*-algebra acting irreducibly upon the Hilbert space $\mathfrak{H}$ and $\{x_i\}$, $\{y_i\}$ are two sets containing n vectors each, the first set consisting of linearly independent vectors, then there exists an operator A in $\mathfrak{A}$ such that $Ax_i = y_i$. In particular, $\mathfrak{A}$ acts algebraically irreducibly upon $\mathfrak{H}$. If $Bx_i = y_i$ for some self-adjoint operator B, then a self-adjoint operator A in $\mathfrak{A}$ may be chosen such that $Ax_i = y_i$.*

Proof: Choosing an orthonormal basis for the space generated by $\{x_i\}$, there will be a set of n vectors such that the class of operators mapping the basis onto this set coincides with the class mapping the x's onto the y's. It suffices, therefore, to deal with the case where $\{x_i\}$ is an orthonormal set.

We note that if $z_1, \ldots, z_n$ are vectors such that $\|z_i\| \leq r$, there is an operator B of norm not exceeding $(2n)^{1/2}r$ (and self-adjoint, in case $z_i = Tx_i$, for some self-adjoint operator T) such that $Bx_i = z_i$. In fact, let $x_1, \ldots, x_n, x_{n+1}, \ldots, x_m$ be an extension of the set $x_1, \ldots, x_n$ to an orthonormal basis of the space $\mathfrak{V}$ generated by $x_1, \ldots, x_n, z_1, \ldots, z_n$. With T and S linear transformations on $\mathfrak{V}$, the mapping which assigns trace (S^*T) to the pair (T, S) defines an inner product and hence a norm—the norm, $[T] = [\text{trace } (T^*T)]^{1/2}$, of T is $(\sum_{i,j} |\alpha_{ij}|^2)^{1/2}$, where the α_{ij}'s are the matrix coordinates of T relative to any orthonormal basis of $\mathfrak{V}$. The matrix whose first n columns (and conjugate first n rows, in the self-adjoint case) are $z_1, \ldots, z_n$—or, rather, their co-ordinate representation, relative to $x_1, \ldots, x_m$—and zeros elsewhere represents an operator B such that $Bx_i = z_i$, and $[B] \leq (2n)^{1/2}r$. It is an elementary fact, however, that $\|B\| \leq [B]$, where $\|B\|$ denotes the usual operator bound of B. Defining B to be 0 on the orthogonal complement of $\mathfrak{V}$, we have an operator with the desired properties.

We proceed now to the construction of an operator A in $\mathfrak{A}$ such that $Ax_i = y_i$. Choose B_0 such that $B_0x_i = y_i$, and A_0 in $\mathfrak{A}$ such that $\|A_0x_i - B_0x_i\| = \|A_0x_i - y_i\| \leq [2(2n)^{1/2}]^{-1}$. (Recall that since $\mathfrak{A}$ is irreducible, its strong closure is the set of all bounded operators on $\mathfrak{H}$.) Next choose B_1 such that $B_1x_i = y_i - A_0x_i$, with $\|B_1\| \leq 1/2$. Kaplansky's theorem[2] guarantees the existence of an operator A_1 in $\mathfrak{A}$ with $\|A_1\| \leq 1/2$ such that $\|A_1x_i - B_1x_i\| \leq [4(2n)^{1/2}]^{-1}$. Suppose now that B_k has been constructed with $\|B_k\| \leq 1/2^k$, $B_kx_i = y_i - A_0x_i - A_1x_i - \ldots - A_{k-1}x_i$. Choose A_k in $\mathfrak{A}$ with $\|A_k\| \leq 1/2^k$ such that $\|A_kx_i - B_kx_i\| \leq \frac{1}{2^{k+1}(2n)^{1/2}}$, and B_{k+1} with $\|B_{k+1}\| \leq 1/2^{k+1}$, $B_{k+1}x_i = y_i - A_0x_i - \ldots - A_kx_i$. Note that if $Tx_i = y_i$, for some self-adjoint operator T, the results of the preceding paragraph allow us to choose self-adjoint B_k's and hence, by Kaplansky's theorem, self-adjoint A_k's. The sum $\sum_{k=0}^{\infty} A_k$ converges in norm to an operator A in $\mathfrak{A}$, and

$$y_i - Ax_i = y_i - \sum_{k=0}^{\infty} A_kx_i = \lim_k (y_i - A_0x_i - \ldots - A_kx_i) = \lim_k B_{k+1}x_i = 0.$$

3. *States and Ideals.*—A state of a C^*-algebra is a linear functional which is 1 at the identity operator I and non-negative on positive operators. The set of states of a C^*-algebra is a convex set whose extreme points generate the set. These extreme points are called "pure states." Each state ρ of $\mathfrak{A}$ gives rise to a representation[3] φ of $\mathfrak{A}$ by means of the following process. The set of elements A in $\mathfrak{A}$ such

that $\rho(A^*A) = 0$ is a left ideal $\mathcal{J}$ in $\mathfrak{A}$—the so-called "left kernel of ρ"—it is the largest left ideal in the null space of ρ. Let $\mathcal{H}_0$ be the vector-space quotient $\mathfrak{A}/\mathcal{J}$. The mapping which assigns $\rho(B^*A)$ to the pair $(A + \mathcal{J}, B + \mathcal{J})$ is well defined and a positive-definite bilinear form on $\mathcal{H}_0$, so that the completion $\mathcal{H}$ of $\mathcal{H}_0$ relative to the metric induced by this inner product is a Hilbert space. The operator $\varphi(A)$ defined on $\mathcal{H}_0$ by $\varphi(A)[B + \mathcal{J}] = AB + \mathcal{J}$ is continuous and has a unique extension as a bounded operator on $\mathcal{H}$ (which we again denote by $\varphi(A)$). The mapping $A \to \varphi(A)$ is the desired representation of $\mathfrak{A}$ as a C^*-algebra. It is well known[3] that ρ is a pure state if and only if $\varphi(\mathfrak{A})$ is an irreducible C^*-algebra. With the notation just established, we can state

COROLLARY 1. *If ρ is a pure state, $\mathfrak{A}/\mathcal{J}$ is complete relative to the inner product induced by ρ.*

Proof: From Theorem 1, $\varphi(\mathfrak{A})$ acts algebraically irreducibly upon $\mathcal{H}$, whence $\mathcal{H}_0 = \{\varphi(A)[I + \mathcal{J}]:A \text{ in } \mathfrak{A}\} = \mathcal{H}$.

COROLLARY 2. *The null space $\mathfrak{N}$ of ρ is $\mathcal{J} + \mathcal{J}^*$ if and only if ρ is a pure state.*

Proof: If $\mathfrak{N} = \mathcal{J} + \mathcal{J}^*$ and $\rho = (\rho_1 + \rho_2)/2$, with ρ_1 and ρ_2 states of $\mathfrak{A}$, then, for A in $\mathcal{J}$, $0 = \rho(A^*A) = \rho_1(A^*A) = \rho_2(A^*A)$; and $\rho_1(A) = \rho_2(A) = 0$, by Schwarz's inequality. Thus ρ_1 and ρ_2 annihilate $\mathfrak{N}$, and $\rho_1 = \rho_2 = \rho$, so that ρ is a pure state.

Assume now that ρ is a pure state. In the Hilbert space $\mathfrak{A}/\mathcal{J}$, the subspace $\mathfrak{N} + \mathcal{J}$ is the orthogonal complement of the vector $I + \mathcal{J}$. The ideal $\mathcal{J}$ is the set of elements A in $\mathfrak{A}$ such that $\varphi(A)$ annihilates this same vector. With B in $\mathfrak{N}$, Theorem 1 tells us that there exists a self-adjoint operator $\varphi(A)$ in $\varphi(\mathfrak{A})$ (so that A may be chosen self-adjoint) which annihilates $I + \mathcal{J}$ and leaves $B + \mathcal{J}$ fixed; i.e., A lies in $\mathcal{J}$ and $AB - B = -C$, with C in $\mathcal{J}$. If B is self-adjoint, then $B = C^* + BA$, so that B lies in $\mathcal{J} + \mathcal{J}^*$. Since $\mathfrak{N} = \mathfrak{N}^*$, it follows that $\mathcal{J} + \mathcal{J}^*$ contains $\mathfrak{N}$. Clearly, $\mathfrak{N}$ contains $\mathcal{J} + \mathcal{J}^*$, whence $\mathfrak{N} = \mathcal{J} + \mathcal{J}^*$.

THEOREM 2. *The left kernel $\mathcal{J}$ of ρ is a maximal left ideal in $\mathfrak{A}$ if and only if ρ is a pure state, in which case ρ is the unique state whose null space contains $\mathcal{J}$. Each closed left ideal in $\mathfrak{A}$ is the intersection of the maximal left ideals containing it.*

Proof: If ρ is a pure state, $\varphi(\mathfrak{A})$ acts algebraically irreducibly upon $\mathfrak{A}/\mathcal{J}$, and a proper left ideal containing $\mathcal{J}$ would give rise to a proper, invariant, linear manifold in $\mathfrak{A}/\mathcal{J}$ under $\varphi(\mathfrak{A})$. Thus $\mathcal{J}$ is maximal.

If $\mathcal{L}$ is a closed left ideal and $\mathcal{K}$ a left ideal (possibly $\mathfrak{A}$) containing $\mathcal{L}$ which is annihilated by each state which annihilates $\mathcal{L}$, then $\mathcal{K} = \mathcal{L}$. In fact, with A a positive operator in $\mathcal{K}$, the set S of states at which A is not less than $1/n^2$ is w^*-compact. The hypothesis, compactness, and the definition of the w^*-topology guarantee the existence of a finite open covering $\{U_i\}$, $i = 1, \ldots, m$, of S and elements A_i in $\mathcal{L}$ such that A_i (and hence $A_i^*A_i$, by the Schwarz inequality) does not vanish on U_i. Some positive multiple T_n^2 of $A_1^*A_1 + \ldots + A_m^*A_m$ exceeds A on S, whence $T_n^2 + I/n^2 \geq A$, where T_n, the positive square root of T_n^2, being a uniform limit of polynomials in T_n^2 without constant terms, by the Weierstrass approximation theorem, lies in $\mathcal{L}$. Now

$$\|A^{1/2}(T_n + I/n)^{-1}T_n - A^{1/2}\|^2 = \|(T_n + I/n)^{-1}A(T_n + I/n)^{-1}\|/n^2 \leq \|(T_n + I/n)^{-2}(T_n^2 + I/n^2)\|/n^2 \leq 1/n^2.$$

Thus $A^{1/2}$, and hence A, lies in $\mathcal{L}$. With B an arbitrary element in $\mathcal{K}$, B^*B and $(B^*B)^{1/2}$ lie in $\mathcal{L}$, from the above, while[4]

$$\|B[(B^*B)^{1/2} + I/n]^{-1}(B^*B)^{1/2} - B\| \leq 1/n,$$

as above, so that B lies in $\mathcal{L}$ and $\mathcal{K} = \mathcal{L}$.

From the foregoing, it follows, in particular, that if $\mathcal{L}$ is a proper ideal, the set $\mathcal{S}$ of states which annihilate $\mathcal{L}$ is non-null—this set is convex and w^*-compact and so the closed, convex hull of its set $\mathcal{E}$ of extreme points. The points of $\mathcal{E}$ are trivially seen to be extreme in the set of all states, i.e., pure states, and the hull property guarantees that an operator which annihilates $\mathcal{E}$ annihilates $\mathcal{S}$. The intersection of the left kernels of states in $\mathcal{E}$ is a left ideal, containing $\mathcal{L}$ and annihilating $\mathcal{S}$, so that this intersection is $\mathcal{L}$. The first paragraph of the proof assures us, however, that these left kernels are maximal left ideals, and the last assertion of the theorem is proved.

If $\mathcal{J}$ is a maximal left ideal, there is a pure state η which annihilates it and for which it must therefore be the left kernel. Corollary 2 tells us that $\mathcal{J} + \mathcal{J}^*$ is the null space of η, hence of ρ, so that $\rho = \eta$ and the proof is complete.

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[1] Few technicians would have entertained seriously the possibility of the truth of the results contained in Theorem 1 and Corollary 1, and even the validity of the results of Corollary 2 and Theorem 2 seemed highly doubtful. R. Prosser has been developing a duality theory for C^*-algebras, by topological linear space methods, which, when complete, should put the results of Theorem 2 in its appropriately general setting. It was the convincing nature of Prosser's program which led the author to re-examination of this area and thence to the results of this note. The author wishes to record his gratitude to R. Prosser for many stimulating discussions about operator algebras.

[2] I. Kaplansky, "A Theorem on Rings of Operators," *Pacific J. Math.*, **1**, 227–232, 1951.

[3] I. Segal, "Irreducible Representations of Operator Algebras," *Bull. Am. Math. Soc.*, **53**, 73–88, 1947.

[4] In essence, this inequality establishes that a closed left ideal is generated by its positive elements—a fact which seems to have been proved first by I. Segal in "Two-sided Ideals in Operator Algebras," *Ann. Math.*, **50**, 856–865, 1949, using a similar estimate.

ZEROS OF NEIGHBORING HOLOMORPHIC FUNCTIONS

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1. *A Lemma.*—The functions $g(z)$ and $f(z)$ are holomorphic throughout the closed disk $\Delta_\rho = \{z: |z - \zeta_0| \leq \rho\}$ on which $f(z)$ has the sole zero ζ_0, and that of order $p \geq 1$. Let

$$0 < m \leq |f(\zeta_0 + \rho e^{i\theta})| \leq M \qquad (0 \leq \theta \leq 2\pi), \tag{1}$$

and define

$$\|f - g\|_\rho = \max_{z \in \Delta_\rho} |f(z) - g(z)|. \tag{2}$$

So soon as it satisfies the inequality $\|f - g\| < m$, $g(z)$ *is assured of having exactly p*