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OPERATOR ALGEBRAS WITH A FAITHFUL WEAKLY-CLOSED REPRESENTATION

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1. Introduction

The problem of finding natural conditions under which a C*-algebra (uniformly-closed, self-adjoint algebra of operators on a Hilbert space) admits a faithful representation as a ring of operators (weakly-closed C*-algebra acting upon some Hilbert space) has been one center of attention in the study of operator algebras. We cannot claim to have analyzed rings of operators algebraically before we are capable of distinguishing them among the C*-algebras. Much information has been gathered concerning this problem [1, 2, 4, 5, 7, 9, 11]. Notably, Kaplansky in [4, 5] singles out a class of C*-algebras which imitates the rings of operators algebraically (but which, as Kaplansky notes, contains C*-algebras not isomorphic to rings of operators), and carries over much of the theory of rings of operators to this class of operator algebras. Based on the results of Stone [10], Dixmier provides a thorough analysis of the commutative situation in [1]. In the next section (Theorem 1), we answer the general question. The conditions we find are certainly as simple as any set of conditions for which we could have hoped, in view of the commutative case, and, indeed, yield a formulation of the commutative result which is slightly neater (i.e., less measure-theoretic) than, though entirely equivalent to, the customary one.

2. W*-algebras

The following definition singles out the class of C*-algebras which we shall subsequently prove to be the class of C*-algebras which have a faithful, weakly-closed representation.

DEFINITION 1. *A C*-algebra $\mathfrak{A}$ in which each bounded, monotone-increasing, directed sequence of self-adjoint elements (A_γ) has a least upper bound A in $\mathfrak{A}$ and which has a separating family of states $\{\omega_\alpha\}$* (i.e., *a family such that a positive element B in $\mathfrak{A}$ is zero if $\omega_\alpha(B) = 0$ for each α*) *such that $\{\omega_\alpha(A_\gamma)\}$ has the directed limit $\omega_\alpha(A)$, for directed sequences such as (A_γ), will be called a W*-algebra. States such as ω_α will be called normal states of $\mathfrak{A}$ (following the terminology of* [1]).

A key to the proof of our main theorem is the:

LEMMA 1. *If $\mathfrak{A}$ is a C*-algebra acting on the Hilbert space $\mathcal{H}$ and the strong limit of each bounded, monotone-increasing, directed sequence of self-adjoint operators in $\mathfrak{A}$ lies in $\mathfrak{A}$ then $\mathfrak{A}$ is a ring of operators.*

PROOF. Denote by $\mathfrak{A}^-$ the weak (and, therefore, strong) closure of $\mathfrak{A}$. To prove our lemma, it suffices, by The Spectral Theorem, to show that each projection in $\mathfrak{A}^-$ lies in $\mathfrak{A}$. (Recall that $\mathfrak{A}$ is uniformly closed and that each self-adjoint operator is a uniform limit of linear combinations of its spectral projections). Working with the directed sequence obtained by subtracting each term from the identity, we observe that bounded, monotone-decreasing, directed sequences in $\mathfrak{A}$ have their strong limits in $\mathfrak{A}$. The projection on the closure of the range of an operator A in $\mathfrak{A}$ lies in $\mathfrak{A}$. (We shall call this projection "the range projection of A"). Indeed, the range projection of A is the same as the range projection of AA^* (since A^* and AA^* have the same null space). If f_n denotes the continuous, real-valued function defined by $f_n(t) = 1$ for $t \geqq 1/n$, $f_n(t) = nt$ for $0 \leqq t \leqq 1/n$, and $f_n(t) = 0$ for $t \leqq 0$, then $f_n(AA^*)$ is a monotone-increasing sequence of operators in $\mathfrak{A}$ (bounded above by I) whose strong limit is, by spectral theory, the range projection of AA^*, which, by hypothesis, lies in $\mathfrak{A}$. If E and F are projections in $\mathfrak{A}$, the range projection of $E + F$ is the union of E and F since the null space of $E + F$ is clearly the intersection of $I - E$ and $I - F$. Thus the union (and, hence, intersection) of any finite set of projections in $\mathfrak{A}$ lies in $\mathfrak{A}$. The projections obtained from the finite unions (intersections) formed from an arbitrary set of projections in $\mathfrak{A}$ yield a bounded, monotone-increasing (decreasing) directed sequence of projection in $\mathfrak{A}$ (directed by the finite subsets, partially-ordered by inclusion) whose strong limit is the union (intersection) of the given set of projections in $\mathfrak{A}$. Thus the union and intersection of an arbitrary set of projections in $\mathfrak{A}$ lie in $\mathfrak{A}$. Since each projection in $\mathfrak{A}^-$ is a union of projections in $\mathfrak{A}^-$ cyclic under the commutant of $\mathfrak{A}$, it suffices to show that each cyclic projection in $\mathfrak{A}^-$ lies in $\mathfrak{A}$. For this, it suffices to prove that for each vector in the orthogonal complement of the given cyclic projection there is a projection in $\mathfrak{A}$ annihilating this vector and containing the cyclic projection (or, equivalently, containing its generating vector).

Let E be a cyclic projection in $\mathfrak{A}^-$ with unit generating vector x, and let y be a unit vector in $I - E$. According to [6], there is a sequence of self-adjoint operators $A_1', A_2', \cdots$ in $\mathfrak{A}$ with norm not exceeding 1 such that $A_n'x$ tends strongly to Ex ($= x$) and $A_n'y$ tends strongly to Ey ($= 0$). With $\varepsilon > 0$ given, there exists a $\delta > 0$ such that if B is a self-adjoint operator, $\| B \| \leqq 2$, and $\| Bz \| < \delta$ for a unit vector z, then $\| B^+z \| < \varepsilon$, where B^+ is the 'positive part' of B (i.e., $B^+ = \frac{1}{2}(B + | B |)$ and $B^- = \frac{1}{2}(B - | B |)$ is the 'negative part' of B). In fact, with f continuous and 0 at 0, by the Weierstrass Approximation Theorem, we can choose a polynomial $p(t) = a_nt^n + \cdots + a_1t$ approximating f uniformly to within $\frac{1}{2}\varepsilon$ on $[-a, a]$. Let $k = | a_n | a^{n-1} + | a_{n-1} | a^{n-2} + \cdots + | a_1 |$. Choose δ equal to $\varepsilon/2k$. Then $\| f(B) - p(B) \| < \varepsilon$ with B self-adjoint and $\| B \| \leqq a$; so that, if $\| Bz \| < \delta$ and $\| z \| \leqq 1$, then

$$\| p(B)z \| \leqq k \| Bz \| < \tfrac{1}{2}\varepsilon,$$

and

$$\| f(B)z \| \leqq \| f(B)z - p(B)z \| + \| p(B)z \| < \varepsilon.$$

Denote by δ_n the δ corresponding to $\varepsilon = (\frac{1}{2})^n$, $f(B) = B^+$, and $a = 2$, just constructed. Choose $N_1, N_2, \cdots$ a monotone-increasing sequence of positive integers such that $\| A_n'z - A_m'z \| < \delta_h$ if $n, m \geqq N_h$, where z denotes either of the vectors x, y. Let $A_n = A'_{N_n}$, and let $B_1 = A_1$, $B_n = A_n - A_{n-1}$ for $n = 2, 3, \cdots$. We assert that $\sum B_n^+ z$ is absolutely convergent. Indeed,

$$\| B_n z \| = \| A'_{N_n} z - A'_{N_{n-1}} z \| < \delta_{n-1},$$

so that $\| B_n^+ z \| < (\frac{1}{2})^{n-1}$, which establishes the assertion. Choose A a weak limit point of $\{A_n\}$, and observe that $Ax = x$, $Ay = 0$ since $(A_n x)$ and $(A_n y)$ converge strongly (and hence, weakly) to x and 0, respectively.

Let $T_k = (I + \sum_{n=1}^k B_n^+)^{-1}$, and note that since $0 \leqq T_k^{-1} \leqq T_{k+1}^{-1}$, $0 \leqq T_{k+1} \leqq T_k$, so that the strong limit T of (T_k) lies in $\mathfrak{A}$ and is positive. For each k, $T^{\frac{1}{2}}(\sum_{n=1}^k B_n^+)T^{\frac{1}{2}} \leqq I$. In fact, for fixed z, $(T^{\frac{1}{2}}(\sum_{n=1}^k B_n^+)T^{\frac{1}{2}}z, z)$ is approximated by

$$(\{\textstyle\sum_{n=1}^k B_n^+\} (I + \sum_{n=1}^r B_n^+)^{-\frac{1}{2}}z, (I + \sum_{n=1}^r B_n^+)^{-\frac{1}{2}}z)$$
$$\leqq (\{I + \textstyle\sum_{n=1}^r B_n^+\}^{-\frac{1}{2}}(\sum_{n=1}^r B_n^+)(I + \sum_{n=1}^r B_n^+)^{-\frac{1}{2}}z, z) \leqq (z, z)$$

for all suitably large r, the last inequality following from spectral theory once we observe that $(I + \sum_{n=1}^r B_n^+)^{-\frac{1}{2}}$ and $\sum_{n=1}^r B_n^+$ commute. Moreover,

$$T^{\frac{1}{2}}(\textstyle\sum_{n=1}^k B_n^+)T^{\frac{1}{2}}$$

is monotone-increasing with k, and so has the strong limit C_1 in $\mathfrak{A}$. The same is true for $T^{\frac{1}{2}}(\sum_{n=t}^k B_n^+)T^{\frac{1}{2}}$, of course, and we denote its strong limit in $\mathfrak{A}$ by C_t. We have

$$T^{\frac{1}{2}}(\textstyle\sum_{n=1}^k B_n^+)T^{\frac{1}{2}} + C_{k+1} = C_1,$$

so that

$$C_1 + T^{\frac{1}{2}}(\textstyle\sum_{n=1}^k B_n^-)T^{\frac{1}{2}} = T^{\frac{1}{2}}(\sum_{n=1}^k B_n^+ + B_n^-)T^{\frac{1}{2}} + C_{k+1}$$
$$= T^{\frac{1}{2}}(\textstyle\sum_{n=1}^k B_n)T^{\frac{1}{2}} + C_{k+1} = T^{\frac{1}{2}}A_k T^{\frac{1}{2}} + C_{k+1}$$

which is monotone decreasing and bounded, since both $T^{\frac{1}{2}}A_k T^{\frac{1}{2}}$ and C_{k+1} are uniformly bounded with respect to k. Now $T^{\frac{1}{2}}A_k T^{\frac{1}{2}}$ has the weak limit point $T^{\frac{1}{2}}AT^{\frac{1}{2}}$ and C_t is clearly positive and monotone decreasing with t and so has a strong limit in $\mathfrak{A}$. Thus, in particular, $T^{\frac{1}{2}}A_k T^{\frac{1}{2}} + C_{k+1}$ has a strong (and hence, weak) limit in $\mathfrak{A}$ which must be the sum of $T^{\frac{1}{2}}AT^{\frac{1}{2}}$ and the strong limit of C_{k+1} (which lies in $\mathfrak{A}$), so that $T^{\frac{1}{2}}AT^{\frac{1}{2}}$ lies in $\mathfrak{A}$.

Note next that $(T(x + \sum B_n^+ x), z) = (\{x + \sum B_n^+ x\}, Tz)$ is approximated as closely as we wish by $(\{x + \sum_{n=1}^k B_n^+ x\}, (I + \sum_{n=1}^k B_n^+)^{-1}z) = (x, z)$ for each z, and $T(x + \sum B_n^+ x) = x$. Similarly $T(y + \sum B_n^+ y) = y$, so that the range projection F of T, lies in $\mathfrak{A}$ and contains x and y. If $\mathfrak{A}$ is commutative then AT lies in $\mathfrak{A}$ and has $AT(x + \sum B_n^+ x) = Ax = x$ in its range, while $ATy = TAy = 0$, so that the range projection of AT lies in $\mathfrak{A}$, contains x, and annihilates y. From the previous work then, our result follows for $\mathfrak{A}$ commutative. In particular, the maximal abelian (self-adjoint) subalgebras of $\mathfrak{A}$ are weakly closed, so that, along with each self-adjoint operator which $\mathfrak{A}$ contains, the weakly-closed algebra (spectral projections, and so forth) generated by this operator lies in $\mathfrak{A}$.

Now TAT is in $\mathfrak{A}$ as is $R_n = g_n(T)$, where g_n is the real-valued function defined by $g_n(t) = 1/t$ for $t \geqq 1/n$ and $g_n(t) = 0$ for $t < 1/n$. Thus $R_nTATR_m = F_nAF_m$ lies in $\mathfrak{A}$, where F_j is a projection in $\mathfrak{A}$ (the spectral projection for T corresponding to the half-infinite interval with left end-point $1/j$). We have that (F_m) is a monotone-increasing sequence with strong limit F. Let $G_1 = F_1$ and $G_n = F_n - F_{n-1}$ for $n = 2, 3, \cdots$, so that $\sum G_n = F$, and note that G_nAG_m lies in $\mathfrak{A}$. Now,

$$(G_mAG_n + G_n)(G_nAG_m + G_n) = G_mAG_nAG_m + G_mAG_n + G_nAG_m + G_n \geqq 0,$$

and $(G_mAG_n + G_n)(G_nAG_m + G_n)$ lies in $\mathfrak{A}$, so that

$$\begin{aligned} &\textstyle\sum_n \{G_mAG_nAG_m + G_mAG_n + G_nAG_m + G_n\} \\ &\qquad = G_mAFAG_m + G_mAF + FAG_m + F \end{aligned}$$

lies in $\mathfrak{A}$. Moreover, $G_mAFAG_m = \sum_n G_mAG_nAG_m$ lies in $\mathfrak{A}$, since $G_mAG_nAG_m$ is positive and lies in $\mathfrak{A}$. Thus $G_mAF + FAG_m$ and $G_m(G_mAF + FAG_m) = G_mAF + G_mAG_m$ lie in $\mathfrak{A}$. Since G_mAG_m is in $\mathfrak{A}$, G_mAF and hence FAG_m lie in $\mathfrak{A}$. As before, $\sum_m FAG_mAF = FAFAF$ lies in $\mathfrak{A}$. But, $FAFAFx = x$ and $FAFAFy = FAFAy = 0$, so that the range projection G of $FAFAF$ contains x, annihilates y, and lies in $\mathfrak{A}$. Thus G is a projection with the desired properties, and the proof is complete.

THEOREM 1. *A C^*-algebra $\mathfrak{A}$ has a faithful representation as a ring of operators if and only if it is a W^*-algebra.*

PROOF. Let $\{\omega_\alpha\}_{\alpha \text{ in } \Gamma}$ be the family of normal states of $\mathfrak{A}$, and let ϕ_α be the representation [3, 8] due to ω_α on the Hilbert space $\mathcal{H}_\alpha$ with wave function x_α (in $\mathcal{H}_\alpha$). Let ϕ be the direct sum of the representations $\{\phi_\alpha\}$ (ϕ represents $\mathfrak{A}$ as operators acting upon $\mathcal{H} = \sum \oplus \mathcal{H}_\alpha$ and $\phi(A)$ is defined by $\phi(A)y_\alpha = \phi_\alpha(A)y_\alpha$ for each y_α in $\mathcal{H}_\alpha$ and each α). With $\mathfrak{A}$ a W^*-algebra, the family $\{\omega_\alpha\}$ is separating, by hypothesis, so that ϕ is a $*$-isomorphism. Indeed, if $\phi(A) = 0$ then $\omega_\alpha(A^*A) = (\phi(A^*A)x_\alpha, x_\alpha) = 0$ for each α, and, since $\{\omega_\alpha\}$ is separating, A^*A, and hence A, is zero.

Suppose that $\{\phi(A_\gamma)\}$ is a bounded, monotone-increasing, directed sequence of operators in $\phi(\mathfrak{A})$. Since ϕ is a $*$-isomorphism, the same properties hold for the directed sequence $\{A_\gamma\}$ in $\mathfrak{A}$, and, by hypothesis, $\{A_\gamma\}$ has a least upper bound A in $\mathfrak{A}$. Now, by definition of 'normal state',

$$\lim_\gamma(\phi(A_\gamma)x_\alpha,\ x_\alpha) = \lim_\gamma \omega_\alpha(A_\gamma) = \omega_\alpha(A) = (\phi(A)x_\alpha,\ x_\alpha),$$

for each α. Since the map $B \to T^*BT$ is an order-isomorphism of $\mathfrak{A}$, with T an invertible element in $\mathfrak{A}$, $\{T^*A_\gamma T\}$ is a bounded, monotone-increasing, directed sequence with least upper bound T^*AT in $\mathfrak{A}$, so that

$$\lim_\gamma(\phi(A_\gamma)\phi(T)x_\alpha,\ \phi(T)x_\alpha) = (\phi(A)\phi(T)x_\alpha,\ \phi(T)x_\alpha),$$

for each α. The properties of the directed sequence $\{\phi(A_\gamma)\}$ imply that it has a strong (and hence weak) limit B, whence $(\{\phi(A) - B\}\phi(T)x_\alpha, \phi(T)x_\alpha = 0$

for each α and each invertible operator T in $\mathfrak{A}$. In particular, taking T to be I, we conclude that $(\{\phi(A) - B\}x_\alpha, x_\alpha) = 0$ for each α. With S an arbitrary operator in $\mathfrak{A}$ there is a positive integer N such that $S + nI$ is invertible for all $n \geqq N$, so that

$$0 = (\{\phi(A) - B\}(\phi(S)x_\alpha + nx_\alpha), \phi(S)x_\alpha + nx_\alpha) = (\{\phi(A) - B\}\phi(S)x_\alpha, \phi(S)x_\alpha) + 2n \operatorname{Re} (\{\phi(A) - B\}\phi(S)x_\alpha, x_\alpha) + n^2(\{\phi(A) - B\}x_\alpha, x_\alpha),$$

for $n \geqq N$. Since $0 = n^2(\{\phi(A) - B\}x_\alpha, x_\alpha)$ and $(\{\phi(A) - B\}\phi(S)x_\alpha, \phi(S)x_\alpha)$ is independent of n, we conclude, from the possibility of choosing n arbitrarily large, that $(\{\phi(A) - B\}\phi(S)x_\alpha, \phi(S)x_\alpha) = 0$. But, as S ranges through $\mathfrak{A}$, $\phi(S)x_\alpha$ ranges through a dense subset of $\mathcal{H}_\alpha$ (from the properties of the wave function x_α); so that $(\{\phi(A) - B\}z, z) = 0$ for each z in $\mathcal{H}$, and $\phi(A) = B$. Thus $\phi(\mathfrak{A})$ satisfies the hypothesis of Lemma 1, so that $\phi(\mathfrak{A})$ is a ring of operators, and the proof is complete.

In general, the statement of the preceding theorem will not hold true if the definition of 'W^*-algebra' is weakened so as to encompass only sequences (rather than directed sequences). In fact, the C^*-algebra consisting of those operators whose range has countable dimension acting on an inseparable Hilbert space generates, together with the identity operator, a C^*-algebra $\mathfrak{A}$ which has the property that bounded, monotone-increasing sequences in $\mathfrak{A}$ have least upper bounds (in fact, strong limits) in $\mathfrak{A}$, and each vector state is a 'normal state' of $\mathfrak{A}$, yet $\mathfrak{A}$ is not strongly closed nor does it have a faithful representation as a ring of operators. In fact, if $\mathfrak{A}$ had such a representation each family of projections in $\mathfrak{A}$ would have a projection in $\mathfrak{A}$ as least upper bound which would be orthogonal to each projection in $\mathfrak{A}$ which was, itself, orthogonal to each projection of the family. Thus, if E is a projection such that E and $I - E$ have inseparable ranges, then E is not in $\mathfrak{A}$ yet E is the only possible least upper bound of the one-dimensional projections contained in E (each of which lies in $\mathfrak{A}$), since E is the only projection containing these one-dimensional projections and orthogonal to all the one-dimensional projections contained in $I - E$.

It is possible, however, to restrict attention to sequences if a countability assumption is made on our W^*-algebra.

Definition 2. *A countable W^*-algebra is a C^*-algebra in which each bounded, monotone-increasing sequence of self-adjoint operators has a least upper bound in the algebra, each orthogonal family of projections is at most countable, and which has a separating family of normal states ("normal" relative to sequences).*

Theorem 2. *A C^*-algebra $\mathfrak{A}$ has a faithful representation as a countably decomposable ring of operators if and only if it is a countable W^*-algebra.*

Proof. We note first that if $\mathfrak{A}$ is assumed to be countably decomposable in the hypothesis of Lemma 1, then the conclusion of this lemma holds with the hypothesis changed to apply only to bounded, monotone-increasing sequences. In fact, in the proof of Lemma 1, the only point at which directed sequences were employed was in concluding that the union and intersection of families of projections in $\mathfrak{A}$ are in $\mathfrak{A}$. Under the monotone sequence hypothesis it follows that unions and intersections of countable families of projections in $\mathfrak{A}$ lie in $\mathfrak{A}$. We show that this fact together with countable decomposability of $\mathfrak{A}$ imply that arbitrary unions (and intersections) of projections in $\mathfrak{A}$ lie in $\mathfrak{A}$, so that the altered version of Lemma 1, noted above, is true. Suppose that there is some family of projections in $\mathfrak{A}$ whose union does not lie in $\mathfrak{A}$, and let $\{E_\alpha\}$ be a family with the least cardinality b in the class of such families. We may assume that the index α runs through the ordinals preceding the initial ordinal with cardinal b. If we denote the union of E_β, for β not exceeding α by F_α, then F_α lies in $\mathfrak{A}$, since F_α is the union of a family with cardinality less than b. Letting G_α denote the orthogonal complement in F_α of the union of F_β with β less than α and noting that $\{F_\alpha\}$ is well-ordered by the usual inclusion relation among projections, while the mapping $\alpha \rightarrow F_\alpha$ is order-preserving, we conclude, from elementary considerations, that $\{G_\alpha\}$ is an orthogonal family in $\mathfrak{A}$ (reasoning as with the F_α) with sum (union) equal to the union of F_α which is, of course, the union of the E_α. However, countable-decomposability of $\mathfrak{A}$ implies that there are at most a countable number of non-zero G_α, so that their sum, and hence the union of the E_α, lies in $\mathfrak{A}$, contrary to assumption. Thus arbitrary unions and intersections of projections in $\mathfrak{A}$ lie in $\mathfrak{A}$, and the modified version of Lemma 1 holds.

To complete the proof of the present theorem, one simply notes that the proof of Theorem 1 applies verbatim with the term 'directed' removed, the modified version of Lemma 1 employed in place of Lemma 1, and the statement that, since $\mathfrak{A}$ is countably decomposable and ϕ is a *-isomorphism, $\phi(\mathfrak{A})$ is countably decomposable, added.

A more easily proved but less cogent abstract characterization of rings of operators is the following:

THEOREM 3. *A C^*-algebra $\mathfrak{A}$ has a faithful representation as a ring of operators if and only if it possesses a separating family $\mathcal{S}$ of states such that if ω is in $\mathcal{S}$ and T is an invertible element in $\mathfrak{A}$ then the positive linear functional ω' on $\mathfrak{A}$ defined by $\omega'(A) = \omega(T^*AT)$ is a multiple of some state in $\mathcal{S}$, and such that the unit sphere of $\mathfrak{A}$ is compact in the weak $\mathcal{S}$-topology on $\mathfrak{A}$.*

PROOF. We proceed exactly as in the proof of Theorem 1, constructing a faithful representation ϕ of $\mathfrak{A}$ by taking the direct sum of the representations engendered by each state ω_α in $\mathcal{S}$. (We adopt the notation of the proof of Theorem 1.) Now, according to [6], each operator A' of norm 1 in $\phi(\mathfrak{A})^-$, the weak closure of $\phi(\mathfrak{A})$, is a strong (and hence, weak) limit point of the unit sphere in $\phi(\mathfrak{A})$. Let X be the direct product of intervals $[-1, 1]$ indexed by $\{\alpha\}$, the indexing family for $\mathcal{S}$, taken in the product topology, so that X is a compact-Hausdorff space, and let Y be the image in X of the unit sphere of $\mathfrak{A}$ under the mapping $B \rightarrow \{\omega_\alpha(B)\}_\alpha$. By definition of the weak $\mathcal{S}$-topology, this mapping is a homeomorphism, so that Y is compact and hence closed in X. Since A' in $\phi(\mathfrak{A})^-$ is a weak limit point of the unit sphere in $\phi(\mathfrak{A})$ and $(\phi(B)x_\alpha, x_\alpha) = \omega_\alpha(B)$,

$$\{(A'x_\alpha, x_\alpha)\}_\alpha$$

lies in Y, so that there is an operator A in $\mathfrak{A}$ such that $(\phi(A)x_\alpha, x_\alpha) = (A'x_\alpha, x_\alpha)$ for all α. Let T be an invertible element in $\mathfrak{A}$, and, for some fixed α, let $\omega_\alpha(T^*BT) = t\omega_\beta(B)$. Choose a directed sequence of operators $\{A_\gamma\}$ in the unit sphere of $\mathfrak{A}$ such that $\{\phi(A_\gamma)\}$ tends weakly to A'. Then

$$(\phi(A_\gamma)\phi(T)x_\alpha, \phi(T)x_\alpha) = t(\phi(A_\gamma)x_\beta, x_\beta)$$

tends to $(A'\phi(T)x_\alpha, \phi(T)x_\alpha)$ and to

$$t(A'x_\beta, x_\beta) = t(\phi(A)x_\beta, x_\beta) = (\phi(A)\phi(T)x_\alpha, \phi(T)x_\alpha)$$

for each invertible T in $\mathfrak{A}$ and all α. We conclude now, exactly as in Theorem 1, that $\phi(A) = A'$, and the proof is complete.

With regard to the relation between Theorems 1 and 3, we remark that the set of normal states of a W^*-algebra has the required invariance property of Theorem 3 and that the least upper bound condition of Theorem 1 replaces the compactness assumption of Theorem 3.

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