Report on Operator Algebras

By Richard V. Kadison

The Arden House Conference was concerned with recent research in operator theory, group representations, and their interconnections. In addition to the formal twenty and forty minute addresses, there was considerable informal discussion. It would be difficult to report accurately on the informal portion of the conference and the results arising therefrom.

This section of the conference report will be devoted to a survey of the theory of operator algebras and to the relation of those formal addresses which dealt primarily with operator algebras to the broader aspects of this theory. We shall omit all bibliographical references, since we could not hope to include reference to all the papers which have directly contributed to the state of our present knowledge about operator algebras in a report of reasonable size.

A C^*-algebra is an algebra of operators on a Hilbert space which is closed under the operation of taking the adjoint and closed in the operator bound (uniform) topology. A C^*-algebra is the natural infinite-dimensional analogue of a finite-dimensional algebra of complex matrices closed under the operation of taking the conjugate transpose (the topological conditions which might be imposed, in the finite–dimensional ease, are automatically satisfied by virtue of the finite–dimensionality). These finite-dimensional matrix algebras are, of course, special cases of C^*-algebras. Their structure is completely described by the Wedderburn theory (algebraically they are direct sums of total complex matrix algebras of various orders and, with regard to their specific action on the underlying spare, they are direct sums of n_i-fold copies of total matrix algebras of order $m_i, i = 1, \cdots, k$). In general terms, the central problem in the study of C^*-algebras is that of finding a structure theory which will do for these algebras what the Wedderburn theory does for the finite-dimensional C^*-algebras.

Aside from their intrinsic interest as a natural class of infinite–dimensional, semi–simple algebras, C^*-algebras find application in the study of group representations, mathematical formulations of physical situations, and certain phases of ergodic theory. If we denote by G a locally compact group (assumed unimodular, for the sake of simplicity), by $L_1(G), L_2(G)$ the integrable and square integrable functions on G relative to Haar measure, respectively, then the functions of $L_1(G)$ acting by (left) convolution on $L_2(G)$ give rise to a family of bounded operators on the (Hilbert) space $L_2(G)$ closed under the adjoint operation. This family of operators and its closures (all of which are C^*-algebras) in the various operator topologies serve as generalizations of the complex group algebra of a finite group. These group algebras play a crucial role in the study of the group representations of G. The measure-theoretic properties of groups of measurability preserving transformations on a measure space, can be studied by investigating the structure of the various C^*-algebras obtained from the operators derived from the action of the group on the Hilbert space of square integrable functions over the measure space together with the operators arising from multiplication by essentially bounded measurable functions on this space of square integrable functions.

The methods used in the study of C^*-algebras are quite diverse. Of course, the techniques derived from modern algebra are employed extensively. While algebraic techniques are sufficient, almost exclusively, for dealing with the finite-dimensional situation, they don‘t begin to give the full picture in the case of infinite-dimensional C^*-algebras. The continuous as well as the discrete (e.g., with regard to spectra) arises in the infinite-dimensional case, while it is not present in the finite-dimensional situation. These considerations make the tools of analysis, notably, complex function theory and abstract measure theory, invaluable for the investigation of C^*-algebras. In addition to these methods, a special brand of point set topology which fashions a topological structure to the algebraic and intrinsic geometrical structure has proved quite useful in the study of C^*-algebras.

It should be remarked that we seem to be not too close to a final structure theory for C^*-algebras. We have no guesses as to how the general C^*-algebra is constructed from a “canonical set” of fully understood C^*-algebras. Aside from this lack of a general theory, however, the subject bristles with simply phrased, quite specific, “yes” or “no” questions for which we have neither the answer nor reasonable guesses as to the answer.

A well-known result of Gelfand-Neumark tells us that a C^*-algebra has an independent algebraic existence, viz., a Banach algebra with a $*$-operation having the usual formal algebraic properties and satisfying, in addition, $\|aa^*\| = \|a\|^2$ is isometrically $*$-isomorphic with a C^*-algebra. Some years ago, M. H. Stone proved a theorem about commutative C^*-algebras which gave the algebraic portion of the spectral theorem a very cogent form. He showed that each commutative C^*-algebra is algebraically isomorphic to the algebra of all continuous, complex-valued functions on some compact-Hausdorff space (derived from the algebraic structure of the C^*-algebra) with the *-operation in the C^*- algebra going into complex-conjugation of functions. He showed, moreover, that the C^*-algebra is determined to within algebraic isomorphism by the homeomorphism type of the associated compact-Hausdorff space. The function ring on each compact-Hausdorff space is easily seen to be a (commutative) C^*-algebra, so that the distinct classes of algebraically isomorphic, commutative C^*-algebras are in $1-1$ correspondence with the homeomorphism classes of compact-Hausdorff spaces. For the purposes of operator theory, this is an adequate algebraic description of such operator algebras. To a non-commutative C^*-algebra. one can again associate a structurally derived compact-Hausdorff space and, this time, a distinguished linear subspace of continuous, complex- valued functions on this compact-Hausdorff space. However, in this case, we do not know canonical forms for the linear subspace taken together with the compact-Hausdorff space, although the system characterizes the C^*-algebra.

A commutative C^*-algebra together with its action on its underlying Hilbert space can be described by its associated compact space and a well-ordered chain of ideals of Borel sets in the space (each, the family of null sets of some measure). Again, we do not have canonical forms for such a construct, but the problems involved in obtaining such canonical forms are in the province of pure measure theory and are already inherent in the classical unitary equivalence description of the action of a single self—adjoint operator on a Hilbert space by Hellinger- Hahn (of which the commutative C^*-algebra result is an extension). Aside from the original Hellinger-Hahn theory, Wecken, Plessner, Rohlin, Segal, Nakano, and Halmos have contributed important techniques to this final formulation of commutative multiplicity theory. It has become possible, recently, to make an analogous study of the action of a not necessarily commutative C^*-algebra on its underlying Hilbert space, assuming the algebraic structure known. This theory inherits, of course, all the problems of the commutative theory, but seems, at this stage, to have no others.

The class of C^*-algebras has several important subclasses which have received special attention. Notable among these is the class of "rings of operators" (also called "W^*-algebras"—those closed in the weak operator topology, i.e., the weakest (coarsest) topology on the bounded operators in which all the linear functionals of the form $A \mapsto (Ax, y)$ are continuous). The assumption that a C^*-algebra be weakly closed produces deep effects upon its structure, and the additional algebraic and geometrical properties visible enable us to subject this class of C^*-algebras to a much more detailed analysis (though, by no means, a definitive analysis, at this point of development of the subject). In particular, rings of operators (containing the identity operator) contain, along with each self-adjoint operator, its complete spectral resolution. J. von Neumann has exhibited rings of operators as "direct integrals" (measure-theoretic generalization of direct sum) of basic constituents called "factors" (rings of operators whose center consists of scalar multiples of the identity operator). Murray and von Neumann have studied these factors in detail. By comparing the relative sizes of the ranges of orthogonal projections in a given factor, M, a relative dimension function D is defined on the projections in M (having the customary properties of a dimension function) and is shown to be unique to within a positive multiplicative constant. With the aid of this dimension function, the factors are separated into three classes. The first class comprises the factors of type I_n, those having minimal projections in which the (normalized) dimension function takes the values $1, 2, \cdots, n$ (n finite or infinite). The second class constitute the factors of type II_1 and II_∞, in which the dimension function takes all values in $[0, 1]$ and $[0, \infty]$, respectively. These are the factors having no minimal projections and containing a non-zero projection of relative dimension different from ∞. The final class consists of the factors of type III in which the dimension of each non-zero projection is ∞. The factors of type I_n are shown to be algebraically $*$-isomorphic to the algebra of all bounded operators on an n-dimensional Hilbert space. Associated with each factor on a Hilbert space, one has the set of operators which commute with it, which is again a factor of type I, II, or III according as the original factor is of type I, II, or III, respectively. If M is of type I_n, M' (the commutant of M) of type I_m, N is of type I_n, and N' of type I_m, then M and N are unitarily equivalent. In general, if M is a factor of type I or II with commutant M' there is associated with M a con- stant, the so called "coupling constant". If x is a non-zero vector in the underlying Hilbert space upon which M acts, the orthogonal projections E and E' on the closures of the linear manifolds spanned by the images of x under operators in M' and M, respectively, lie in M and M', respectively. The ratio of the dimensions of E and E', relative to M and M', respectively, is the coupling constant just mentioned (it is shown to be independent of the vector x chosen). If N is another factor algebraically $*$-isomorphic to M, with commutant N' and coupling constant equal to that of M and M', then M and

N are unitarily equivalent, and, moreover, the given algebraic isomorphism can be implemented by a unitary transformation. This result does not apply *per se* to the case where M is of type II_∞ and M' of type II_1 . This last case can be handled, however, by suitable modifications of the above mentioned techniques. Recently, E L. Griffin has shown that (at least in the case of separable Hilbert space) each $*$-isomorphism between factors of type III can be implemented by a unitary transformation between the underlying Hilbert spaces. The problems then, in the study of factors and rings of operators, are largely ones of the algebraic nature of these operator algebras.

By considering the weakly closed group algebras of various locally compact topological groups, examples can be constructed of each of the various types of factors. In point of fact, however, factors of type III were constructed, only after much effort, by considering groups of measurability preserving transformations acting on measure spaces which do not admit group invariant measures.

In terms of the dimension function constructed, a trace function with the usual properties can be introduced in factors of types I_n and II_1 . In terms of this trace function a topology can be imposed on the factor which is useful for the study of its structural properties.

Current research in the theory of operator algebras centers about the study of factors of type II_1. A broad class of factors of type II_1, the socalled "approximately finite factors" in which any finite set of operators can be approximated as closely as desired in the trace topology by operators lying in a subring of finite linear dimension, have been shown to be algebraically $*$-isomorphic to each other. On the other hand, it has also been shown that there are factors of type II_1 which are not of the same algebraic type as the approximately finite factors. This is effected by showing that the approximately finite factors of type II_1 possess an approximate (relative to the trace topology) commutativity property which the weakly closed group algebra of certain groups does not have (e.g., the free group on two generators).

With regard to the study of factors and, more generally, rings of operators, one of the important projects involves the analysis of the structure preserving maps. At the Arden House Conference, I. M. Singer presented some of his recent results concerning the automorphisms of factors of type II_1. He considered factors of type II_1 arising from groups of measure-preserving transformations acting ergodically upon a finite, non-atomic measure space. Roughly speaking, the measure preserving transformations induce unitary operators on the Kronecker product of the Hilbert space of square integrable functions on the group with the Hilbert space of square integrable functions on the measure space. This group of unitary operators taken together with the algebra A of operators obtained from the multiplication action of essentially bounded measurable functions on the measure space generate a factor M of type II_1. The subalgebra A of M can easily be shown to be a maximal abelian subalgebra of M. Singer studies the group G of $*$-automorphisms of M which leave A setwise-invariant and its normal subgroup G_a consisting of these automorphisms in G which leave A elementwise invariant. He describes G in terms of the original group of measure-preserving transformations. In particular, he proves that G is the semi-direct product of G_0 and another group described in terms of the original constructions. A neat statement of these results in cohomological terms was presented. By these means, Singer can show that, in many cases, where the outer automorphisms themselves are not apparent, the factor in question must admit $*$-automorphisms which are not inner. The present author had raised the question of whether or not a factor of type II_1 (or, more generally, a ring of operators) obeys some sort of Galois theorem relative to its group of $*$-automorphisms (such is the case for rings of type I). Singer answers this question negatively on the basis of his general techniques with specific examples.

Relating to the question of structure preserving maps of operator algebras, I. Kaplansky presented results concerning derivations of certain classes of $C*$—algebras. It is appropriate, at this point, to note another trend in current research on operator algebras. Various subclasses of C*-algebras more accessible, structurally, than the general $C*$-algebra are considered. One of the main proponents of this approach is I. Kaplansky who has developed a reasonably detailed structure theory for a class of C^*-algebras he calls "CCR algebras"(these admitting sufficiently many representations by algebras of completely continuous operators). He has introduced a class of algebras he calls AW^*-algebras (abstract W^*-algebras). This class of C^*-algebras embodies the main algebraic features of W^*-algebras while being algebraically defined (it is a broader class than the W^*-algebras). Kaplansky and others have pursued the program of carrying over to the AW^*-algebras the known algebraic properties of W^*- algebras, as well as trying to extend the known theory of W^*-algebras in terms of AW^*-algebras. In his conference talk, I. Kaplansky introduced a construct which he calls a "C^*-module". It is a module with an abelian C^*-algebra as operator ring and an "inner product" with values in the abelian C^*-algebra This construct may prove to be a very convenient tool for the investigation of operator algebras and for providing new examples of operator algebras (especially, if the general theory can be extended to not necessarily commutative rings of operators). Kaplansky discussed the general theory of his C^*-modules but specialized, in a short time, to the case where his C^*-algebra was an AW^*- algebra and the module over this algebra satisfies two additional algebraic assumptions. Such C^*-modules, he calls AW^*-modules, and, for these, he carries the general theory much further. With the aid of this new device. Kaplansky then settles an open existence question for certain classes of AW^*-algebras. Among other things, he proves that each derivation of an AW^*-algebra of type I is inner, basing his argument on a lemma due to Singer.

The study of factors leads one to the study of various algebraic structures attached to these factors. In particular, the group of all unitary operators in a factor and the group of all invertible operators in a factor have attracted a certain amount of attention recently. Henry A. Dye talked on the unitary group in a factor of type II_1, and showed that certain isomorphisms between the unitary groups of such factors give rise to $*$-isomorphisms or $*$-anti-isomorphisms between the factors. In this connection, I. Singer has shown that a Lie algebra isomorphism between factors of type II_1 satisfying certain slight continuity conditions implies the existence of a $*$-isomorphism between the factors. I. Kaplansky has relaxed these conditions somewhat. The present author talked on the structure of the unitary and general linear groups of a factor. A complete list of the uniformly closed normal subgroups was given.

It might be remarked that these groups are a natural generalization of the classical groups. L. Loomis talked on a general ordered structure resembling the order structure of the projections in a factor. For such structures, Loomis is able to develop a dimension theory, but, without the added structure of a factor, his techniques must be more delicate than those employed by Murray and von Neumann to define a dimension function on factors.

Another important trend in current research on operator theory is the global investigation of rings of operators. As noted earlier, a ring of operators admits a type of measurable decomposition into factors, relative to its center. This focuses attention on the study of factors. In reality. however, the passage from information about the factors in a decomposition to information about the ring from which they derive is rarely smooth, involving, as it generally does, thorny difficulties of a measure-theoretic and operator-theoretic nature. Since rings rather than factors arise in applications, it is desirable to have some global techniques for dealing with them rather than passing to the factor decomposition. Dixmier, Dye, Godement, Griffin, Kaplansky, Segal, and others have developed such techniques. Dixmier systematically investigated the center- valued trace in rings of operators. Kaplansky's work on AW^*-algebras contributed heavily to our global techniques. The methods used are a rather interesting mixture of classical measure theory and modern operator theory, which have their roots in the early work of Murray and von Neumaun. I. E. Segal formalized this interrelation between measure theory and operator theory in a non-commutative integration theory. It should be noted that the measure space rather than the range of values is the non-commutative object (the measurable sets corresponding to the projections in a ring of operators and the integration process corresponding to a trace like, linear functional). Surmounting considerable technical difficulties, Segal proves non-commutative analogues to the Riesz-Fischer and Fubini Theorems as well as other classical measure-theoretic theorems. At the conference, Segal talked on an extension of dimension theory to arbitrary rings of operators without a finiteness assumption. He discussed a cardinal-valued integration theory appropriate to this extension. Segal also discussed non-commutative extensions of probability theory. He defined a (not necessarily commutative) abstract probability space and proved, among other things, the (non-commutative) analogue of the Kolmolgoroff theorem concerning the existence of random variables having preassigned joint distributions (satisfying certain necessary consistency conditions). In the process, Segal, gives a systematic treatment of direct limits of rings of operators.

It would be rash to say that we are confident of an early solution to the central problems still facing us. Nevertheless, though these problems seem quite difficult and recent progress slow, many of us have hope for a useful structure theory for self-adjoint operator algebras in the not too distant future.

Columbia University
New York, New York