THE GENERAL LINEAR GROUP OF INFINITE FACTORS

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1. **Introduction.** In [3], we determined all the uniformly closed, normal subgroups of $\mathfrak{M}_g$, the group of all invertible operators in the factor $\mathfrak{M}$. Our information was complete for factors of all types with the exception of those of type II_∞. This note is devoted to supplying the missing information in the II_∞ case. We defined $\mathfrak{M}_{gf}$ to be the uniform closure in $\mathfrak{M}_g$ of the set of invertible operators which act as a scalar multiple of the identity operator on the orthogonal complement of a subspace of finite relative dimension and $\mathfrak{M}_{gf(1)}$ to be the uniform closure of the set of those operators for which this scalar is 1. We showed, in [3], that $\mathfrak{M}_{gf}$ is a closed, normal subgroup of $\mathfrak{M}_g$ (proper and non-central when $\mathfrak{M}$ is of type I_∞ or II_∞) and that $\mathfrak{M}_{gf}$ is the direct product of $\mathfrak{M}_{gf(1)}$ and the group of non-zero, complex scalars. In Lemma 6 of [3], we showed that each uniformly closed, proper, non-central, normal subgroup, $\mathcal{G}$, of $\mathfrak{M}_g$ contains $\mathfrak{M}_{gf(1)}$ (and that each normal operator in $\mathcal{G}$ lies in $\mathfrak{M}_{gf}$). We completed the determination of the subgroups $\mathcal{G}$ for factors $\mathfrak{M}$ of type I_∞, in Theorem 4 of [3], by showing that each such subgroup, $\mathcal{G}$, is the direct sum of $\mathfrak{M}_{gf(1)}$ and some closed subgroup of the scalars. It was strongly presumed that this same result holds for factors of type II_∞. However, the proof given, failed, at one point, to encompass the II_∞ case. In the following section, we shall supply a new proof which covers both the I_∞ and II_∞ cases, thereby completing the determination of all the closed normal subgroups of the general linear groups of the various types of factors.

We are indebted to I. Kaplansky for pointing out to us the advisability of taking the quotient of our factor by the unique closed, two-sided ideal in the I_∞ and II_∞ cases.

2. **The normal subgroups.** The following result will be needed for the final determination of the closed, normal subgroups. We give three proofs, the first is based on Fuglede's Theorem [1] and the second, due to I. Kaplansky, makes use of Putnam's generalization [6] of Fuglede's Theorem. The result in question does not lie as deep as Fuglede's Theorem. The third proof avoids such considerations and shows that the result is valid in a Banach algebra with a symmetric *-operation (i.e., a^*a has positive spectrum).

Lemma 1. *If $\mathfrak{A}$ is a C^*-algebra, an operator A in $\mathfrak{A}$ lies in the center of $\mathfrak{A}$ if and only if each inner transform of A, $P^{-1}AP$ (P in $\mathfrak{A}$), is a normal operator.*

Proof I. If A lies in the center of $\mathfrak{A}$ then so does A^*, so that $AA^* = A^*A$ and A is normal as is $P^{-1}AP = A$. Assume, now, that $P^{-1}AP$ is normal for each invertible operator P in $\mathfrak{A}$. Then $P^*A^*P^{*-1}$ is normal for each invertible P

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in $\mathfrak{A}$, so that $P^{-1}A^*P$ is normal. Moreover, A is normal (choosing P to be I), so that $(P^{-1}AP)\,(P^{-1}A^*P) = P^{-1}AA^*P = P^{-1}A^*AP = (P^{-1}A^*P)\,(P^{-1}AP)$, i.e., $P^{-1}AP$ and $P^{-1}A^*P$ are commuting normal operators. Thus both $P^{-1}A_1P$ and $P^{-1}A_2P$ are normal, where $A_1 = (A + A^*)/2$, $A_2 = (A - A^*)/2i$ (this follows at once from Fuglede's Theorem [1]). Now A_1 and A_2 are self-adjoint, and $A = A_1 + iA_2$; so that, if we had the result for self-adjoint operators, A_1 and A_2 would lie in the center of $\mathfrak{A}$ as would A. We may assume, therefore, that A is self-adjoint. Now with P self-adjoint and regular, $P^{\ 1}AP$ is normal and has real spectrum (since A, being self-adjoint, has real spectrum). Thus $P^{-1}AP$ is self-adjoint, so that

$$P^{-1}AP = PAP^{-1},$$

and

$$AP^2 = P^2A.$$

Hence A commutes with each positive, regular operator in $\mathfrak{A}$. Now, the positive regular operators in $\mathfrak{A}$ are dense in the set of positive operators in $\mathfrak{A}$, so that A commutes with each positive operator in $\mathfrak{A}$. But each self-adjoint operator in $\mathfrak{A}$ is the difference of two positive operators in $\mathfrak{A}$. Hence A commutes with each self-adjoint operator in $\mathfrak{A}$, so that A lies in the center of $\mathfrak{A}$.

Proof II. $P^{-1}AP = B$ is normal, as is A. Thus $AP = PB$ and, by [6], $P^*A = BP^*$ so that $PP^*A = PBP^* = APP^*$. Again A commutes with each positive, regular operator in $\mathfrak{A}$, and, therefore, lies in the center of $\mathfrak{A}$.

Proof III. Let H be a regular self-adjoint element in $\mathfrak{A}$. Then $H^{-1}AH$ is normal so that $(H^{-1}AH)\,(H^{-1}AH)^* = (H^{-1}AH)^*(H^{-1}AH)$ or $AH^2A^*H^{-2} = H^2A^*H^{-2}A$. Thus $AKA^*K^{-1} = KA^*K^{-1}A$ for each positive, regular K in $\mathfrak{A}$. In particular, for all small real t, $A(I - tH)A^*\,(I - tH)^{-1} = (I - tH)A^*\,(I - tH)^{-1}A$ or

$$A(I - tH)A^*\left(\sum_{n=0}^{\infty} t^nH^n\right) = (I - tH)A^*\left(\sum_{n=0}^{\infty} t^nH^n\right)A,$$

and, comparing coëfficients of t: $AA^*H - AHA^* = A^*HA - HA^*A$. Thus $A(A^*H - HA^*) - (A^*H - HA^*)A = 0$. If we denote by $\mathfrak{D}_C$ inner derivation by C on $\mathfrak{A}$ (i.e., $\mathfrak{D}_C(B) = CB - BC$), the last equation can be rewritten as $\mathfrak{D}_A\mathfrak{D}_{A^*}(H) = 0$, for each self-adjoint H in A. Thus

$$\begin{aligned} 0 = \mathfrak{D}_A\mathfrak{D}_{A^*}(H^2) &= \mathfrak{D}_A(\mathfrak{D}_{A^*}(H)\cdot H + H\cdot\mathfrak{D}_{A^*}(H)) = \mathfrak{D}_A\mathfrak{D}_{A^*}(H)\cdot H \\ &\quad + \mathfrak{D}_{A^*}(H)\cdot\mathfrak{D}_A(H) + \mathfrak{D}_A(H)\cdot\mathfrak{D}_{A^*}(H) + H\cdot\mathfrak{D}_A\mathfrak{D}_{A^*}(H) \\ &= \mathfrak{D}_{A^*}(H)\cdot\mathfrak{D}_A(H) + \mathfrak{D}_A(H)\cdot\mathfrak{D}_{A^*}(H) \\ &= -(AH - HA)^*(AH - HA) - (AH - HA)(AH - HA)^*. \end{aligned}$$

Thus, $AH = HA$, and A lies in the center of $\mathfrak{A}$.

We proceed to the proof of the main theorem.

THEOREM 1. *If $\mathfrak{M}$ is a factor of type I_∞ or II_∞, $\mathfrak{M}_g$ its group of invertible operators, and $\mathfrak{M}_{gf(1)}$ the uniform closure in $\mathfrak{M}_g$ of the set of invertible operators which act as the identity operator on the orthogonal complement of some subspace of finite relative dimension, then each uniformly closed, non-central, proper, normal subgroup, $\mathcal{G}$, of $\mathfrak{M}_g$ is the direct product of $\mathfrak{M}_{gf(1)}$ and some closed subgroup of the group of non-zero, complex scalars $\{\lambda I\}$.*

Proof. In view of the last paragraph of the proof of Theorem 4 of [3], and [3; Lemma 6], we need only prove that $\mathcal{G}$ is contained in $\mathfrak{M}_{gf}$. Let $\mathcal{I}$ be the uniform closure of the set of operators in $\mathfrak{M}$ the projections on the closure of whose ranges have finite relative dimension. According to [2; Theorem 2], $\mathcal{I}$ is the unique, closed, two-sided ideal in $\mathfrak{M}$, and, by [5; Theorem 3], $\mathfrak{M}/\mathcal{I} = \mathfrak{A}$ is again a C^*-algebra. Since $\mathcal{I}$ is closed under the adjoint operation, it is also the closure of the collection of operators in $\mathfrak{M}$ the orthogonal complements of whose null spaces have finite relative dimension. It is easy to see that $\mathfrak{M}_{gf(1)}$ is contained in $I + \mathcal{I}$. In fact, if A is in $\mathfrak{M}_{gf(1)}$, then A is the uniform limit of a sequence of operators A_n such that A_n acts as the identity operator on a subspace E_n, where $I - E_n$ has finite relative dimension. Thus $A_n - I$ lies in $\mathcal{I}$ and tends to $A - I$. Since $\mathcal{I}$ is closed, A lies in $I + \mathcal{I}$. Under the natural map, η, of $\mathfrak{M}$ onto $\mathfrak{A}$, $\mathfrak{M}_{gf(1)}$, therefore, maps onto I, and $\mathfrak{M}_{gf}$ maps onto $\{\lambda I\}$, the group of non-zero scalars in $\mathfrak{A}$ (since $\mathfrak{M}_{gf}$ is the direct product of this group of scalars and $\mathfrak{M}_{gf(1)}$). Now, according to Lemma 7 of [3], $UHU^{-1}H^{-1}$ is in $\mathfrak{M}_{gf(1)}$ for each $A = UH$ in $\mathcal{G}$, where U unitary, H positive is the polar decomposition of A. Thus $\eta(A) = \eta(U)\eta(H)$ and $\eta(A^*) = \eta(HU^{-1}) = \eta(H)\eta(U)^{-1} = \eta(A)^*$ commute; for $I = \eta(U)\eta(H)\eta(U)^{-1}\eta(H)^{-1}$, so that $\eta(U)$ and $\eta(H)$ commute, and hence $\eta(U)^{-1}$ and $\eta(H)$ commute. With T an invertible operator in $\mathfrak{M}$ and A in $\mathcal{G}$, we have $T^{-1}AT$ is in $\mathcal{G}$ so that $\eta(T)^{-1}\,\eta(A)\eta(T)$ is normal. We assert that $T'^{-1}\eta(A)T'$ is normal for each invertible operator T' in $\mathfrak{A}$. It suffices to establish this for positive, invertible T', for if $K^{-1}\eta(A)K$ is normal, with $T' = KV$, the polar decomposition of T', then $T'^{-1}\eta(A)T'$ is normal, being a unitary transform, $V^{-1}K^{-1}\eta(A)KV$, of $K^{-1}\eta(A)K$. Note that, with T' positive, it is possible to choose T in $\mathfrak{M}$ such that T is positive and $\eta(T) = T'$. In fact, choose T_0 in $\mathfrak{M}$ such that $\eta(T_0) = T'$. Then, with $T = |\,1/2\,(T_0 + T_0^*)\,|$, $\eta(T) = |\,1/2\,(\eta(T_0) + \eta(T_0)^*)\,| = |\,T'\,| = T'$. Now $\eta(T + \epsilon I)^{-1}\,\eta(A)\eta(T + \epsilon I) = (T' + \epsilon I)^{-1}\eta(A)(T' + \epsilon I)$ is normal, for each $\epsilon > 0$. Taking the limit, which exists, as ϵ tends to 0, we have $T'^{-1}\eta(A)T'$ is normal (for the set of normal operators in a C^*-algebra, being the inverse image of 0 under the continuous map $B \rightarrow BB^* - B^*B$ is uniformly closed). It now follows from Lemma 1 that $\eta(A)$ lies in the center of $\mathfrak{A}$. But $\mathfrak{A}$ is simple, for if $\mathfrak{A}$ contained a proper, two-sided ideal, the closure of this ideal would be proper and the inverse image of this closure under η would be a proper, closed, two-sided ideal containing $\mathcal{I}$ properly. But there are no such ideals [2]. Now $\mathfrak{A}$, being simple, has center consisting of scalars [4; Theorem 5.10], and, thus, $\eta(A) = \lambda I$ for each A in $\mathcal{G}$. For such an A, then, $A - \lambda I$ is in $\mathcal{I}$ so that $A = \lim\,(\lambda I + B_n)$ where B_n has as null space a subspace F_n whose complement has finite relative dimension. With

A invertible, $\lambda I + B_n$ is invertible for large n, and, from the description of B_n, is clearly seen to lie in $\mathfrak{M}_{gf}$. Hence A lies in $\mathfrak{M}_{gf}$, and the proof is complete.

Dixmier has communicated to me a modification of the proof of Theorem 4 in [3] which makes that proof valid for factors of type II_∞. From 'Suppose that $\cdots$' on the last line of [3; 83] to '$\cdots$ is infinite' on the first line of [3; 85], replace by:

'We show that some P_t contains an infinite subspace such that E' acting on this infinite subspace has a bounded inverse (i.e., $\| E'x \|$ stays above a fixed positive constant as x ranges over the unit vectors in the subspace). In fact, let $\mathcal{J}$ be the unique closed two-sided ideal in $\mathfrak{M}$. Then, since E' has infinite relative dimension, $E' = E'P_1 + \cdots + E'P_k$ is not in $\mathcal{J}$, so that some $E'P_t$ is not in $\mathcal{J}$, say $E'P_1$. Let $E'P_1 = V_1K_1$ be the polar decomposition of $E'P_1$. Since $\mathcal{J}$ is an ideal, K_1 is not in $\mathcal{J}$. Now $K_1 \geq 0$, so that K_1 is a uniform limit of finite linear combinations of its spectral projections corresponding to intervals bounded below by a positive constant. Since K_1 is not in $\mathcal{J}$, one of these spectral projections must have infinite relative dimension. But $K_1 = (P_1E'P_1)^{1/2}$, so that this spectral projection is contained in P_1. Moreover, with x a unit vector in the range of this spectral projection $\| E'x \|^2 = (P_1E'P_1\, x, x) = \| K_1x \|^2$, so that E' has a bounded inverse on this spectral projection. Thus E' acting on some infinite subspace of some P_t has a bounded inverse.'

REFERENCES

1. BENT FUGLEDE, *A commutativity theorem for normal operators*, Proceedings of the National Academy of Sciences, vol. 36(1950), pp. 35–40.
2. I. GELFAND AND M. NEUMARK, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Recueil Mathématique, Matematiceskii Sbornik, vol. 12(1943), (vol. 54 of the collection), pp. 197–213.
3. R. V. KADISON, *Infinite general linear groups*, Transactions of the American Mathematical Society, vol. 76(1954), pp. 66–91.
4. C. E. RICKART, *Banach algebras with an adjoint-operation*, Annals of Mathematics, Second Series, vol. 47(1946), pp. 528–550.
5. I. E. SEGAL, *Two-sided ideals in operator algebras*, Annals of Mathematics, Second Series, vol. 50(1949), pp. 856–865.
6. C. R. PUTNAM, *On normal operators in Hilbert space*, American Journal of Mathematics, vol. 73(1951), pp. 357–362.

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