$$G(x) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{k \leqq \log_{\sqrt[\alpha]{2}} x} \dfrac{2^{[(2/\alpha) - 1]k}}{1 + 2^{2k/\alpha}}, & x \geqq 0, \\ -G(-x), & x \leqq 0. \end{cases} \tag{14}$$

An easy calculation shows that

$$\log \varphi_\alpha(t) = \psi_\alpha(t) = \sum_{k=-\infty}^{\infty} \frac{\cos t2^{k/\alpha} - 1}{2^k}. \tag{15}$$

If $a = b = (1/2)^{1/\alpha}$, the hypotheses of the conjecture are certainly satisfied, but the distribution implied by equation (15) is not stable.

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[1] Linnik, Yu. V., "Linear Statistics and the Normal Distribution Law," *Doklady Akad. Nauk SSSR* (N.S.), **83**, 353–355, 1952; "On Some Identically Distributed Statistics," *ibid.*, **89**, 9–11, 1953.

[2] Doob, J. L., *Stochastic Processes* (New York: John Wiley & Sons, 1953), chap. iii.

ON THE ADDITIVITY OF THE TRACE IN FINITE FACTORS

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1. *Introduction.*—The additivity of the trace has always seemed to those of us familiar with the technical details of the subject of rings of operators to be the most difficult single point in the basic theory. The proof of this fact given by Murray and von Neumann,[1] mysteriously enough, seems to require much more effort than so basic a point merits; and the fact that several other natural attempts to define the trace and establish its additivity lead to simple statements which appear to require the additive trace itself for their proofs only serves to deepen the mystery.

In the next section we present a completely natural and elementary proof of the additivity of the trace. In fact, our point of departure is the dimension theory of projections in a factor—we establish the existence of the trace and its additivity independently of the "semi-authenticated" trace.[2] Our proof was inspired by that of Murray and von Neumann,[1] the last paragraph of our proof being a direct adaptation, in modern dress, of their "local linear approximability" argument.[3]

2. *The Trace.*—If φ is a state of a factor M of type II_1 such that $\varphi(B^*B) = \varphi(BB^*)$ for each B in M, then φ is the trace on M. In fact, φ is the same on equivalent projections from the foregoing equality, and this, combined with the state properties of φ, implies that φ is the dimension function on projections in M; whence our assertion. It suffices, of course, to establish that $\varphi(B^*B) \leq \varphi(BB^*)$ for each B in M (substituting B^* for B, we conclude equality). If we can locate a state φ_n of M, for each positive integer n, such that $\varphi_n(B^*B) \leq (n + 1/n)\varphi_n(BB^*)$, for each B in M, then, by weak compactness of the set of states, we can choose a weak limit φ of some subsequence of (φ_n) which will of course satisfy $\varphi(B^*B) \leq \varphi(BB^*)$, for each B in M. Suppose now that we have located a positive, linear functional ρ

on M such that $\rho(A^*A) \leq (n + 1/n)\rho(AA^*)$, for each A in EME, with E some nonzero projection in M. We may assume that $D(E) = 1/m$, with m some positive integer (choosing a subprojection of E in place of E, if necessary). Let $E = E_1, \ldots, E_m$ be m, orthogonal, equivalent projections in M, and let V_i be a partial isometry in M such that $V_i^*V_i = E_1$, $V_iV_i^* = E_i$. We define φ_n by $\varphi_n(B) = \sum_{i=1}^{m} \rho(V_i^*BV_i)$. Then

$$\varphi_n(B^*B) = \sum_i \rho(V_i^*B^*(\sum_j E_j)\, BV_i) = \sum_{i,j} \rho(V_i^*B^*V_jV_j^*BV_i)$$
$$= \sum_{i,j} \rho[(E_1V_i^*B^*V_jE_1)(E_1V_i^*B^*V_jE_1)^*] \leq \frac{n+1}{n} \sum_{i,j} \rho[(V_j^*BV_i)\,(V_i^*B^*V_j)]$$
$$= \frac{n+1}{n} \sum_j \rho(V_j^*B(\sum_i E_i)B^*V_j) = \frac{n+1}{n} \varphi_n(BB^*).$$

Clearly φ_n is a positive, linear functional on M, and multiplying it by a positive scalar does not change its properties, so that we may assume that φ_n is a state.

It remains to locate a projection such as E and a positive, linear functional such as ρ. Suppose that we have found a positive, linear functional ρ and a nonzero projection E such that

$$D(F) \leq \rho(F) \leq \frac{n+1}{n} D(F) \qquad (*)$$

for each projection F in E. We write M in place of EME. Then we may replace F by an equivalent projection G in the left and right terms of (*) and conclude that

$$\rho(F) \leq \frac{n+1}{n} D(G) \leq \frac{n+1}{n} \rho(G). \qquad (**)$$

Of course, (**) holds if F and G are replaced by a positive, linear combination of mutually orthogonal projections and its transform via a unitary operator in M, respectively. The Spectral Theorem, uniform continuity of ρ, and this last remark imply that $\rho(A) \leq (n + 1/n)\rho(U^*AU)$, for each positive A and unitary U in M. In particular, $\rho(A^*A) \leq (n + 1/n)\rho(AA^*)$, since A^*A and AA^* are unitarily equivalent in M (use the polar decomposition of A, and recall that partial isometries in finite factors have unitary extensions).

We locate E and ρ such that (*) holds. Let η be any normal state of M (such as a vector state), and define $s(G)$ and $i(G)$ to be sup $\eta(F)/D(F)$ and inf $\eta(F)/D(F)$, respectively, as F ranges over the nonzero subprojections of G. There exists a nonzero subprojection F of G such that $i(F) \geq \eta(G)/D(G)$ (and, by precisely the same proof with each inequality sign reversed, there exists a subprojection F' such that $s(F') \leq \eta(G)/D(G)$). In fact, choose, by Zorn's Lemma, a maximal orthogonal family of projections (G_α) such that $\eta(G_\alpha)/D(G_\alpha) < \eta(G)/D(G)$. If (G_α) is void, G may be taken as F. If not, $G - \sum_\alpha G_\alpha \neq 0$, for otherwise

$$\frac{\eta(G)}{D(G)} = \frac{\Sigma\eta(G_\alpha)}{\Sigma D(G_\alpha)} < \frac{\Sigma D(G_\alpha)\eta(G)/D(G)}{\Sigma D(G_\alpha)} = \frac{\eta(G)}{D(G)},$$

a contradiction. Clearly $G - \sum_\alpha G_\alpha$ serves as the desired F in this case. Since

$\eta(I) = 1 = D(I)$, we can, from the above, find a nonzero projection G with $i(G) = a \geq 1$. Choose a nonzero projection F in G with $\eta(F)/D(F) \leq (n + 1)a/n$. Once more, we can find a nonzero projection E in F such that $s(E) \leq (n + 1)a/n$. Certainly $a \leqq i(E)$, so that, for each nonzero projection P in E, $D(P) \leq \eta(P)/a \leq (n + 1)D(P)/n$. Taking ρ to be η/a, the proof is complete.

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[1] F. J. Murray and J. von Neumann, "On Rings of Operators. II," *Trans. Am. Math. Soc.*, **41**, 208–248, 1937; see, especially, pp. 210–217.

[2] F. J. Murray and J. von Neumann, "On Rings of Operators," *Ann. Math.*, **37**, 116–229, 1936.

[3] It has come to the author's attention that J. Dixmier has made an analogous remark in a forthcoming book on "rings of operators," which, incidentally, introduces many elegant simplifications in the subject.

REPRESENTATION OF A CLASS OF STOCHASTIC PROCESSES

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This note is a review of results obtained from a new approach to the study of certain time-homogeneous Markoff processes. The processes studied are of three types: (*a*) random walks, (*b*) birth and death processes, and (*c*) diffusion processes. Only the one-dimensional cases have been considered, although the method is not limited to these. The property which is common to the above processes and which makes them all amenable to the same treatment is the local nature of the changes that occur. This property can be characterized in terms of the semigroup of operators related to the process. Specifically, the infinitesimal generators of the semigroups in question are second-order differential or difference operators.

A random walk is the motion along a line executed by a particle which at each unit time can either stand still, or move one unit to the right, or move one unit to the left, the probabilities of these transitions depending perhaps on the position of the particle but not on the time. The set of possible positions (or states) of the particle is thus identifiable with a finite or infinite set of integers. The probability that if the particle starts at position i, it will be at position j after n units of time is denoted by P_{ij}^{n}. The fundamental matrix $P = (P_{ij}^{1})$ is given, and one wishes to know various statistical properties of the motion.

A birth and death process is essentially a random walk in which the time parameter has been made continuous. Let $P_{ij}(t)$ be the probability that if the particle was initially in state i, it will be in state j at time t. The process is called a birth and death process if, as $t \rightarrow 0$,

$$\begin{aligned} P_{i,\,i+1}(t) &= \lambda_i t + o(t), \\ P_{i,\,i-1}(t) &= \mu_i t + o(t), \\ P_{i,\,i}(t) &= 1 - (\lambda_i + \mu_i)t + o(t), \end{aligned}$$

where λ_i and μ_i are constants which may be thought of as the rates of absorption from state i into states $i + 1$ and $i - 1$.