ON THE ORTHOGONALIZATION OF OPERATOR REPRESENTATIONS.*

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1. Introduction. The main problem with which we shall be concerned is that of finding conditions under which a group representation is similar to a unitary representation and conditions for a representation of a self-adjoint algebra of operators on a Hilbert space to be similar to an adjoint preserving representation (* representation). These situations are slightly different aspects of the representation orthogonalization process with very close interconnections. With regard to the algebra situation one can phrase the main question in the following way. *Is an algebra of operators on a Hilbert space which is the isomorphic image of a C^*-algebra (uniformly closed self-adjoint algebra of operators) similar to a C^*-algebra in such a way that the composition of the isomorphism and the similarity is an adjoint preserving representation of the C^*-algebra?* The question for group representations takes the following form. *Is every bounded representation of a group by operators on a Hilbert space similar to a unitary representation, where, by "bounded representation" we mean that there exists a constant such that each representing operator is, in norm, less than this constant?* In this form, the group question has been raised before, notably in [1], [2], [5]. The question of when a group admits a mean (or Banach Limit) is the primary consideration of these papers and then, employing the technique of [4], it is shown that for such groups all bounded representations (continuous in the strong topology) are similar to unitary representations (this is done in [4] for the infinite cyclic group and the reals). We outline this technique. A (left, right) mean on a group G is a positive linear functional on the linear space $B(G)$ of bounded continuous functions on G which takes the value 1 at the constant function 1 and which is invariant under (left, right) translations on the group. If G admits a right mean m and $g \rightarrow A_g$ is a strongly continuous representation of the group by operators on a Hilbert space, then, for each x, y in the Hilbert space $g \rightarrow (A_g x, A_g y)$ is a bounded continuous function on G and $\langle x, y \rangle = m((A_g x, A_g y))$ is an inner product on the Hilbert

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space giving rise to a norm equivalent to the original norm and under which the operators A_g are unitary. It is then standard to find the similarity of the original Hilbert space which takes each A_g into a unitary operator. Continuity considerations do not enter into the question of whether all bounded representations of all groups are similar to unitary representations since it is obviously sufficient to settle this for discrete groups. With regard to the technique of means, it is well-known that many groups admit neither a left nor right mean (in the sense noted above, e. g., the free group on two generators; see [1], [2]) so that this method cannot, in itself, give the full answer.

Our concern is not with conditions on the group which imply that bounded representations are similar to unitary representations but rather with restrictions upon the representations which insure that they are similar to unitary representations. We feel that this approach gives hope of settling the full question one way or the other.

The truth of the operator algebra proposition is trivially implied by the truth of the group statement (see the proof of Theorem 7). On the other hand, a bounded representation of a group (discrete) can be extended to a representation of its L_1 algebra (as a self-adjoint algebra of operators acting by convolution on L_2). This algebra of operators need not be closed in the uniform operator topology and the representation need not be extendable to the uniform closure of this algebra; so that it would appear that the truth of the algebra result would imply the group conjecture only for those bounded representations which are extendable to the uniform closure of the L_1 algebra. (The similarity which transforms the representation of the L_1 algebra into a * representation will transform the group representation into a unitary representation). However, it is possible to renorm the L_1 algebra in such a way that the completion of the resulting * algebra is a C^*-algebra to which each bounded group representation is extendable, by assigning to each element of the L_1 algebra the supremum of the norms of its images in each * representation. It follows from the existence of a Banach algebra norm on L_1 in which the * map is isometric (viz., the L_1 norm) that this supremum is not greater than the L_1 norm (and certainly finite). Our extended group representation, being a bounded algebra representation relative to the L_1 norm on the L_1 algebra of the group, is a continuous representation of the L_1 algebra in the C^* norm just constructed. Although the representation extended to this C^*-algebra may not be an isomorphism, the kernel is a closed two-sided ideal so that the factor algebra is a C^*-algebra [8], and the induced representation on this factor C^*-algebra is an isomorphism. Thus we have the complete equivalence of the group and C^*-algebra questions. We are indebted to I. Kaplansky for bringing to our attention the known renorming device used above. It is possible that a more incisive operator algebra result would apply directly to the L_1 algebra (acting by convolution on L_2). In [6], Mackey proves the algebra result in the commutative case by direct methods (this result follows at once, as in the proof of Theorem 7, from the fact that commutative groups have means, in the sense defined above; see [5]).

The origin of the group question can be found in the classical statement which says that each representation of a finite group by (complex, real) matrices is equivalent to a representation by (unitary, orthogonal) matrices and its extension to continuous representations of compact groups [7]. The technique used in these proofs, invariant integration over the group, is almost identical with the technique of means. Using this theorem for compact groups the operator algebra result follows for finite-dimensional operator algebras (applied to the (compact) group of unitary operators in the algebra). Perhaps a more natural way of concluding the algebra result in the finite-dimensional case is thru the semi-simplicity of the image algebra. In this case the various concepts of semi-simplicity coincide so that the semi-simplicity of the original C^*-algebra, interpreted algebraically, is inherited by the image, and this, interpreted spatially, shows that this image is similar to a C^*-algebra. In the infinite-dimensional case it is not at all difficult to construct an algebra of operators which is semi-simple in all the conventional senses but not similar to a C^*-algebra. The following topological difficulty can occur: while each invariant subspace may have a complementary invariant subspace, the greatest of the angles between the given space and all possible invariant complements may tend to 0 for some sequence of invariant subspaces, thus ruling out what we may call the "topological semi-simplicity" of the algebra. For a C^*-algebra and a unitary group, the orthogonal complement of an invariant subspace is invariant. Our similarity problem for the given family of operators (group or algebra) amounts to an orthogonalization process.

In Section 2, we begin by defining concepts of local semi-simplicity and bounded local semi-simplicity of group representations. Theorem 1 states that bounded local semi-simplicity of a group representation is necessary and sufficient for the representation to be similar to a unitary representation. Several different forms of a conjecture concerning the group question are discussed, with the aid of Theorem 1. To study the operator algebra question, we develop a device for measuring the deviation of a set of vectors from being an orthonormal set. After stating a condition for a representation of a C^*-

algebra to be similar to a * representation in terms of the group condition (Theorem 1), we employ this device to give a more delicate criterion for topological semi-simplicity of an algebra of operators. In the concluding section, we discuss some extensions of the results stated, examples, and a class of natural questions an affirmative answer to any of which would yield the fact that all bounded operators on a Hilbert space have non-trivial, closed, invariant subspaces.

2. Conditions for topological semi-simplicity. The following definition contains a description of local behavior of a group representation, which is necessary and sufficient for the group representation to be similar to a unitary representation. The statement and proof of this fact are contained in Theorem 1.

DEFINITION 1. *A representation $g \to A_g$ of the group G by bounded operators A_g on the Banach space $\mathfrak{B}$ is said to be "locally semi-simple" when, for each finite set $x_1, \cdots, x_n$ of vectors in $\mathfrak{B}$ and $g_1, \cdots, g_n$ of elements in G, one can find a linear transformation S defined on the finite-dimensional vector space $\mathfrak{V}$ generated by $x_1, \cdots, x_n$, $A_{g_1}x_1, \cdots, A_{g_n}x_n$ such that $\| Sx_i \| = \| SA_{g_i}x_i \|$; $i = 1, \cdots, n$. If there exists a constant M such that S can always be chosen satisfying*

$$1/M \leqq \inf \{ \| Sx \| : x \in \mathfrak{V}, \| x \| = 1 \};$$
$$M \geqq \sup \{ \| Sx \| : x \in \mathfrak{V}, \| x \| = 1 \},$$

we say that the representation is "boundedly locally semi-simple."

THEOREM 1. *A representation $g \to A_g$ of the group G by bounded operators A_g on a Hilbert space $\mathfrak{H}$ is boundedly locally semi-simple if and only if it is similar to a unitary representation.*

Proof. The necessity of the condition is quite easy. Indeed, suppose that S is a bounded invertible operator on $\mathfrak{H}$ such that SA_gS^{-1} is unitary for each g in G. Let $x_1, \cdots, x_n$ in $\mathfrak{H}$ and $g_1, \cdots, g_n$ in G be given. Since SA_gS^{-1} is unitary we have

$$\| SA_{g_i}x_i \| = \| SA_{g_i}S^{-1}Sx_i \| = \| Sx_i \|; \qquad i = 1, \cdots, n,$$

which establishes the bounded local semi-simplicity of the given representation.

Suppose now that the given group representation is known to be boundedly locally semi-simple and that M is a bounding constant. We establish, in the succeeding lemmas, the existence of an invertible operator

P with $\| P \|$, $\| P^{-1} \|$ not exceeding M and such that $P^{-1}A_gP$ is unitary for each g in G.

LEMMA 2. *If $\mathcal{V}$ is a finite-dimensional subspace of $\mathcal{H}$ there exists a Hilbert space norm $\| \; \|'$ on $\mathcal{V}$ such that $\| A_g x \|' = \| A_{g'} x \|'$ whenever $A_g x$ and $A_{g'} x$ are in $\mathcal{V}$ and such that $1/M \| y \| \leqq \| y \|' \leqq M \| y \|$ for each vector y in $\mathcal{V}$.*

Proof. We endow the conjugate tensor product $\mathcal{V} \otimes \mathcal{V}$ (i.e., the tensor product which is conjugate linear in the second variable, $x \otimes \alpha y = \bar{\alpha}(x \otimes y)$) with the natural inner product derived from the inner product of $\mathcal{H}$, i.e., we define $\langle x \otimes y, z \otimes w \rangle = (x, z)(w, y)$ and extend the domain of definition of this inner product to all of $\mathcal{V} \otimes \mathcal{V}$ by bilinearity. It is well-known that this process gives rise to an inner product (independent) of the representation of the elements involved as a sum of elements of the form $x \otimes y$—(see [3], for example).

Let $\mathcal{V}\otimes$ be the subspace of $\mathcal{V} \otimes \mathcal{V}$ generated by tensors of the form $x \otimes x$, and let $\mathcal{E}\otimes$ be the subspace of $\mathcal{V}\otimes$ generated by vectors of the form $A_g y \otimes A_g y - y \otimes y$, where y and $A_g y$ are in $\mathcal{V}$. Choose a basis

$$A_{g_1} y_1 \otimes A_{g_1} y_1 - y_1 \otimes y_1, \cdots, A_{g_m} y_m \otimes A_{g_m} y_m - y_m \otimes y_m$$

for $\mathcal{E}\otimes$. By the bounded local semi-simplicity of the representation $g \to A_g$, we can find a linear transformation S such that $\| S A_{g_i} y_i \| = \| S y_i \|$; $i = 1, \cdots, m$ and

$$1/M \leqq \inf \{ \| Sy \| : y \text{ in } \mathcal{V}', \| y \| = 1 \};$$
$$M \geqq \sup \{ \| Sy \| : y \text{ in } \mathcal{V}', \| y \| = 1 \},$$

where $\mathcal{V}'$ is the subspace of $\mathcal{V}$ generated by $y_1, \cdots, y_m, A_{g_1} y_1, \cdots, A_{g_m} y_m$. By defining S to be a suitable scalar (between $1/M$ and M) on $\mathcal{V} \ominus \mathcal{V}'$, we obtain a linear transformation, which we denote again by S, defined on all of $\mathcal{V}$ and satisfying the same conditions as above with $\mathcal{V}$ replacing $\mathcal{V}'$ (no difficulties can arise if we choose S so that $S(\mathcal{V}') = \mathcal{V}'$ by composing the original S with a unitary transformation).

Let $\tilde{x} = \sum_{j=1}^{n} S^* x_j \otimes S^* x_j$, where $x_1, \cdots, x_n$ is an orthonormal basis for $S(\mathcal{V})$. Observe that

$$\langle \tilde{x}, A_{g_i} y_i \otimes A_{g_i} y_i - y_i \otimes y_i \rangle = \sum_{j=1}^{n} (S^* x_j, A_{g_i} y_i)(A_{g_i} y_i, S^* x_j)$$
$$- \sum_{j=1}^{n} (S^* x_j, y_i)(y_i, S^* x_j) = \sum_{j=1}^{n} | (x_j, S A_{g_i} y_i) |^2 - \sum_{j=1}^{n} | (x_j, S y_i) |^2$$
$$= \| S A_{g_i} y_i \|^2 - \| S y_i \|^2 = 0,$$

as follows from the fact that $SA_{g_i}y_i$, Sy_i are in $S(\mathcal{V})$, the Parseval equality, and the choice of S. Consequently $\tilde{x}$ is orthogonal to $\mathcal{E}\otimes$. We define a new inner product on $\mathcal{V}$ by means of $(x,y)' = \langle x\otimes y, \tilde{x}\rangle$ so that the new Hilbert norm on $\mathcal{V}$ satisfies

$$\| y \|' = \langle y\otimes y, \tilde{x}\rangle^{\frac{1}{2}} = [\sum_{j=1}^{n} (S^*x_j, y)\,(y, S^*x_j)\,]^{\frac{1}{2}}$$

$$= [\sum_{j=1}^{n} |(x_j, Sy)|^2]^{\frac{1}{2}} = \| Sy \|.$$

It follows at once from this last equality that $1/M \| y \| \leqq \| y \|' \leqq M \| y \|$. If now y and $A_g y$ are in $\mathcal{V}$ then $A_g y \otimes A_g y - y\otimes y$ is in $\mathcal{E}\otimes$ so that

$$0 = (A_g y\otimes A_g y - y\otimes y, \tilde{x}) = (A_g y\otimes A_g y, \tilde{x}) - (y\otimes y, \tilde{x})$$
$$= (\| A_g y \|')^2 - (\| y \|')^2$$

or $\| y \|' = \| A_g y \|'$. If $A_g x$ and $A_{g'}x = A_{g'}A_{g^{-1}}(A_g x)$ are in $\mathcal{V}$, then $\| A_g x \|' = \| A_{g'} x \|'$, which completes the proof of this lemma.

The following lemma allows us to pass from our finite-dimensional information to information about the full space on which the A_g operate.

LEMMA 3. *If $\mathfrak{B}$ is a Banach space and ρ is a partition function on $\mathfrak{B}$ such that on each finite-dimensional subspace $\mathfrak{B}_1$ of $\mathfrak{B}$ one can introduce a norm $\| \ \|'$ in which $\mathfrak{B}_1$ is a Hilbert space, each partition class intersected with $\mathfrak{B}_1$ lies on the shell of some sphere center at 0 in the norm $\| \ \|'$, and there exists a constant M (depending upon ρ) such that*

$$1/M \| x \| \leqq \| x \|' \leqq M \| x \|$$

for each x in $\mathfrak{B}_1$ (where $\| \ \|$ is the norm on $\mathfrak{B}$); then the underlying vector space of $\mathfrak{B}$ admits a norm, equivalent to the original norm, in which it is a Hilbert space and such that the partition classes of ρ each lie on the shell of some sphere center at 0 relative to the new norm.

Proof. We form a product of intervals with $\mathfrak{B}$ as the indexing family. To each point x in $\mathfrak{B}$, we make correspond the closed interval $[\| x \|/M, M\| x \|]$ (thus to 0 in $\mathfrak{B}$ we make correspond the number 0). Denote by X the Cartesian product

$$\prod_{x\in\mathfrak{B}} I_x = \prod_{x\in\mathfrak{B}} [\| x \|/M, M \| x \|].$$

We consider X in its standard product topology, in which it is compact, where each I_x is given its usual metric topology. Let $\mathfrak{B}_1$ be a finite-dimensional subspace of $\mathcal{B}$, and let $X(\mathcal{B}_1)$ be the set of points of X which as functions restricted to $\mathcal{B}_1$ give rise to a Hilbert space norm on $\mathcal{B}_1$ which is constant on the partition classes of ρ intersected with $\mathcal{B}_1$. We shall show presently that $X(\mathcal{B}_1)$ is a closed subset of X. Assume, for the moment, that we have proved this fact. The sets $X(\mathcal{B}_1)$ have the finite intersection property ($\mathcal{B}_1$ ranging over the finite-dimensional subspaces of $\mathcal{B}$). Indeed, let $\mathcal{B}_1, \cdots, \mathcal{B}_n$ be a (finite) set of finite-dimensional subspaces of $\mathcal{B}$ and let $\mathcal{B}_0$ be the (finite-dimensional) subspace they generate. By assumption, we can find a Hilbert space norm $\| \ \|_0$ on $\mathcal{B}_0$ which is constant on the partition classes of ρ intersected with $\mathcal{B}_0$, and which satisfies the inequality

$$1/M \| x \| \leqq \| x \|_0 \leqq M \| x \|$$

for each x in $\mathcal{B}_0$. The function which assigns to each x not in $\mathcal{B}_0$ the value $M \| x \|$ and to each x in $\mathcal{B}_0$ the value $\| x \|_0$ lies in $X(\mathcal{B}_0)$ which is clearly contained in $\bigcap_{i=1}^{n} X(\mathcal{B}_i)$. It now follows from the compactness of X (and our assumption that the sets $X(\mathcal{B}_1)$ are closed in X) that the intersection of all the sets $X(\mathcal{B}_1)$ is not empty. Let $\| \ \|'$ be a function on $\mathcal{B}$ in this intersection. Then, on each finite-dimensional subspace of $\mathcal{B}$, $\| \ \|'$ induces a Hilbert space norm. It is immediate that $\| \ \|'$ satisfies the norm axioms and the Parallelogram Law on $\mathcal{B}$ as well as being constant on the partition classes of ρ, so that $\| \ \|'$ is our desired Hilbert space norm on $\mathcal{B}$. Of course $1/M \| x \| \leqq \| x \|' \leqq M \| x \|$, since $\| \ \|'$ is in X. It remains to prove that the sets $X(\mathcal{B}_1)$ are closed in X. We shall omit this proof, however, since it is a standard approximation argument of the type employed in the proof of the w^*-compactness of the unit sphere in the conjugate space of a normed linear space.

Proof of Theorem 1. As partition function ρ on $\mathcal{H}$ we take the map which assigns to each vector x in $\mathcal{H}$ the set of vectors $\{A_g x : g \text{ in } G\}$. Since the family of operators $\{A_g\}$ forms a group, this map defines a partition function on $\mathcal{H}$. Lemma 2 establishes the hypothesis of Lemma 3 with this partition function and $\mathcal{H}$ for $\mathcal{B}$, so that we can conclude the existence of a norm $\| \ \|'$ on $\mathcal{H}$ in which $\mathcal{H}$ is a Hilbert space and such that $\| A_{g_1} x \|' = \| A_{g_2} x \|'$ for each x in $\mathcal{H}$ and g_1, g_2 in G. In particular $\| x \|' = \| A_g x \|'$ so that each operator A_g is isometric with respect to the norm $\| \ \|'$. Moreover, $\| \ \|'$ can be so chosen that $\| x \|/M \leqq \| x \|' \leqq M \| x \|$ for each vector x in $\mathcal{H}$.

Let $x_1, \cdots$ be an orthonormal basis for $\mathcal{H}$ with respect to the norm $\| \ \|$

(and associated inner product $(\ ,\ )$), and let $y_1, \cdots$ be an orthonormal basis for $\mathcal{H}$ with respect to the norm $\|\ \|'$ (and associated inner product $(\ ,\ )'$). Define a linear transformation P of $\mathcal{H}$ into itself by $Px_i = y_i$; $i = 1, \cdots$. Then

$$(x, y) = (\sum_i (x, x_i) x_i, \sum_i (y, x_i) x_i) = \sum_i (x, x_i)(x_i, y)$$
$$= (\sum_i (x, x_i) y_i, \sum_i (y, x_i) y_i)' = (Px, Py)'.$$

Of course $(P^{-1}x, P^{-1}y) = (x, y)'$, substituting $P^{-1}x$ for x and $P^{-1}y$ for y throughout the above equality. We assert that $P^{-1}A_gP$ is a unitary operator on $\mathcal{H}$ with respect to the norm $\|\ \|$ for each g in G. Indeed,

$$(P^{-1}A_gPx, P^{-1}A_gPy) = (A_gPx, A_gPy)' = (Px, Py)' = (x, y).$$

We note in conclusion that $\| P \|$, $\| P^{-1} \|$ do not exceed M. In fact, if $x = \sum_i \alpha_i x_i$ with $1 = \| x \|^2 = \sum_i | \alpha_i |^2$ is given, then $\| Px \|' = \| \sum_i \alpha_i y_i \|' = 1$ and $\| Px \|/M \leqq \| Px \|' \leqq M \| Px \|$, so that $1/M \leqq \| Px \| \leqq M$.

There are several ways of formulating a conjecture concerning the classical question of whether or not each bounded representation of a group is similar to a unitary representation.

CONJECTURE A. *Every bounded representation of a group by operators on a Hilbert space is similar to a unitary representation.*

CONJECTURE B. *There exists a function f from the positive reals to the positive reals with the property that for each bounded representation $g \to A_g$ with bound M, of a group G by operators on a Hilbert space one can find an invertible operator P such that $P^{-1}A_gP$ is unitary for each g in G and such that $\| P \|$, $\| P^{-1} \|$ do not exceed $f(M)$.*

CONJECTURE C. *Same as* B *with f as the identity transform.*

Each of the above conjectures is clearly stronger than the preceding one. We shall show that B is actually equivalent to A in the next lemma.

LEMMA 4. *Conjecture* A *is equivalent to Conjecture* B.

Proof. Clearly B implies A. Suppose now that A is true. If B is false there exists a sequence of groups $G_1, G_2, \cdots$ and a sequence of representations $g^{(1)} \to A_{g^{(1)}}$, $g^{(2)} \to A_{g^{(2)}}, \cdots$ of these groups, respectively, each with bound M and such that if $N_i = \inf\{\max(\| P_i \|, \| P_i^{-1} \|) : P_i^{-1}A_{g^{(i)}}P_i$ unitary for each $g^{(i)}$ in $G_i\}$ then $\lim_i N_i = \infty$. Let $G = G_1 \otimes G_2 \otimes \cdots$ be the weak direct sum of the groups $G_1, G_2, \cdots$, and let $g \to A_g$ be the direct sum of the representations $g^{(1)} \to A_{g^{(1)}}, \cdots$. The representation $g \to A_g$ of G has bound M. Assuming A, we can find an operator P such that $P^{-1}A_gP$ is unitary for each g in G. Restricted to each direct summand, this similarity induces similarities of all the representations $g^{(i)} \to A_{g^{(i)}}$, each similarity with bound not greater than $\max(\| P \|, \| P^{-1} \|)$—a contradiction. Hence A implies B.

THEOREM 5. *If* B *is true for the free groups on finitely many generators then* B *is true for all groups.*

Proof. Let $g \to A_g$ be a representation of G with bound M. We shall show that this representation is boundedly locally semi-simple with bounding constant $f(M)$. In fact, let $x_1, \cdots, x_n$ in $\mathfrak{H}$ and $g_1, \cdots, g_n$ in G be given. The group G_n generated by $g_1, \cdots, g_n$ is the homomorphic image of F_n, the free group on n generators. Thus the representation $g \to A_g$ of G restricted to G_n gives rise to a representation of F_n with bound M which, by hypothesis, is similar to a unitary representation via an operator P with $\| P \|$, $\| P^{-1} \|$ not exceeding $f(M)$. As in the proof of Theorem 1, we now conclude that $\| PA_{g_i}x_i \| = \| Px_i \|$, $i = 1, \cdots, n$; so that the representation is boundedly locally semi-simple with bounding constant $f(M)$. Hence, by Theorem 1, the representation $g \to A_g$ is similar to a unitary representation via a T such that $\| T \|$, $\| T^{-1} \|$ do not exceed $f(M)$. Thus B follows for all groups.

Note that the proof of Lemma 4 shows that assuming A for the class of groups generated by no more than a countable number of elements implies B for this class (since the group G constructed in the proof would be in this class). Now every group in this class is the homomorphic image of the free group on countably many generators, F_∞, so that assuming A for F_∞ implies A for all groups with a countable number of generators and hence B for the free groups on finitely many generators. With the theorem just proved, this yields:

COROLLARY 6. *If* A *holds for the free group on a countable infinity of generators then* A *and hence* B *holds for all groups.*

We turn our attention now to the question of topological semi-simplicity of algebras of operators. In Theorem 8, we state a necessary and sufficient condition for a representation of a C^*-algebra to be similar to a * representation. Before stating this result, however, it is necessary to introduce some geometrical concepts. In particular we must associate to each configuration of vectors an object which measures its deviation from being an orthonormal set. To this end, we introduce an "inner product" between two sets of n vectors in $\mathcal{H}$. This inner product has as its range of values, operators on $\mathcal{H}$.

DEFINITION 2. *If $\tilde{x} = (x_1, \cdots, x_n)$, $\tilde{y} = (y_1, \cdots, y_n)$ are two n-tuples of vectors in $\mathcal{H}$ with $\mathcal{V}$, $\mathcal{W}$ the spaces generated by $\{x_1, \cdots, x_n\}$; $\{y_1, \cdots, y_n\}$, respectively, we denote by $\langle \tilde{x}, \tilde{y} \rangle$ the operator on $\mathcal{H}$ defined as follows. Let C^n be the space of n-tuples of complex numbers with the usual inner product and let $e_1, \cdots, e_n$ be the basis $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$. Let P be the map of C^n into $\mathcal{V}$ determined by $P(e_i) = x_i$, $i = 1, \cdots, n$, and let Q be the map of C^n into $\mathcal{W}$ determined by $Q(e_i) = y_i$. By Q^* we mean the adjoint map to Q, from $\mathcal{H}$ into C^n (characterized by $(Q^*x, a) = (x, Qa)$, where x is an arbitrary vector in $\mathcal{H}$, a in C^n, the first inner product is taken in C^n, and the second in $\mathcal{H}$). Then $\langle \tilde{x}, \tilde{y} \rangle = PQ^*$.*

We note some of the properties of $\langle\ ,\ \rangle$. As a function on the product of $\mathcal{H}_n$ (the n-fold direct sum of $\mathcal{H}$ with itself) with $\mathcal{H}_n$, this inner product is conjugate bilinear. Indeed with $\tilde{x} = (x_1, \cdots, x_n)$, $\tilde{x}' = (x_1', \cdots, x_n')$, $\tilde{y} = (y_1, \cdots, y_n)$, $\tilde{y}' = (y_1', \cdots, y_n')$, and $\langle \tilde{x}, \tilde{y} \rangle = PQ^*$, $\langle \tilde{x}', \tilde{y} \rangle = P'Q^*$, $\langle \tilde{x}, \tilde{y}' \rangle = PQ'^*$ we have

$$\langle \tilde{x} + \tilde{x}', \alpha\tilde{y} \rangle = (P + P')(\alpha Q)^* = \bar{\alpha}(PQ^* + P'Q^*) = \bar{\alpha}(\langle \tilde{x}, \tilde{y} \rangle + \langle \tilde{x}', \tilde{y} \rangle)$$

and similarly

$$\langle \alpha\tilde{x}, \tilde{y} + \tilde{y}' \rangle = \alpha(\langle \tilde{x}, \tilde{y} \rangle + \langle \tilde{x}, \tilde{y}' \rangle), \qquad \langle \tilde{x}, \tilde{y} \rangle = PQ^* = (QP^*)^* = \langle \tilde{y}, \tilde{x} \rangle^*.$$

With the notation as in the definition, we see that the range of $\langle \tilde{x}, \tilde{y} \rangle$ is contained $\mathcal{V}$ and that the range of $\langle \tilde{x}, \tilde{y} \rangle^* = \langle \tilde{y}, \tilde{x} \rangle = QP^*$ is contained in $\mathcal{W}$. Since the null space of an operator is the complement of the range of its adjoint, we have that the complement of $\mathcal{W}$ in $\mathcal{H}$ is the null space of $\langle \tilde{x}, \tilde{y} \rangle$. Thus, effectively, $\langle \tilde{x}, \tilde{y} \rangle$ is a transformation from $\mathcal{W}$ to $\mathcal{V}$.

We compute the transformation $\langle \tilde{x}, \tilde{y} \rangle$ precisely. With the notation above we have, for z in $\mathcal{H}$:

$$(Q^*z, e_i) = (z, Qe_i) = (z, y_i),$$

so that $Q^*z = \sum_{i=1}^{n} (z, y_i) e_i$, hence $PQ^*z = \sum_{i=1}^{n} (z, y_i) x_i$. Thus $\langle \tilde{x}, \tilde{y} \rangle$ can be expressed symbolically as $(\ , y_1)x_1 + \cdots + (\ , y_n)x_n$. It is immediate from this, that $\langle \tilde{x}, \tilde{x} \rangle$ is a positive operator on $\mathcal{H}$ (as it is from the expression PP^* for $\langle \tilde{x}, \tilde{x} \rangle$), and as such has a (unique) positive square root. We denote this square root by $\langle \tilde{x} \rangle$ and refer to it as "the geometrical norm of $\tilde{x}$ (or of the configuration $x_1, \cdots, x_n$)." The fact that $\langle \tilde{x} \rangle$ is the identity operator on the n-dimensional space $\mathcal{V}$ (so that $x_1, \cdots, x_n$ are linearly independent, in particular) is equivalent to

$$\langle \tilde{x} \rangle^2 = \langle \tilde{x}, \tilde{x} \rangle = (\ , x_1) x_1 + \cdots + (\ , x_n) x_n$$

being the identity transformation on $\mathcal{V}$, which is equivalent to $x_1, \cdots, x_n$ being an orthonormal frame. The spread of the spectrum of $\langle \tilde{x} \rangle$, in general, is a measure of how much $x_1, \cdots, x_n$ deviates from being a scalar multiple of an orthonormal set. In the one-dimensional case, i.e., with x, y vectors in $\mathcal{H}$, we have $\langle x, y \rangle = (\ , y) x$. If we restrict this operator to the one-dimensional space generated by x, it becomes multiplication by (x, y), the usual inner product of x and y.

Suppose, now, that $\mathfrak{A}$ is a C^*-algebra and ϕ is a representation (not necessarily * preserving) of $\mathfrak{A}$ by operators on a Hilbert space $\mathcal{H}$. Employing Theorem 1, we obtain the following criterion for ϕ to be similar to a * representation.

THEOREM 7. *If $\mathcal{U}$ is the unitary group of the C^*-algebra $\mathfrak{A}$, a necessary and sufficient condition for a representation ϕ of $\mathfrak{A}$ by operators on a Hilbert space $\mathcal{H}$ to be similar to a * representation of $\mathfrak{A}$ is that ϕ restricted to $\mathcal{U}$ be a boundedly, locally semi-simple group representation of $\mathcal{U}$.*

Proof. If ϕ is similar to a * representation there exists an operator P on $\mathcal{H}$ such that $P^{-1}\phi(U)P$ is unitary, for each U in $\mathcal{U}$. Thus ϕ restricted to $\mathcal{U}$ is similar to a unitary representation of $\mathcal{U}$; and ϕ is boundedly, locally semi-simple, by Theorem 1. On the other hand, if ϕ is boundedly, locally semi-simple as a representation of $\mathcal{U}$ then, by Theorem 1, there exists an operator P on $\mathcal{H}$ such that $P^{-1}(U)P$ is unitary for each unitary operator U in $\mathfrak{A}$. It now follows that $A \to P^{-1}\phi(A)P$ is a * representation of $\mathfrak{A}$. Indeed, the given map is an algebraic isomorphism of $\mathfrak{A}$. Suppose A is a self-adjoint operator in $\mathfrak{A}$ of norm not exceeding 1. Then $A = \frac{1}{2}(U_1 + U_2)$ where $U_1 = A + i(I - A^2)^{\frac{1}{2}}$ and $U_2 = A - i(I - A^2)^{\frac{1}{2}}$ are unitary operators in $\mathfrak{A}$. Thus $P^{-1}\phi(A)P = \frac{1}{2}[P^{-1}\phi(U_1)P + P^{-1}\phi(U_2)P]$ with $P^{-1}\phi(U_1)P$ and $P^{-1}\phi(U_2)P$ unitary, so that

$$[P^{-1}\phi(U_i)P]^* = [P^{-1}\phi(U_i)P]^{-1} = P^{-1}\phi(U_i^{-1})P = P^{-1}\phi(U_i^*)P; \quad i = 1, 2.$$

Thus

$$[P^{-1}\phi(A)P]^* = P^{-1}\phi(\tfrac{1}{2}(U_1^* + U_2^*))P = P^{-1}\phi(A^*)P = P^{-1}\phi(A)P,$$

so that $A \to P^{-1}\phi(A)P$ takes self-adjoint operators in $\mathfrak{A}$ into self-adjoint operators, and, therefore, is a * representation of $\mathfrak{A}$.

Making use of the foregoing concept of inner product between sets of

n vectors, it is possible to give a more delicate analysis to the question of which representations of C^*-algebras are similar to * representations. If ϕ is a representation of a C^*-algebra $\mathfrak{A}$ by operators on a Hilbert space $\mathcal{H}'$ ($\mathfrak{A}$ acts on $\mathcal{H}$) and $\phi(A)x' = 0$ for some unit vector x' in $\mathcal{H}'$, then for each positive ϵ one can find a unit vector x in $\mathcal{H}$ such that $\| Ax \| < \epsilon$. Indeed, if $\phi(A_i)x' = 0$, $i = 1, \cdots, n$ one can choose the unit vector x so that $\| A_i x \| < \epsilon$, $i = 1, \cdots, n$, i.e., the relations $\phi(A_i)x' = 0$ can be "approximately duplicated" with $\mathfrak{A}$ and $\mathcal{H}$ via ϕ. In fact, the set of operators A such that $\phi(A)x' = 0$ forms a proper left ideal $\mathfrak{J}$ in $\mathfrak{A}$ (proper, since $\phi(I) = I$). If for each unit vector x in $\mathcal{H}$ one of $A_i x$ has norm not less than ϵ then $T = \sum A_i^* A_i \geqq \epsilon I$. But T is then invertible and in $\mathfrak{J}$. This contradiction implies the existence of the desired unit vector x. Given $\epsilon > 0$, vectors $x_1', \cdots, x_n'$ in $\mathcal{H}'$ such that $\sum \| x_i' \|^2 = 1$, and relations $\sum_{i=1}^{n} \phi(A_{hi})x_i' = 0$, $h = 1, \cdots, m$; it is even possible to find vectors $x_1, \cdots, x_n$ in $\mathcal{H}$ with $\sum \| x_i \|^2 = 1$ such that $\| \sum_{i=1}^{n} A_{hi} x_i \| < \epsilon$, $h = 1, \cdots, m$. This can be done by working with the $n \times n$ matrix algebras over $\mathfrak{A}$ and $\phi(\mathfrak{A})$ as we did above with $\mathfrak{A}$ and $\phi(\mathfrak{A})$ themselves. On the other hand, suppose the relations $\sum_{i=1}^{n} \phi(A_{hi})x_i' = 0$, $h = 1, \cdots, m$, subsist with $x_1', \cdots, x_n'$ an orthonormal set in $\mathcal{H}'$; is it possible to choose $x_1, \cdots, x_n$ an orthonormal set in $\mathcal{H}$ such that $\| \sum_{i=1}^{n} A_{hi} x_i \| < \epsilon$, $h = 1, \cdots, m$? This is not necessarily possible on two grounds; a multiplicity consideration, or more simply, the dimension of $\mathcal{H}$ may not be large enough to accommodate an orthonormal set with n vectors, secondly, it is too much to ask for orthonormality of $x_1, \cdots, x_n$ in light of the fact that ϕ may not be a * representation (Theorem 8 shows that if it is possible to choose $x_1, \cdots, x_n$ an orthonormal set then ϕ is already a * representation). The multiplicity question can be avoided by asking whether or not a * representation ψ of $\mathfrak{A}$ can be found (once the relations $\sum_{i=1}^{n} \phi(A_{hi}) x_i' = 0$ and $\epsilon > 0$ are given) such that $\| \sum_{i=1}^{n} \psi(A_{hi}) x_i \| < \epsilon$. As for the orthonormality question, can we at least find bounds, dependent upon the representation ϕ alone, for the distortion of $x_1, \cdots, x_n$ from being an orthonormal set? The technique for measuring this distortion has just been developed. It is not difficult to see that if ϕ is similar to a * representation ψ via an operator P, then ψ will serve for the exact duplication of all relations with the distortion bounded by $\max(\| P \|, \| P^{-1} \|)$ (this will be done in detail in the necessity portion of Theorem 8). These considerations lead us to:

DEFINITION 3. *Let ϕ be a representation of the C^*-algebra $\mathfrak{A}$ by operators on the Hilbert space $\mathcal{H}$, and let $\tilde{x} = (x_1, \cdots, x_n)$ be an n-tuple consisting of vectors $x_1, \cdots, x_n$ which form an orthonormal set in $\mathcal{H}$ such that $(\phi(A_{ij}))\tilde{x} = 0$, where $(\phi(A_{ij}))$ is an $n \times n$ matrix whose entries are operators in $\phi(\mathfrak{A})$. Let ψ be a * representation of $\mathfrak{A}$ by operators on a Hilbert space $\mathcal{H}'$, and let $\tilde{x}' = (x_1', \cdots, x_n')$ be an n-tuple of vectors in $\mathcal{H}'$ such that $\| (\psi(A_{ij})\tilde{x}' \| < \epsilon$, where ϵ is some positive number and the spectrum of $\langle \tilde{x}' \rangle$, as an operator on the space generated by $x_1', \cdots, x'_n$, is contained in the interval $[k, K]$. We say, then, that "$\| (\psi(A_{ij}))\tilde{x}' \| < \epsilon$ is a self-adjoint ϵ cover of the relations $(\phi(A_{ij}))\tilde{x} = 0$ with distortion in $[k, K]$." If there exist constants k, K, $(K > k > 0)$ such that each relation of the form $(\phi(A_{ij}))\tilde{x} = 0$, with $\tilde{x}$ as above and (A_{ij}) a positive operator (in the C^*-algebra consisting of $n \times n$ matrices over $\mathfrak{A}$), has, for each positive ϵ, a self-adjoint ϵ cover with distortion in $[k, K]$, we say that "the representation ϕ has a self-adjoint cover (with distortion in $[k, K]$)."*

We have not made the definition of a representation having a self-adjoint cover as restrictive as we might, in that we require only relations coming from positive $n \times n$ matrices to have self-adjoint ϵ covers. This is all that is needed for each relation to have a cover. It might seem more natural to use the phrase "ϕ has a self-adjoint cover" to mean that for each ϵ there is a self-adjoint representation which serves as a self-adjoint ϵ cover of ϕ for all relations. That this actually follows from the weaker condition used and, indeed, that there is a self-adjoint representation which works for all positive ϵ and all relations is the substance of:

THEOREM 8. *A necessary and sufficient condition for a representation ϕ of a C^*-algebra $\mathfrak{A}$ by operators on a Hilbert space $\mathcal{H}$ to be similar to a * representation is that ϕ have a self-adjoint cover. If the distortion is in $[k, K]$ then a similarity can be effected by a positive operator with spectrum in $[k, K]$.*

Proof. The necessity presents little difficulty. Suppose that there exists an invertible operator T on $\mathcal{H}$ such that $A \to T\phi(A)T^{-1}$ is a * representation of $\mathfrak{A}$, and let $M = \max(\| T \|, \| T^{-1} \|)$. If $(\phi(A_{ij}))\tilde{x} = 0$ is some relation, with $\tilde{x} = (x_1, \cdots, x_n)$, $x_1, \cdots, x_n$ an orthonormal set, then $(T\phi(A_{ij})T^{-1})\tilde{x}' = 0$ is a self-adjoint ϵ cover for this relation (all $\epsilon > 0$), where $\tilde{x}' = (Tx_1, \cdots, Tx_n)$, and where the distortion lies in $[1/M, M]$. Indeed, that $(T\phi(A_{ij})T^{-1})\tilde{x}' = 0$ with the given $\tilde{x}'$ is immediate. Let P be the linear transformation from C^n

into $\mathcal{H}$ defined by $Pe_i = x_i$ (see Definition 2), and let E be the projection on the space generated by $x_1, \cdots, x_n$. Then

$$\langle \tilde{x}', \tilde{x}' \rangle = TPP^*T^* = TET^* = (TE)(TE)^*.$$

Thus $\| \langle \tilde{x}', \tilde{x}' \rangle \| = \| TE \|^2 \leqq \| T \|^2 \leqq M^2$. Moreover

$$\begin{aligned}\inf\{(\langle \tilde{x}', \tilde{x}' \rangle x, x) : \| x \| = 1, x \text{ in } TE\mathcal{H}\} &= \inf\{\| (TE)^* x \|^2\} \\ = \inf\{\| ET^*Ty \|^2 : y \text{ in } E\mathcal{H}, \| Ty \| = 1\} &\geqq \inf\{(T^*Ty, y/\| y \|)^2\} \\ = \inf\{\| Ty \|^4/\| y \|^2\} = \inf\{1/\| y \|^2 : y \text{ in } E\mathcal{H}, \| Ty \| = 1\} &\geqq 1/M^2.\end{aligned}$$

Thus the spectrum of $\langle \tilde{x}', \tilde{x}' \rangle$ as an operator on $TE\mathcal{H}$ lies in $[1/M^2, M^2]$ so that the spectrum of $\langle \tilde{x}' \rangle$ lies in $[1/M, M]$, and $(T\phi(A_{ij})T^{-1})\tilde{x}' = 0$ is a self-adjoint ϵ cover (all $\epsilon > 0$) for $(\phi(A_{ij}))\tilde{x} = 0$ with distortion in $[1/M, M]$. In connection with foregoing inequalities, note that y is in $E\mathcal{H}$ so that the length of the projection of T^*Ty upon $E\mathcal{H}$ is not less than the length of the projection of T^*Ty upon the subspace generated by y (this length being $(T^*Ty, y/\| y \|)$).

Suppose now that the map ϕ has a self-adjoint cover. As in Theorem 1, we show that each finite-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ admits a Hilbert space norm $\| \ \|'$ such that $\| \phi(U)x \|' = \| \phi(V)x \|'$ when U, V are unitary operators in $\mathfrak{A}$ with $\phi(U)x, \phi(V)x$ in $\mathcal{V}$, and such that $k \| y \| \leqq \| y \|' \leqq K \| y \|$ for each y in $\mathcal{V}$. Following Theorem 1, form the conjugate tensor product $\mathcal{V} \otimes \mathcal{V}$ of $\mathcal{V}$ with itself and endow it with the unitary structure described in that theorem. Let $\mathcal{V}\otimes$ be the subspace of $\mathcal{V} \otimes \mathcal{V}$ generated by tensors of the form $x \otimes x$ and $\mathcal{E}\otimes$ the subspace generated by elements $\phi(U)x \otimes \phi(U)x - x \otimes x$, where U is a unitary operator in $\mathfrak{A}$ and $x, \phi(U)x$ are in $\mathcal{V}$. Choose a basis $\phi(U_1)y_1 \otimes \phi(U_1)y_1 - y_1 \otimes y_1, \cdots, \phi(U_m)y_m \otimes \phi(U_m)y_m - y_m \otimes y_m$ for $\mathcal{E}\otimes$ and an orthonormal basis $x_1, \cdots, x_n$ for $\mathcal{V}$. We have

$$\phi(U_i)y_i = \sum_{j=1}^{n} \beta'_{ij}x_j \text{ and } y_i = \sum_{j=1}^{n} \beta_{ij}x_j; \qquad i = 1, \cdots, m,$$

so that

$$0 = \sum_{j=1}^{n} (\beta_{ij}\phi(U_i) - \beta'_{ij})x_j = \sum_{j=1}^{n} \phi(\beta_{ij}U_i - \beta'_{ij}I)x_j; \qquad i = 1, \cdots, m.$$

Let a positive integer r and a positive number δ be given. We wish to establish the existence of a * representation ψ of $\mathfrak{A}$ as operators on a Hilbert space $\mathcal{H}'$ and vectors $x_1', \cdots, x_n'$ in $\mathcal{H}'$ such that $\langle \tilde{x}' \rangle$ has its spectrum in $[k, K]$, with $\tilde{x}' = (x_1', \cdots, x_n')$, and such that

$$\| \sum_{j=1}^{n} \psi(\beta_{ij}U_i - \beta'_{ij}I)x_j' \| < \delta; \qquad i = 1, \cdots, m.$$

We write A_{hj} for $\beta_{hj}U_h - \beta'_{hj}I$; $h = 1, \cdots, m$ and $\phi(A)^{\sim}$ for the $n \times n$ matrix whose i, j entry is the operator $\sum_{h=1}^{m} \phi(A_{hi}^*)\phi(A_{hj})$. Now $\phi(A)^{\sim}\tilde{x} = 0$, where $\tilde{x} = (x_1, \cdots, x_n)$, for

$$\phi(A)^{\sim} = \sum_{h=1}^{m} \begin{bmatrix} \phi(A_{h1}^*) & 0 \cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \phi(A_{hn}^*) & 0 \cdots 0 \end{bmatrix} \cdot \begin{bmatrix} \phi(A_{h1}) & \cdots & \phi(A_{hn}) \\ 0 & \cdots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \phi(A_{h1}) & \cdots & \phi(A_{hn}) \\ 0 & \cdots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{bmatrix} \tilde{x} = 0,$$

$h = 1, \cdots, m$. By hypothesis on ϕ, the relation $\phi(A)^{\sim}\tilde{x} = 0$ has a self-adjoint δ cover with distortion in $[k, K]$. Let ψ be a representation of $\mathfrak{A}$ by operators on the Hilbert space $\mathcal{H}'$ and let $\tilde{x}' = (x_1', \cdots, x_n')$ be a vector such that $\langle \tilde{x}' \rangle$ has spectrum in $[k, K]$ and $\| \psi(A)^{\sim}\tilde{x}' \| < \delta^2 / n^{\frac{1}{2}}K$ where $\psi(A)^{\sim}$ is the $n \times n$ matrix whose i, j entry is $\sum_{h=1}^{m} \psi(A_{hi}^*)\psi(A_{hi})$. In particular then,

$$(\psi(A)^{\sim}\tilde{x}', \tilde{x}') \leqq \| \psi(A)^{\sim} \tilde{x}' \| \cdot \| \tilde{x}' \| < \delta^2$$

for

$$\| \tilde{x}' \| = (\sum_{i=1}^{n} \| x_i' \|^2)^{\frac{1}{2}} \leqq (\sum_{i=1}^{n} K^2)^{\frac{1}{2}} = n^{\frac{1}{2}}K.$$

In fact, since $\langle \tilde{x}' \rangle$ has spectrum in $[k, K]$, $\langle \tilde{x}', \tilde{x}' \rangle = \langle \tilde{x}' \rangle^2$ has spectrum in $[k^2, K^2]$, so that

$$\| \langle \tilde{x}', \tilde{x}' \rangle \| = \| PP^* \| = \| P \|^2 \leqq K^2 \quad \text{and} \quad \| x_i' \| = \| Pe_i \| \leqq K$$

(notation as in Definition 2). Now $(\psi(A)^{\sim}\tilde{x}', \tilde{x}') =$

$$\sum_{h=1}^{m} (\begin{bmatrix} \psi(A_{h1}^*) & 0 \cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \psi(A_{hn}^*) & 0 \cdots 0 \end{bmatrix} \cdot \begin{bmatrix} \psi(A_{h1}) & \cdots & \psi(A_{hn}) \\ 0 & \cdots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{bmatrix} \tilde{x}', \tilde{x}')$$

$$= \sum_{h=1}^{m} \| \begin{bmatrix} \psi(A_{h1}) & \cdots & \psi(A_{hn}) \\ 0 & \cdots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{bmatrix} \tilde{x}' \|^2.$$

Thus $\| \sum_{j=1}^{n} \psi(A_{hj}) x_j' \| < \delta$; $h = 1, \cdots, m$. Let $\mathcal{V}'$ be the subspace of $\mathcal{H}'$ generated by $x_1', \cdots, x_n'$, and let $y_1', \cdots, y_n'$ be an orthonormal basis for $\mathcal{V}'$. Denote by S the linear transformation determined by $Sy_i' = x_i'$. We assert that $SS^* = \langle \tilde{x}', \tilde{x}' \rangle = \sum_{i=1}^{n} (\ , x_i') x_i'$. Indeed, $(S^* x_i', y_j') = (x_i', Sy_j') = (x_i', x_j')$, so that $S^* x_i' = \sum_{j=1}^{n} (x_i', x_j') y_j'$ and

$$SS^* x_i' = \sum_{j=1}^{n} (x_i', x_j') S y_j' = \sum_{j=1}^{n} (x_i', x_j') x_j' = \langle \tilde{x}', \tilde{x}' \rangle (x_i').$$

Thus, since $x_1', \cdots, x_n'$ span $\mathcal{V}'$, $SS^* = \langle \tilde{x}', \tilde{x}' \rangle$ as asserted. It follows that SS^* has its spectrum in $[k^2, K^2]$ from which, $S^{-1*} S^{-1}$ has its spectrum in $[1/K^2, 1/k^2]$, so that $\| S \| \leqq K$ and $\| S^{-1} \| \leqq 1/k$. Let $S^* y_j' = \sum_{i=1}^{n} \alpha_{ij} y_i'$, so that the matrix of the transformation S^* relative to orthonormal basis $\{y_i'\}$ is (α_{ij}). In $\mathcal{V}$ define $y_j'' = \sum_{i=1}^{n} \alpha_{ij} x_i$. We set up a unitary transformation between $\mathcal{V}$ and $\mathcal{V}'$ by means of the map $x_i \to y_i'$. Under this map, we see that $y_j'' \to S^* y_j'$, so that

$$\begin{aligned}
&| (\sum_{j=1}^{n} y_j'' \otimes y_j'', \phi(U_i) y_i \otimes \phi(U_i) y_i - y_i \otimes y_i) | \\
&\quad = | (\sum_{j=1}^{n} S^* y_j' \otimes S^* y_j', [\sum_{h=1}^{n} \beta_{ih}' y_h'] \otimes [\sum_{h=1}^{n} \beta_{ih}' y_h'] - [\sum_{h=1}^{n} \beta_{ih} y_h'] \otimes [\sum_{h=1}^{n} \beta_{ih} y_h']) | \\
&\quad = | \sum_{j=1}^{n} (S^* y_j', \sum_{h=1}^{n} \beta_{ih}' y_h') (\sum_{h=1}^{n} \beta_{ih}' y_h', S^* y_j') \\
&\qquad - \sum_{j=1}^{n} (S^* y_j', \sum_{h=1}^{n} \beta_{ih} y_h') (\sum_{h=1}^{n} \beta_{ih} y_h', S^* y_j') | \\
&\quad = | \sum_{j=1}^{n} \{ | (y_j', \sum_{h=1}^{n} \beta_{ih}' x_h') |^2 - | (y_j', \sum_{h=1}^{n} \beta_{ih} x_h') |^2 \} | \\
&\quad = | \, \| \sum_{h=1}^{n} \beta_{ih}' x_h' \|^2 - \| \sum_{h=1}^{n} \beta_{ih} x_h' \|^2 | = | \, \| \sum_{h=1}^{n} \beta_{ih}' x_h' \|^2 - \| \psi(U_i) (\sum_{h=1}^{n} \beta_{ih} x_h') \|^2 | \\
&\quad \leqq \| \sum_{h=1}^{n} (\psi(U_i) \beta_{ih} x_h' - \beta_{ih}' x_h') \| \cdot [\| \sum_{h=1}^{n} \beta_{ih}' x_h' \| + \| \sum_{h=1}^{n} \psi(U_i) \beta_{ih} x_h' \|] \\
&\qquad \leqq \| \sum_{h=1}^{n} \psi(A_{ih}) x_h' \| (2n\beta K) \leqq 2n\beta K \delta,
\end{aligned}$$

where $\beta = \max \{ | \beta_{ih} |, | \beta_{ih}' | ; \ i = 1, \cdots, m; \ h = 1, \cdots, n \}$.

In connection with the above inequality, note that we have proved that $\| x_i' \| \leqq K$, and that $\psi(U_i)$ is a unitary operator on $\mathcal{H}'$ since ψ is a * representation (with $\psi(I)=I$). We now specify the choice of δ as $1/2n\beta Kr$ (all the constants that appear in this choice were determined before the introduction of δ). Our inequality becomes then:

$$|(\sum_{j=1}^{n} y_j''\otimes y_j'', \phi(U_i)y_i\otimes\phi(U_i)y_i - y_i\otimes y_i)| \leqq 1/r; \qquad i=1,\cdots,m.$$

We write $y_j(r)$ for y_j'' to indicate the dependence of the y_j'' upon r. Observe that $y_j(r)$ lies in the sphere of radius K and outside the sphere of radius k, center at 0, in $\mathcal{V}$ since $y_j(r)$ is the image of S^*y_j' under our unitary map between $\mathcal{V}$ and $\mathcal{V}'$ (and $\| y_j' \| = 1$, $\| S^* \| \leqq K$, $\| S^{*-1} \| \leqq 1/k$). By compactness, one can choose a subsequence $\{r_h\}$ of r's such that $\lim_h y_j(r_h) = z_j$; $j=1,\cdots,n$. Clearly

$$(\sum_{j=1}^{n} z_j\otimes z_j, \phi(U_i)y_i\otimes\phi(U_i)y_i - y_i\otimes y_i) = 0; \qquad i=1,\cdots,m,$$

and the z_j lie between the spheres of radii k and K with center at 0 in $\mathcal{V}$. We consider the norm $\|\ \|'$ induced on $\mathcal{V}$ by means of the definition:

$$(\| x \|')^2 = (\sum_{j=1}^{n} z_j\otimes z_j, x\otimes x) = \sum_{j=1}^{n} |(z_j, x)|^2.$$

We have just proved that $\sum_{j=1}^{n} z_j\otimes z_j$ is orthogonal to $\mathcal{E}\otimes$ so that $\| \phi(U)y \|' = \| y \|'$ if both $\phi(U)y$ and y are in $\mathcal{V}$. Thus, if $\phi(U)y$ and $\phi(V)y = \phi(VU^*)\phi(U)y$ are in $\mathcal{V}$ then $\| \phi(U)y \|' = \| \phi(V)y \|'$. For x an arbitrary vector in $\mathcal{V}$, we have $(\| x \|')^2 =$

$$\lim_h (\sum_{j=1}^{n} y_j(r_h)\otimes y_j(r_h), x\otimes x) = \lim_h (\sum_{j=1}^{n} S_h^* y_j'(h)\otimes S_h^* y_j'(h), x'(h)\otimes x'(h))$$

$$= \lim_h \sum_{j=1}^{n} |(y_j'(h), S_h x'(h))|^2 = \lim_h \| S_h x'(h) \|^2,$$

where $y_j'(h)$ are the vectors corresponding to y_j' in the foregoing discussion (with r_h now replacing r) S_h is the S of that discussion and $x'(h)$ is the image of x under the unitary map between $\mathcal{V}$ and $\mathcal{V}_h'$ of the present discussion. Now

$$\lim_h \| S_h x'(h) \|^2 \leqq \lim_h K^2 \| x'(h) \|^2 = \lim_h K^2 \| x \|^2 = K^2 \| x \|^2,$$

$$\lim_h \| S_h x'(h) \|^2 \geqq \lim_h k^2 \| x'(h) \|^2 = \lim_h k^2 \| x \|^2 = k^2 \| x \|^2,$$

so that $k \| x \| \leqq \| x \|' \leqq K \| x \|$.

If we take as partition classes in $\mathcal{H}$ the sets $\{\phi(U)x: U$ a unitary operator in $\mathfrak{A}\}$, we arive at a situation satisfying the hypotheses of Lemma 3, so that $\mathcal{H}$ admits a Hilbert space norm in which $\phi(U)$ is unitary for U a unitary operator in $\mathfrak{A}$ (this norm equivalent to the original norm with constants k, K). Thus, as at the end of the proof of Theorem 1, we can find an operator P with $P^{-1}\phi(U)P$ unitary for each unitary operator U in $\mathfrak{A}$ and $\| P \|$, $\| P^{-1} \|$ do not exceed $\max(K, 1/k)$. It now follows, as in the proof of Theorem 7, that the representation $A \rightarrow P^{-1}\phi(A)P$ is a * representation of $\mathfrak{A}$. Writing the polar decomposition HU, U unitary, $H = (PP^*)^{\frac{1}{2}}$ for P, we have $A \rightarrow H^{-1}\phi(A)H$ is a * representation of $\mathfrak{A}$ with H positive and having spectrum in $[k, K]$.

3. Concluding remarks. The discussion preceding Theorem 8 and Definition 3, concerning the approximate duplication of relations draws very heavily upon the fact that the initial algebra is a C^*-algebra (in particular, is uniformly closed) for the fact that an invertible operator in the algebra has its inverse in the algebra. On the other hand, Definition 3 and Theorem 8 apply as they stand to self-adjoint (not necessarily closed) algebras (although they are not stated this way). It follows immediately from this that:

COROLLARY 9. *A representation of a group by bounded operators on a Hilbert space is similar to a unitary representation if and only if the extension of this representation to the* (*finite, translation*) *group algebra* (*acting on* L_2 *of the group*) *has a self-adjoint cover.*

Despite the applicability of Definition 3 and Theorem 8 to self-adjoint algebras which are not uniformly closed, it should not be felt that the general conjecture about operator algebras has application to the non-closed, self-adjoint algebras. That is, examples are easily constructed of algebras which are not similar to self-adjoint algebras but are algebraically isomorphic to non-closed, self-adjoint algebras (not the continuous image, of course). In fact, let $x_1, x_2, \cdots$ be a sequence of linearly independent unit vectors which tend (strongly) to x and which span the Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be the algebra of bounded operators A on $\mathcal{H}$ which have the form $Ax_i = \alpha_i x_i$ for some sequence $\{\alpha_i\}$ of complex numbers, and let $\mathcal{S}$ be the set of sequences which arise in this manner ($\mathcal{S}$ contains all sequences which have only a finite number of non-zero terms). Let $y_1, y_2, \cdots$ be an orthonormal basis for $\mathcal{H}$ and let $\mathfrak{A}'$ be the algebra of operators B of the form $By_i = \alpha_i y_i$ where $\{\alpha_i\}$ is in $\mathcal{S}$. Then $\mathfrak{A}'$ is a self-adjoint algebra containing I, for $B^* y_i = \bar{\alpha}_i y_i$ and $\{\bar{\alpha}_i\}$ is in $\mathcal{S}$ if and only if $\{\alpha_i\}$ is in $\mathcal{S}$ (the x_i's being so chosen that the transformation $\alpha_1 x_1 + \cdots + \alpha_n x_n \rightarrow \bar{\alpha}_1 x_1 + \cdots + \bar{\alpha}_n x_n$ is bounded). Moreover the map $A \rightarrow B$ of $\mathfrak{A}$ onto $\mathfrak{A}'$ where $Ax_i = \alpha_i x_i$ and $By_i = \alpha_i y_i$ is an algebraic isomorphism (which is continuous, since $\| A \| \geqq \sup_i | \alpha_i | = \| B \|$). For each invertible operator P and each operator A in $\mathfrak{A}$, the operator $P^{-1}AP$ has $P^{-1}x_i$ as eigenvectors and these converge to $P^{-1}x$. Now the algebra $\mathfrak{A}$ is commutative (as is $P^{-1}\mathfrak{A}P$) so that, if $P^{-1}\mathfrak{A}P$ is self-adjoint then $P^{-1}AP$ is normal for each A in $\mathfrak{A}$. Given $i \neq j$ we can easily find a sequence $\{\alpha_p\}$ in $\mathcal{S}$ with $\alpha_i \neq \alpha_j$ (let $\alpha_i = 1$, $\alpha_p = 0$ for $p \neq i$). Let A be the operator in $\mathfrak{A}$ with sequence $\{\alpha_p\}$. If $P^{-1}AP$ is normal then $P^{-1}x_i$ and $P^{-1}x_j$ are orthogonal. Thus if $P^{-1}\mathfrak{A}P$ is self-adjoint it follows that $P^{-1}x_i$, $i = 1, 2, \cdots$ is a set of mutually orthogonal vectors, which we have just seen cannot be the case.

We commented briefly, in the introduction, on the topological difficulty present in the infinite-dimensional case concerning the geometrical interpretation of semi-simplicity. By making suitable corrections for this difficulty, one arrives at a geometrical condition which might suffice for an algebra of operators to be similar to a self-adjoint algebra of operators. The conjecture obtained is quite natural in that it corrects for all the immediately visible difficulties which occur in passing from the finite to the infinite-dimensional case. For the moment, we specifically avoid describing the topology in which the operator algebra in question is closed.

Let $\mathfrak{A}$ be an algebra of operators on a Hilbert space with the property that there exists a positive δ such that if $\mathcal{V}$ is a closed subspace (setwise) invariant under the operators of $\mathfrak{A}$ then there exists a complementary closed invariant subspace $\mathcal{W}$ (i. e., *$\mathcal{V} + \mathcal{W}$ is the whole space and $\mathcal{V} \cap \mathcal{W} = (0)$*) *which makes an angle greater than δ with $\mathcal{V}$. Is $\mathfrak{A}$ similar to a self-adjoint algebra?*

Note that since the angle between $\mathcal{W}$ and $\mathcal{V}$ is assumed to be positive their linear sum is closed. Let us assume that the answer to this question is yes (with any closure assumption on $\mathfrak{A}$) and that A is a bounded operator on the Hilbert space $\mathcal{H}$ with no closed invariant subspaces other than (0) and $\mathcal{H}$. Let $\mathfrak{A}$ be the (commutative) algebra generated by A and the identity operator (the closure taken in the appropriate topology). Since $\mathfrak{A}$ has no closed non-trivial invariant subspaces, the hypothesis is vacuously satisfied and there exists an operator P such that $P^{-1}\mathfrak{A}P$ is self-adjoint. Since $P^{-1}\mathfrak{A}P$ is commutative, it consists of normal operators. In particular $P^{-1}AP$ is normal and has an abundance of non-trivial, closed, invariant subspaces. If $\mathcal{V}$ is such a subspace then $P\mathcal{V}$ is non-trivial, closed and invariant under

A—a contradiction. Again, if $\mathfrak{A}$ is an irreducible algebra of operators then the hypothesis are trivially satisfied, and an affirmative answer to the question would imply that $\mathfrak{A}$ is similar to a self-adjoint algebra. Making use of this remark, we can answer the question in the uniformly closed case negatively. Our own approach to this counter-example rested upon producing a uniformly closed irreducible operator algebra containing an invertible operator whose inverse didn't lie in the operator algebra (note that this can't occur in an algebra which is similar to a C^*-algebra). A much more cogent device was suggested to us by I. Kaplansky. Using the completely continuous operators as a basic irreducible set of operators, build a closed operator algebra over it whose quotient by the completely continuous operators is a (finite-dimensional), non-semi-simple algebra. The larger algebra is not even the isomorphic image of a C^*-algebra, for a quotient algebra of a C^*-algebra is again a C^*-algebra [8] and therefore semi-simple. A concrete example is obtained by taking as our algebra the algebra generated by the completely continuous operators, the identity operator, and a nilpotent operator of index two (say a partial isometry between an infinite-dimensional subspace and its orthogonal complement).

In conclusion, we note the simple fact that a representation of a group by uniformly bounded operators each of which is normal is itself a unitary representation. In fact, an invertible normal operator all of whose powers form a set which is uniformly bounded in norm must have its spectrum on the unit circle and is therefore unitary.

Added in proof (June 1, 1955): In a recent note, (*Proceedings of the National Academy of Sciences,* vol. 41 (1955)) F. Mautner and L. Ehrenpreis announce that the group question has a negative answer, i. e., they produce a group and a bounded representation of it which is not similar to a unitary representation. Presumably, then, the "distortion continuity" condition of Theorem 8 cannot be removed. Restricting attention to relations involving n or fewer vectors, we can discuss representations satisfying an "n-distortion continuity" condition—the boundedness of a group representation (or continuity of a C^*-algebra representation) amounts to "1-distortion continuity." We feel that there are groups and representations of them which have n but not $n+1$ distortion continuity.

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